AN ALGEBRA OF DISTRIBUTIONS ON AN OPEN INTERVAL

BY

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ABSTRACT. Let \((a, b)\) be any open subinterval of the reals which contains the origin and let \(\mathcal{B}\) denote the family of all distributions on \((a, b)\) which are regular in some interval \((\epsilon, 0)\), where \(\epsilon < 0\). Then \(\mathcal{B}\) is a commutative algebra: Multiplication is defined so that, when restricted to those distributions on \((a, b)\) whose supports are contained in \([0, b)\), it is ordinary convolution. Also, \(\mathcal{B}\) can be injected into an algebra of operators; this family of operators is a sequentially complete locally convex space. Since it preserves multiplication, this injection serves as a generalization (there are no growth restrictions) of the two-sided Laplace transformation.

In [6] there is introduced a new algebra \(\mathcal{B}'\) of distributions on \((-\infty, \infty)\), closed under convolution and containing the space of distributions having support in \([0, \infty)\) as well as all locally integrable functions. No growth or support restrictions are placed on the elements of \(\mathcal{B}'\). There is also defined a one-to-one transformation of \(\mathcal{B}'\) into a commutative algebra of operators (somewhat analogous to the Fourier transformation). In the present article we generalize these results in obtaining a space \(\mathcal{B}\) of distributions on \(\Omega\), where \(\Omega\) is any open subinterval of the reals which contains the origin. A distribution \(F\) on \(\Omega\) belongs to \(\mathcal{B}\) if and only if \(F\) is regular in some interval \((\epsilon, 0)\), where \(\epsilon < 0\). Convolution is defined and \(\mathcal{B}\) is shown to be closed under this operation. It is also shown that the algebra \(\mathcal{B}\) into which \(\mathcal{B}\) can be injected is a sequentially complete locally convex space in which convergence is defined simply in terms of the ordinary pointwise convergence of functions.

0. Preliminaries. Throughout we assume \(-\infty < a < 0 < b < \infty\) and set \(\Omega = (a, b)\). We define \(L\) to be the space of all the complex-valued functions which are Lebesgue integrable on each compact subinterval of \(\Omega\). We denote by \(L_+\) (respectively, \(L_-\)) the subspace consisting of those elements of \(L\) which vanish on \((a, 0)\) (respectively, \((0, b)\)). If \(f\) and \(g\) belong to \(L\) then the function \(f \wedge g\) defined by the equation

\[
(0.01) \quad f \wedge g(t) = \int_0^t f(t - u)g(u)\,du \quad (t \in \Omega)
\]
also belongs to \( L \), moreover, if we identify functions which are equal almost everywhere on \( \Omega \) then

\[
f' \wedge g = g \wedge f
\]

(see [4]). For any \( f \) in \( L \) we define

\[
f_+(t) = \begin{cases} f(t), & 0 \leq t < b, \\ 0, & t < 0, \end{cases} \quad \text{and} \quad f_-(t) = \begin{cases} 0, & t \geq 0, \\ f(t), & a < t < 0. \end{cases}
\]

If \( \Omega_0 \) is an open subinterval of the reals we denote by \( \mathcal{D}(\Omega_0) \) the space of complex-valued infinitely differentiable functions defined on the reals which vanish outside of a compact subset of \( \Omega_0 \). If \( \phi \in \mathcal{D}(\Omega_0) \) we define the support of \( \phi \), denoted \( \text{supp} \phi \), to be the closure of the set \( \{ t : \phi(t) \neq 0 \} \). Then \( \text{supp} \phi \subseteq \Omega_0 \) for all \( \phi \) in \( \mathcal{D}(\Omega_0) \).

As usual, the dual of \( \mathcal{D}(\Omega_0) \), that is, the space of distributions on \( \Omega_0 \), is denoted by \( \mathcal{D}'(\Omega_0) \). If \( R \) belongs to \( \mathcal{D}'(\Omega_0) \) and \( \phi \) belongs to \( \mathcal{D}(\Omega_0) \) the scalar which \( R \) assigns to \( \phi \) will be written \( \langle R(x), \phi(x) \rangle \). If \( f \) belongs to the family of locally integrable functions on \( \Omega_0 \) and \( m \) is a nonnegative integer we shall write \( \partial^m f \) for the element of \( \mathcal{D}'(\Omega_0) \) defined by

\[
\langle \partial^m f(x), \phi(x) \rangle = (-1)^m \int_{\Omega_0} f(x) \phi^{(m)}(x) \, dx \quad (\phi \in \mathcal{D}(\Omega_0)).
\]

In particular, \( \partial^0 f \) is the regular distribution corresponding to the function \( f \). The support of a distribution \( R \) on \( \Omega_0 \) (denoted \( \text{supp} R \)) is defined to be the complement with respect to \( \Omega_0 \) of the largest open set on which \( R \) vanishes.

1. The algebra \( \mathfrak{B} \). We denote by \( \mathcal{D}'_b \) the space of elements in \( \mathcal{D}'((-\infty, b)) \) having support in \([0, b)\). We denote by \( \mathcal{D}'_a \) the space of elements in \( \mathcal{D}'((a, \infty)) \) having support in \((a, 0]\).

1.01. Definition. Suppose \( \{b_n\} \) and \( \{a_n\} \) are sequences of real numbers, that \( \{F_n\} \) and \( \{K_n\} \) are sequences of nonnegative integers and that \( \{F_n\} \) and \( \{G_n\} \) are sequences in \( L \). If the ordered pair \( (R, S) \) belongs to the cartesian product \( \mathcal{D}'_b \times \mathcal{D}'_a \) we say that the sequence \( \{(F_n, b_n, J_n, G_n, a_n, K_n)\} \) belongs to \( \Sigma_{R,S} \) if

\[
\begin{align*}
(1.01.1) & \quad a_0 < a_2 < a_1 < a_0 = 0 = b_0 < b_1 < b_2 < \cdots \rightarrow b; \\
(1.01.2) & \quad F_n \text{ vanishes on } (-\infty, b) \quad \text{and} \quad G_n \text{ vanishes on } (a, \infty); \\
(1.01.3) & \quad R = \sum_{n=0}^{\infty} \partial^n F_n \quad \text{and} \quad S = \sum_{n=0}^{\infty} \partial^n K_n G_n.
\end{align*}
\]

1.02. Theorem. Given any \( (R, S) \) in \( \mathcal{D}'_b \times \mathcal{D}'_a \) and any sequences \( \{b_n\} \) and \( \{a_n\} \) satisfying (1.01.1) there exists an element \( \{(F_n, b_n, J_n, G_n, a_n, K_n)\} \) of \( \Sigma_{R,S} \).
Proof. By [8, 2.17] there exists a sequence \( \{F_n\} \) in \( L \) such that the equation
\[
R = \sum_{n=0}^{\infty} \partial^n F_n
\]
holds for some sequence \( \{n\} \) of nonnegative integers. We define an element \( T \) of \( \mathcal{D}'((\infty, -a)) \) as follows:

(1) \[
\langle T(x), \phi(x) \rangle = \langle S(x), \phi(-x) \rangle \quad (\phi \in \mathcal{D}'((\infty, b, -a))).
\]

Then, since \( \text{supp } S \subset (a, 0) \), the distribution \( T \) has support contained in \( [0, -a) \).

Since \( 0 = -a_0 < -a_1 < \cdots < -a \) we may infer from [8, 2.17] the existence of a sequence \( \{H_n\} \) in \( L^{1\infty}((\infty, -a)) \) such that \( H_n \) vanishes on \( (-\infty, -a_n) \) and such that the equation

(2) \[
T = \sum_{n=0}^{\infty} \partial^n H_n
\]
holds for some sequence \( \{K_n\} \) of nonnegative integers. If we define
\[
G_n(x) = (-1)^{K_n} H_n(-x)
\]
then \( G_n \) vanishes on \( (a_n, \infty) \) and we may combine (1) and (2) to obtain
\[
S = \sum_{n=0}^{\infty} \partial^n G_n.
\]

Therefore, \( \{(F_n, b_n, j_n, G_n, a_n, K_n)\} \in \Sigma_{R, s} \).

1.03. Definition. For each \( \phi \) in \( \mathcal{D}'((\infty, b)) \) we define \( [\phi]^+ \) to be the family of infinitely differentiable functions \( \lambda \) on the reals such that \( \lambda \) is equal to 1 on a neighborhood of \( [0, \infty) \) and vanishes on some interval \( (-\infty, a') \), where \( \text{supp } \phi \subset (-\infty, a' + b) \). For each \( \phi \) in \( \mathcal{D}'((a, \infty)) \) we define \( [\phi]^+ \) to be the family of infinitely differentiable functions \( \mu \) on the reals such that \( \mu \) is equal to 1 on a neighborhood of \( (-\infty, 0) \) and vanishes on some interval \( (b', \infty) \), where \( \text{supp } \phi \subset (a + b', \infty) \).

1.04. Theorem. Suppose \( (r, s) \) and \( (R, S) \) belong to \( \mathcal{D}'_b \times \mathcal{D}'_a \). If
\[
\{(f_n, b_n, j_n, g_n, a_n, k_n)\} \text{ belongs to } \Sigma_{r,s} \text{ and } \{(F_n, b_n, j_n, G_n, a_n, K_n)\} \text{ belongs to } \Sigma_{R,S},
\]
then for any \( \phi \) in \( \mathcal{D}'((\infty, b)) \) the equation
\[
(1.04.1) \quad \langle r(y), \langle R(x), \lambda(y)\phi(x + y) \rangle \rangle = \lim_{N \to \infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \langle \partial^{j_n + l_n}(f_m \wedge F_n)(x), \phi(x) \rangle
\]
holds for all \( \lambda \) in \( [\phi]^+ \), and for any \( \phi \) in \( \mathcal{D}'((a, \infty)) \) the equation

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holds for all $\mu$ in $[\phi]^+$. 

Proof. Suppose $\phi \in \mathcal{D}(-\infty, b))$ and $\lambda \in [\phi]^+$. There exist numbers $\beta$ and $a'$ such that 

$$
\text{supp } \phi \subset (-\infty, \beta] \subset (-\infty, a' + b)
$$

and such that $\lambda$ vanishes on $(-\infty, a')$. For any $y$ the function $x \mapsto \lambda(y) \phi(x + y)$ is infinitely differentiable. From (1) it follows that its support is contained in $(-\infty, \beta - y]$. Thus, for $y \geq a'$, its support is contained in $(-\infty, \beta - a']$ and therefore in $(-\infty, b)$. And, for $y < a'$, it vanishes identically (since $\lambda(y) = 0$). Consequently, the function $x \mapsto \lambda(y) \phi(x + y)$ belongs to $\mathcal{D}((-\infty, b))$ and has support in $(-\infty, \beta - a')$ for all $y$. There exists $N$ such that $b_{N+1} > \beta - a'$ and therefore 

$$
\langle R(x), \lambda(y) \phi(x + y) \rangle = \sum_{n=0}^{\infty} (-1)^n \int_0^b F_n(x) \lambda(y) \phi^{(n)}(x + y) \, dx
$$

for all $y$ (recall that $F_n$ vanishes on $(-\infty, b_n)$). From (2) and [2, 250] it follows that the function 

$$
y \mapsto \langle R(x), \lambda(y) \phi(x + y) \rangle
$$

is infinitely differentiable. From (1) comes the equality 

$$
\langle R(x), \lambda(y) \phi(x + y) \rangle = 0 \quad (\text{all } y > \beta)
$$

(recall supp $R \subset [0, b)$). And 

$$
\langle R(x), \lambda(y) \phi(x + y) \rangle = 0 \quad (\text{all } y < a')
$$

since $\lambda$ vanishes on $(-\infty, a')$. Consequently, the function (3) belongs to $\mathcal{D}((-\infty, b))$. Moreover, we may combine (4) and the inequality $b_{N+1} > \beta$ to obtain 

$$
\int_{0^{+}}^b \mu(y) \langle R(x), \lambda(y) \phi(x + y) \rangle \, dy = 0 \quad (\text{all } m > N).
$$

Therefore,
\((r(y), (R(x), \lambda(y)\phi(x + y)))\)

\[
= \sum_{m=0}^{N} (-1)^m \int_{0}^{b} \frac{d^m}{dy^m}(R(x), \lambda(y)\phi(x + y)) \, dy
\]

\[
= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{m+n} \int_{0}^{b} \left( \int_{0}^{b} F_n(x) \phi^{(m+n)}(x + y) \, dx \right) \, dy;
\]

the last equality is from (2), [2, 250] and the fact that \(\lambda = 1\) on \([0, \infty)\). We may now use the change of variable \(t = x + y\) and [2, 283] to obtain

\[
(r(y), (R(x), \lambda(y)\phi(x + y)))
\]

\[
= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{m+n} \int_{0}^{b} \left( \int_{0}^{b} F_n(t) \phi^{(m+n)}(t) \, dt \right) \, dy
\]

\[
= \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{m+n} \int_{0}^{b} \left( \int_{0}^{b} F_n(t) \phi^{(m+n)}(t) \, dt \right) \, dy.
\]

We need only observe now that

\[
\int_{0}^{t} f_m(y) F_n(t - y) \, dy = 0 \quad (0 \leq t \leq \beta)
\]

for \(m > N\) and \(n > N\) to obtain (1.04.1). Suppose now that \(\phi \in \mathcal{D}((a, \infty))\) and \(\mu \in [\phi]^\ominus\). There exist numbers \(a\) and \(b'\) such that

\[
\text{supp } \phi \subset [a, \infty) \subset (a + b', \infty)
\]

and such that \(\mu\) vanishes on \((b', \infty)\). For any \(y\) the function \(x \mapsto \mu(y)\phi(x + y)\) is infinitely differentiable. From (5) it follows that its support is contained in \([a - y, \infty)\). Thus, for \(y \leq b'\), its support is contained in \([a - b', \infty)\) and therefore in \((a, \infty)\). And, for \(y > b'\), it vanishes identically (since \(\mu(y) = 0\)). Consequently, the function \(x \mapsto \mu(y)\phi(x + y)\) belongs to \(\mathcal{D}((a, \infty))\) and has support in \([a - b', \infty)\) for all \(y\). There exists \(N\) such that \(a_{n+1} < a - b'\) and therefore

\[
(S(x), \mu(y)\phi(x + y)) = \sum_{n=0}^{\infty} (-1)^n \int_{a}^{0} G_n(x) \mu(y) \phi^{(n)}(x + y) \, dx
\]

\[
= \sum_{n=0}^{N} (-1)^n \int_{a}^{0} G_n(x) \mu(y) \phi^{(n)}(x + y) \, dx
\]

for all \(y\) (recall that \(G_n\) vanishes on \((a_n, \infty)\)). From (6) and [2, 250] it follows that the function
is infinitely differentiable. From (5) comes the equality

\[ \langle S(x), \mu(y)\phi(x + y) \rangle = 0 \quad (\text{all } y < a) \]

(recall \( \text{supp } S \subset (a, 0] \)). And

\[ \langle S(x), \mu(y)\phi(x + y) \rangle = 0 \quad (\text{all } y > b') \]

since \( \mu \) vanishes on \((b', \infty)\). Consequently, the function (7) belongs to \( \mathcal{D}'((a, \infty)) \).

Moreover, we may combine (8) and the inequality \( a_{N+1} < \alpha \) to obtain

\[ \int_a^0 g_m(y) \langle S(x), \mu(y)\phi(x + y) \rangle \, dy = 0 \quad (\text{all } m > N). \]

Therefore,

\[ \langle s(y), \langle S(x), \mu(y)\phi(x + y) \rangle \rangle \]

\[ = \sum_{m=0}^{N} (-1)^m \int_a^0 g_m(y) \frac{d^m}{dy^m} \langle S(x), \mu(y)\phi(x + y) \rangle \, dy \]

\[ = \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{m+N} \int_a^0 g_m(y) \left( \int_a^0 G_n(x) \phi^{(m+N)}(x + y) \, dx \right) \, dy; \]

the last equality is from (5), \([2, 250]\) and the fact that \( \mu = 1 \) on \((-\infty, 0]\). We may now use the change of variable \( t = x + y \) and \([2, 283]\) to obtain

\[ \langle s(y), \langle S(x), \mu(y)\phi(x + y) \rangle \rangle \]

\[ = \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{m+N} \int_a^0 \left( \int_a^y g_n(m)(t - y) \phi^{(m+N)}(t) \, dt \right) \, dy \]

\[ = \sum_{m=0}^{N} \sum_{n=0}^{N} (-1)^{m+N} \int_a^0 \left( \int_t^y g_n(m)(t - y) \phi^{(m+N)}(t) \, dt \right) \, dy. \]

We need only observe now that

\[ \int_a^0 g_m(y) G_n(t - y) \, dy = 0 \quad (a \leq t \leq 0) \]

for \( m > N \) and \( n > N \) and that

\[ - \int_t^0 g_m(y) G_n(t - y) \, dy = g_n \wedge G_n(t) \]

to obtain (1.04.2).
1.05. **Corollary.** Suppose that \((r, s)\) and \((R, S)\) belong to \(\mathcal{D}'_b \times \mathcal{D}'_a\). For any \(\phi\) in \(\mathcal{D}((- \infty, b))\) the family
\[
\{(r(y), (R(x), \lambda(y)\phi(x + y))): \lambda \in [\phi]^+\}
\]
contains a unique element, which will be denoted by \(\langle r \ast R(x), \phi(x) \rangle\). For any \(\phi\) in \(\mathcal{D}((a, \infty))\) the family
\[
\{(s(y), (S(x), \mu(y)\phi(x + y))): \mu \in [\phi]^-\}
\]
contains a unique element, which will be denoted by \(\langle s \ast S(x), \phi(x) \rangle\).

1.06. **Definition.** Let \((r, s)\) and \((R, S)\) belong to \(\mathcal{D}'_b \times \mathcal{D}'_a\). We denote by \(r \ast R\) the functional that assigns to any \(\phi\) in \(\mathcal{D}((- \infty, b))\) the number \(\langle r \ast R(x), \phi(x) \rangle\); we denote by \(s \ast S\) the functional that assigns to any \(\phi\) in \(\mathcal{D}((a, \infty))\) the number \(\langle s \ast S(x), \phi(x) \rangle\).

1.07. **Remark.** It follows from 1.04 and (0.02) that \(r \ast R = R \ast r\) and \(s \ast S = S \ast s\).

1.08. **Corollary.** If \((r, s)\) and \((R, S)\) belong to \(\mathcal{D}'_b \times \mathcal{D}'_a\) then \((r \ast R, s \ast S)\) belongs to \(\mathcal{D}'_b \times \mathcal{D}'_a\).

**Proof.** It follows from (1.04.1) and the sequential completeness of \(\mathcal{D}'((- \infty, b))\) (see [1, Proposition 2, p. 315]) that \(r \ast R\) belongs to \(\mathcal{D}'((- \infty, b))\). Moreover, since \(r \ast R\) is the limit of a sequence of distributions on \((- \infty, b)\) all having support in \([0, b)\), it too must have support in \([0, b)\), i.e. \(r \ast R \in \mathcal{D}'_b\). Similarly, \(s \ast S \in \mathcal{D}'_a\).

1.09. **Remark.** If \(f\) belongs to \(L\) then \(\partial^0 f_+ \in \mathcal{D}'_b\) and \(\partial^0 f_- \in \mathcal{D}'_a\). We may deduce from 1.01 and 1.04 that the equations \(\partial^0 f_+ \ast \partial^0 g_+ = \partial^0 (f_+ \wedge g_+)\) and \(- \partial^0 f_- \ast \partial^0 g_- = \partial^0 (f_- \wedge g_-)\) hold for all \(f_+\) and \(g_+\) in \(L\).

1.10. **Definition.** We denote by \(\mathcal{B}_-\) the linear subspace consisting of those elements of \(\mathcal{D}'_a\) which are regular in a neighborhood of the origin.

1.11. **Remark.** Thus \(S \in \mathcal{B}_-\) if and only if \(S = \partial^0 f_+ + T\) where \(f_+ \in L_+\) and \(T \in \mathcal{D}'_a\) with \(\text{supp } T \subset (a, 0]\). In particular, \(\partial^0 f_- \in \mathcal{B}_-\) for all \(f_- \in L_-\).

1.12. **Lemma.** Suppose \((R, S)\) and \((R_1, S_1)\) belong to \(\mathcal{D}'_b \times \mathcal{B}_-\). If the elements \(R + S\) and \(R_1 + S_1\) of \(\mathcal{D}'(\Omega)\) are equal then \(R = R_1\) and \(S = S_1\).

**Proof.** Since \(S\) and \(S_1\) both vanish on \((0, b)\) we have
\[
S = S_1 \quad \text{on } (0, b).
\]
Since \(R\) and \(R_1\) both vanish on \((a, 0)\) it follows from \(R + S = R_1 + S_1\) that
\[
S = S_1 \quad \text{on } (a, 0).
\]
Now, there exists $\epsilon > 0$ and elements $f$ and $f_1$ of $L_-$ such that $S = \partial^0 f$ and $S_1 = \partial^0 f_1$ on $(-\epsilon, \epsilon)$. From (1) and (2) it follows that $\partial^0 f = \partial^0 f_1$ on $(0, \epsilon)$ and on $(-\epsilon, 0)$. Thus, by [9, p. 224] we have $f = f_1$ almost everywhere on $(-\epsilon, \epsilon)$, from which it follows that $\partial^0 f = \partial^0 f_1$ on $(-\epsilon, \epsilon)$. Therefore,

(3) \[ S = \partial^0 f = \partial^0 f_1 = S_1 \quad \text{on} \quad (-\epsilon, \epsilon). \]

We may now combine (1), (2) and (3) to conclude that $S = S_1$ on $(a, b)$ (see [9, Theorem 24.1]).

1.13. Definition. We denote by $\mathcal{B}$ the linear subspace consisting of those elements of $\mathcal{D}'(\Omega)$ of the form $R + S$ where $R \in \mathcal{D}_b'$ and $S \in \mathcal{B}_-$. 

1.14. Remark. Thus, $F \in \mathcal{B}$ if and only if $F \in \mathcal{D}'(\Omega)$ and is regular in some neighborhood $(\epsilon, 0)$, where $a < \epsilon < 0$. In particular, $\partial^0 f \in \mathcal{B}$ for all $f \in L$.

1.15. Theorem. If $F$ belongs to $\mathcal{B}$ there exists a unique element of $\mathcal{D}'_b \times \mathcal{B}_-$, denoted $(F_+, F_-)$, such that $F = F_+ + F_-$. 

Proof. Immediate from 1.12.

1.16. Corollary. The mapping $F \mapsto (F_+, F_-)$ is an isomorphism of $\mathcal{B}$ into $\mathcal{D}'_b \times \mathcal{D}'_a$.

Proof. One may easily verify that the mapping $F \mapsto (F_+, F_-)$ is linear. The corollary then follows from 1.15.

1.17. Lemma. If $V$ and $V_1$ belong to $\mathcal{D}'_a$ with $\text{supp} V \subset (a, \epsilon)$ for some $\epsilon < 0$ then $\text{supp} V \ast V_1 \subset (a, \epsilon)$.

Proof. Let $\phi \in \mathcal{D}((a, \infty))$ and have support in $[\epsilon', \infty)$, where $\epsilon < \epsilon' < 0$. Then, for $y < \epsilon' - \epsilon$ the function $x \mapsto \phi(x + y)$ has support contained in $(\epsilon, \infty)$. Therefore,

$$\langle V_1(x), \mu(y)\phi(x + y) \rangle = 0 \quad \text{for all} \quad \mu \in [\phi]^-$$

for all $y < \epsilon' - \epsilon$. Thus, the function $y \mapsto \langle V_1(x), \mu(y)\phi(x + y) \rangle$ has support contained in $(0, \infty)$ for all $\mu \in [\phi]^-$. Since $V$ vanishes on this interval it follows that

$$\langle V \ast V_1(x), \phi(x) \rangle = \langle V(y), \langle V_1(x), \mu(y)\phi(x + y) \rangle \rangle = 0$$

for all $\mu \in [\phi]^-$. Therefore, $V \ast V_1$ vanishes on $(\epsilon, \infty)$.

1.18. Theorem. If $F$ and $G$ belong to $\mathcal{B}$ then $F_+ \ast G_+ - F_- \ast G_-$ belongs to $\mathcal{B}$ with $(F_+ \ast G_+ - F_- \ast G_-)_- = -F_- \ast G_-$. 

Proof. It suffices to show that $F_- \ast G_- \in \mathcal{B}_-$. By 1.11 there exist $f$ and $g$ in $L_-$ and $T$ and $U$ in $\mathcal{D}'_a$ such that $F_- = \partial^0 f + T$ and $G_- = \partial^0 g + U$ with
supp $T \subset (a, \epsilon]$ and supp $U \subset (a, \epsilon]$ for some $\epsilon < 0$. Therefore,

$$F_- \ast G_- = (\partial^0 f + T) \ast (\partial^0 g + U) = (\partial^0 f) \ast (\partial^0 g) + d^0 f \ast U + T \ast (\partial^0 g) + T \ast U.$$  

From 1.09 and 1.07 it follows that

$$F_- \ast G_- = \partial^0 (f \wedge g) + (\partial^0 f) \ast U + (\partial^0 g) \ast T + T \ast U.$$  

If we set $S = (\partial^0 f) \ast U + (\partial^0 g) \ast T + T \ast U$ we may infer from 1.17 that supp $S \subset (a, \epsilon]$ and therefore $F_- \ast G_- \in \mathcal{B}_-$.

1.19. Definition. If $F$ and $G$ belong to $\mathcal{B}$ we denote the element $F_+ \ast G_+ - F_- \ast G_- \in \mathcal{B}$ by $F \wedge G$.

1.20. Remark. As a consequence of 2.23, the space $\mathcal{B}$, with multiplication defined by 1.19, is a commutative algebra.

1.21. Theorem. The equation $\partial^0 (f \wedge g) = (\partial^0 f) \wedge (\partial^0 g)$ holds for all $f$ and $g$ in $L$.

Proof. Since $f \wedge g = f_+ \wedge g_+ + f_- \wedge g_-$ we may use 1.09 to obtain

$$\partial^0 (f \wedge g) = \partial^0 (f_+ \wedge g_+) + \partial^0 (f_- \wedge g_-) = \partial^0 f_+ \ast \partial^0 g_+ - \partial^0 f_- \ast \partial^0 g_-$$

$$= (\partial^0 f)_+ \ast (\partial^0 g)_+ - (\partial^0 f)_- \ast (\partial^0 g)_- = (\partial^0 f) \wedge (\partial^0 g).$$

2. The algebra of operators. Let $W$ be the space of all the complex-valued infinitely differentiable functions $w$ on $\Omega$ such that $w^{(k)}(0) = 0$ for $k \geq 0$. In [4] it is shown that $f \wedge w$ belongs to $W$ with

$$\langle f \wedge w \rangle = \langle f \rangle \wedge \langle w \rangle$$

whenever $f$ belongs to $L$ and $w$ belongs to $W$. We denote by $\langle f \rangle$ the operator which assigns to each $w$ in $W$ the function $f \wedge w$ in $W$. Thus, $\langle f \rangle w = f \wedge w$ (all $w$ in $W$). Let $\mathcal{A}$ be the set of all the operators $A$ mapping $W$ into itself such that

$$A(w_1 \wedge w_2) = (Aw_1) \wedge w_2$$

for all $w_1$ and $w_2$ in $W$. We make $\mathcal{A}$ into a vector space by defining addition and scalar multiplication in the usual way. We define the product of two operators to be the composition of the operators. Then $\mathcal{A}$ is a commutative algebra which contains the identity operator $I$ and the differentiation operator $D$; moreover, the mapping $f \mapsto \langle f \rangle$ is a linear injection of $L$ into $\mathcal{A}$ and

$$\langle f \wedge g \rangle = \langle f \rangle \langle g \rangle$$

for all $f$ and $g$ in $L$ (see [4]).

2.04. Theorem. If $\{F_n, b_n, J_n, G_n, a_n, K_n\}$ belongs to $\Sigma_{R,S}$ for some $(R, S)$ in $\mathcal{D}_b^' \times \mathcal{D}_a^'$ then the equation
\[
Aw(t) = \sum_{n=0}^{\infty} \left( (F_n \land w)^{(J_n)}(t) + (G_n \land w)^{(K_n)}(t) \right) \quad (t \in \Omega, w \in W)
\]

defines an element of \( \mathfrak{A} \).

**Proof.** Let \( w \in W \) and \( a < \alpha < \beta < b \). Then there exists a positive integer \( N \) such that \( a_{N+1} < \alpha < \beta < b_{N+1} \). Since \( F_N \) vanishes on \((a, b)\) we may infer that \( F_N \land w \) vanishes on \((a, \beta)\) for all \( n > N \); since \( G_N \) vanishes on \((a, b)\) we may infer that \( G_N \land w \) vanishes on \((a, \beta)\) for all \( n > N \). Consequently,
\[
Aw(t) = \sum_{n=0}^{N} \left( (F_n \land w)^{(J_n)}(t) + (G_n \land w)^{(K_n)}(t) \right) \quad (a < t < \beta).
\]

Since each \( F_n \land w \) and each \( G_n \land w \) is infinitely differentiable on \((a, \beta)\) it follows from (1) that \( Aw \) is infinitely differentiable on \((a, \beta)\); and, clearly, every derivative of \( Aw \) vanishes at the origin since the same is true of each term on the right-hand side of (1). Since \((a, \beta)\) was an arbitrary open subinterval of \( \Omega \) we may conclude that \( Aw \in W \). There remains to show that the equation \( A(w_1 \land w_2) = (Aw_1) \land w_2 \) holds for all \( w_1 \) and \( w_2 \) in \( W \). But, using (2.01) and the fact that \( \langle F_n \rangle \) and \( \langle G_n \rangle \) belong to \( \mathfrak{A} \) we may deduce that
\[
A(w_1 \land w_2) = \sum_{n=0}^{\infty} \left( (F_n \land (w_1 \land w_2))^{(J_n)} + (G_n \land (w_1 \land w_2))^{(K_n)} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( (F_n \land w_1)^{(J_n)} \land w_2 + (G_n \land w_1)^{(K_n)} \land w_2 \right)
\]
\[
= \left( \sum_{n=0}^{\infty} ((F_n \land w_1)^{(J_n)} + (G_n \land w_1)^{(K_n)}) \right) \land w_2 = (A_w_1) \land w_2.
\]

For each \( w \) in \( W \) and each \( t \) in \( \Omega \) the equation \( \rho_{w,t}(A) = |Aw(t)| \) defines a seminorm on the space \( \mathfrak{A} \). Let \( \mathfrak{A} \) be endowed with the locally convex topology defined by the family of seminorms \( \{\rho_{w,t}: t \in \Omega, w \in W\} \).

2.05. **Remark.** If \( \{A_n\} \) is a sequence in \( \mathfrak{A} \) then \( A_0 = \lim A_n \) if and only if \( A_0 w(t) = \lim A_n w(t) \) for all \( w \) in \( W \) and all \( t \) in \( \Omega \).
2.06. Definition. If $0 < t < b$ we denote by $[t]$ the set of all infinitely differentiable functions $p$ which assume the value $1$ on a neighborhood of $[0, \infty)$ and which vanish on some interval $(-\infty, \alpha_p)$, where $t - b < \alpha_p$. If $a < t < 0$ we let $[t]$ denote the set of all infinitely differentiable functions $q$ which assume the value $1$ on a neighborhood of $(-\infty, 0]$ and which vanish on some interval $(\beta_q, \infty)$, where $\beta_q < t - a$.

2.07. Definition. If $w$ belongs to $W$ and $0 < t < b$ we define

$$w_t(x) = \begin{cases} w(t - x), & t - b < x < t, \\ 0, & \text{otherwise}; \end{cases}$$

if $w$ belongs to $W$ and $a < t < 0$ we define

$$w_t(x) = \begin{cases} w(t - x), & t < x < t - a, \\ 0, & \text{otherwise}. \end{cases}$$

2.08. Remark. If $w \in W$ and $0 < t < b$, the function $w_t$ is infinitely differentiable on $(t - b, \infty)$ and vanishes on $(t, \infty)$; thus, if $p \in [t]$, the function $pw_t$ (the pointwise product of the functions $p$ and $w_t$) belongs to $\mathcal{D}((0, \infty))$ with $\text{supp } pw_t \subseteq (-\infty, t]$. If $w \in W$ and $a < t < 0$, the function $w_t$ is infinitely differentiable on $(-\infty, t - a)$ and vanishes on $(-\infty, t)$; thus, if $q \in [t]$, the function $qw_t$ belongs to $\mathcal{D}((t, \infty))$ with $\text{supp } qw_t \subseteq [t, \infty)$.

2.09. Lemma. If $f$ belongs to $L$ and $m$ is a nonnegative integer, the equation

$$\langle \partial^m f(x), p(x)w_t(x) \rangle = (f \wedge w)^{(m)}(t)$$

holds for any $p$ in $[t]$ and any $w$ in $W$ and the equation

$$- \langle \partial^m f(x), q(x)w_t(x) \rangle = (f \wedge w)^{(m)}(t)$$

holds for any $q$ in $[t]$ and any $w$ in $W$.

Proof. Let $0 < t < b$ and $p \in [t]$. For any $w \in W$,

$$\langle \partial^m f(x), p(x)w_t(x) \rangle = (-1)^m \int_0^b f(x)[pw_t]^{(m)}(x) \, dx.$$
By observing that \((-1)^m[w_t]^m(x) = w^m(t - x)\) for \(x > t - b\), we may use (2.01) to obtain

\[
\langle \partial^m f(x), p(x)w_t(x) \rangle = \int_0^t f(x)w^m(t - x)\,dx = (f \wedge w)^m(t).
\]

Now, let \(a < t < 0\) and \(q \in [t]\). For any \(w \in W\),

\[
\langle \partial^m f(x), q(x)w_t(x) \rangle = (-1)^m \int_a^t f(x)[qw_t]^m(x)\,dx
\]

\[
= (-1)^m \int_a^t f(x)[w_t]^m(x)\,dx = \int_a^t f(x)w^m(t - x)\,dx
\]

\[
= -\int_a^t f(x)w^m(t - x)\,dx = -(f \wedge w)^m(t).
\]

2.10. Theorem. If \((R, S)\) belongs to \(D'_b \times D'_a\) there exists an element \(A\) of \(\mathfrak{A}\) such that the equation

(2.10.1) \(\langle R(x), p(x)w_t(x) \rangle = Aw(t)\) \((0 < t < b)\)

holds for any \(p\) in \([t]\) and any \(w\) in \(W\) and such that the equation

(2.10.2) \(-\langle S(x), q(x)w_t(x) \rangle = Aw(t)\) \((a < t < 0)\)

holds for any \(q\) in \([t]\) and any \(w\) in \(W\). If \(\{(F_n, b_n, J_n, G_n, a_n, K_n)\}\) belongs to \(\Sigma_{R, S}\) then

(2.10.3) \(A = \sum_{n=0}^{\infty} (D^J_{n+1} F_n + D^K_{n+1} G_n)\).

Proof. Let \(\{(F_n, b_n, J_n, G_n, a_n, K_n)\} \in \Sigma_{R, S}\). Then, by 2.09,

(1) \(\langle R(x), p(x)w_t(x) \rangle = \sum_{n=0}^{\infty} \langle \partial^J_n F_n(x), p(x)w_t(x) \rangle = \sum_{n=0}^{\infty} (F_n \wedge w)^J(t)\)

for \(0 < t < b\), any \(p \in [t]\) and any \(w\) in \(W\). Similarly,

(2) \(-\langle S(x), q(x)w_t(x) \rangle = \sum_{n=0}^{\infty} -\langle \partial^K_n G_n(x), q(x)w_t(x) \rangle = \sum_{n=0}^{\infty} (G_n \wedge w)^K(t)\)

for \(a < t < 0\), any \(q \in [t]\) and any \(w\) in \(W\). If we define

\[
Aw(t) = \sum_{n=0}^{\infty} ((F_n \wedge w)^J(t) + (G_n \wedge w)^K(t))\)

\((t \in \Omega, w \in W)\).
then $A \in \mathcal{A}$ by 2.04. Moreover, since each $F_n \in L_+$ and each $G_n \in L_-$ we have

$$
A \omega(t) = \begin{cases} 
\sum_{n=0}^{\infty} (F_n \wedge \omega)^{(J_n)}(t) & \text{for } 0 \leq t < b, \\
\sum_{n=0}^{\infty} (G_n \wedge \omega)^{(K_n)}(t) & \text{for } a < t < 0,
\end{cases}
$$

from which follows the theorem.

2.11. Corollary. Suppose that $(R, S)$ belongs to $\mathcal{D}_b \times \mathcal{D}_a$ and $\omega \in \mathcal{W}$. If $0 \leq t < b$, the family $\{(R(x), p(x)\omega(x)) : p \in [i]\}$ contains a unique element. If $a < t < 0$, the family $\{(S(x), q(x)\omega(x)) : q \in [i]\}$ contains a unique element. If we define

$$
\langle (R, S) \rangle \omega(t) = \begin{cases} 
\langle R(x), p(x)\omega(x) \rangle & (p \in [i], 0 \leq t < b) \\
- \langle S(x), q(x)\omega(x) \rangle & (q \in [i], a < t < 0)
\end{cases}
$$

and denote by $\langle (R, S) \rangle$ the mapping $\omega \mapsto \langle (R, S) \rangle \omega$, then $\langle (R, S) \rangle \in \mathcal{A}$.

2.12. Corollary. If $(R, S) \in \mathcal{D}_b \times \mathcal{D}_a$ and $\{(F_n, b_n, J_n, G_n, a_n, K_n) : n \in \mathbb{N}\} \in \Sigma_{R, S}$, then

$$
\langle (R, S) \rangle = \sum_{n=0}^{\infty} (D^J_n \langle F_n \rangle + D^K_n \langle G_n \rangle).
$$

Proof. Immediate from (2.10.3).

2.13. Corollary. The equation $\langle \partial^m f_+ + \partial^n f_- \rangle = D^m \langle f_+ \rangle + D^n \langle f_- \rangle$ holds for all $f$ in $L$ and all nonnegative integers $m$ and $n$.

Proof. Immediate from 2.09.

2.14. Lemma. There exists a sequence $\{w_n\}$ in $\mathcal{W}$ such that

$$
A = \lim_{n \to \infty} \langle A \omega_n \rangle
$$

for all $A$ in $\mathcal{A}$.

Proof. Choose $w_n$ in $\mathcal{W}$ satisfying

1. $w_n \geq 0$ on $(0, b)$,
2. $w_n \leq 0$ on $(a, 0)$,
3. $w_n(t) = 0$ for $|t| \geq 1/n$,
4. $\int_0^b w_n(x) \, dx = 1 = -\int_0^a w_n(x) \, dx$
Let $A \in \mathcal{A}$ and $w \in \mathcal{W}$. Then, for $0 < t < b$ and $n$ sufficiently large so that $t - 1/n > 0$,

$$(Aw) w_n(t) = \int_{t-1/n}^{t} Aw(x)w_n(t-x)\,dx.$$ 

And, by (3)–(4), we may write

$$Aw(t) = \int_{t-1/n}^{t} Aw(t)w_n(t-x)\,dx.$$ 

Consequently,

$$| (Aw) w_n(t) - Aw(t) | = \left| \int_{t-1/n}^{t} [Aw(x) - Aw(t)]w_n(t-x)\,dx \right|$$

$$\leq \sup_{|x-t| \leq 1/n} |Aw(x) - Aw(t)| \int_{t-1/n}^{t} w_n(t-x)\,dx$$

$$= \sup_{|x-t| \leq 1/n} |Aw(x) - Aw(t)|.$$ 

Using (5) and the continuity of $Aw$ at $t$ we obtain

$$Aw(t) = \lim_{n \to \infty} (Aw) w_n(t) \quad (0 < t < b).$$

For $a < t < 0$ and $n$ sufficiently large so that $t + 1/n > 0$,

$$(Aw) w_n(t) = - \int_{t}^{t+1/n} Aw(x)w_n(t-x)\,dx.$$ 

And, by (3)–(4), we may write

$$Aw(t) = - \int_{t}^{t+1/n} Aw(t)w_n(t-x)\,dx.$$ 

Consequently,

$$| (Aw) w_n(t) - Aw(t) | = \left| - \int_{t}^{t+1/n} [Aw(x) - Aw(t)]w_n(t-x)\,dx \right|$$

$$\leq \sup_{|x-t| \leq 1/n} |Aw(x) - Aw(t)| \left( - \int_{t}^{t+1/n} w_n(t-x)\,dx \right)$$

$$= \sup_{|x-t| \leq 1/n} |Aw(x) - Aw(t)|.$$ 

Using (7) and the continuity of $Aw$ at $t$ we obtain
\( Aw(t) = \lim_{n \to \infty} (Aw)_n(t) \quad (a < t < 0). \)

Observing that \( (Aw) \wedge w_n = (Aw_n) \wedge w \) and the fact that \( (Aw_n) \wedge w(0) = 0 = Aw(0) \)
we infer from (6) and (8) that \( Aw(t) = \lim_{n \to \infty} (Aw_n) \wedge w(t) \) (all \( t \in \Omega \)) and there-
therefore that \( A = \lim (Aw_n). \)

2.15. Remark. It follows from 2.14 that each \( A \) in \( \mathcal{A} \) is linear.

2.16. Lemma. If \( \{ (R_n, S_n) \} \) is a sequence in \( \mathcal{D}'_b \times \mathcal{D}'_a \) and if \( R_0 = \lim R_n \)
and \( S_0 = \lim S_n \) then \( \langle (R_0, S_0) \rangle = \lim \langle (R_n, S_n) \rangle. \)

Proof. Let \( w \in \mathcal{W} \). If \( R_0 = \lim R_n \) and \( S_0 = \lim S_n \) then
\[
\langle R_0(x), \phi(x) \rangle = \lim_{n \to \infty} \langle R_n(x), \phi(x) \rangle \quad (\text{all } \phi \in \mathcal{D}((-\infty, b))),
\]
\[
\langle S_0(x), \phi(x) \rangle = \lim_{n \to \infty} \langle S_n(x), \phi(x) \rangle \quad (\text{all } \phi \in \mathcal{D}((a, \infty))).
\]
Therefore, for \( 0 < t < b, \)
\[
\langle (R_0, S_0) \rangle w(t) = \langle R_0(x), p(x)w_t(x) \rangle
\]
\[
= \lim_{n \to \infty} \langle R_n(x), p(x)w_t(x) \rangle = \lim_{n \to \infty} \langle (R_n, S_n) \rangle w(t) \quad (\text{all } p \in [t]),
\]
and, for \( a < t < 0, \)
\[
\langle (R_0, S_0) \rangle w(t) = -\langle S_0(x), q(x)w_t(x) \rangle
\]
\[
= \lim_{n \to \infty} -\langle S_n(x), q(x)w_t(x) \rangle = \lim_{n \to \infty} \langle (R_n, S_n) \rangle w(t) \quad (\text{all } q \in [t]).
\]

2.17. Definition. For any \( \phi \) in \( \mathcal{D}((-\infty, \infty)) \) and any real \( t \) we define
\( \phi_t(x) = \phi(t - x) \) for all \( x. \)

2.18. Theorem. The mapping \( (R, S) \mapsto \langle R, S \rangle \) is a linear bijection of
\( \mathcal{D}'_b \times \mathcal{D}'_a \) onto \( \mathcal{A}. \)

Proof. It is easily seen that the mapping is linear. We show first that it is
"onto." Let \( A \in \mathcal{A} \) and define
\[
\phi_t(x) = \begin{cases} 
Aw_n(x) & \text{for } 0 \leq x < b, \\
g_n(x) & \text{for } a < x < 0, \\
0 & \text{for } x < 0;
\end{cases} \quad \phi_t(x) = \begin{cases} 
Aw_n(x) & \text{for } a < x < 0, \\
0 & \text{for } x > 0.
\end{cases}
\]
(see 2.14). Then \( \partial_0^t \psi_n \in \mathcal{D}'_b \) and \( \partial_0^t g_n \in \mathcal{D}'_a \). For any \( \phi \) in \( \mathcal{D}((-\infty, b)) \) there
exists \( t \in (0, b) \) such that \( \text{supp } \phi \subset (-\infty, t] \) and therefore \( \phi_t \in \mathcal{W}. \) Thus,
(1) \[ \langle \partial^0 f_n(x), \phi(x) \rangle = \langle \partial^0 f_n(x), (\phi_t)_\tau(x) \rangle = \langle f_n \rangle \phi_t(t). \]

Combining (1) and 2.14 we have \( \lim_{n \to \infty} \langle \partial^0 f_n(x), \phi(x) \rangle = A(\phi_t)(t) \). Thus the sequence \( \{ \langle \partial^0 f_n(x), \phi(x) \rangle \} \) converges for all \( \phi \) in \( \mathcal{D}((-\infty, b)) \). By [1, Proposition 2, p. 315] there exists \( R \in \mathcal{D}((-\infty, b)) \) such that \( R = \lim \partial^0 f_n \); it is easily seen that \( R \in \mathcal{D}_a \). For any \( \phi \) in \( \mathcal{D}((a, \infty)) \) there exists \( t \in (a, 0) \) such that \( \text{supp} \phi \subset [t, \infty) \) and therefore \( \phi_t \in \mathcal{W} \). Thus,

(2) \[ \langle \partial^0 g_n(x), \phi(x) \rangle = \langle \partial^0 g_n(x), (\phi_t)_\tau(x) \rangle = \langle g_n \rangle \phi_t(t). \]

Combining (2) and 2.14 we have \( \lim_{n \to \infty} \langle \partial^0 g_n(x), \phi(x) \rangle = A(\phi_t)(t) \). Thus the sequence \( \{ \langle \partial^0 g_n(x), \phi(x) \rangle \} \) converges for all \( \phi \) in \( \mathcal{D}((a, \infty)) \). We may similarly infer the existence of \( S \) in \( \mathcal{D}_a \) such that \( S = \lim \partial^0 g_n \). We may now use 2.16, 2.13 and 2.14 to obtain

\[
\langle (R, S) \rangle = \left( \lim_{n \to \infty} \partial^0 f_n, \lim_{n \to \infty} \partial^0 g_n \right)
= \lim_{n \to \infty} \langle \partial^0 f_n, \partial^0 g_n \rangle = \lim_{n \to \infty} \langle A w_n \rangle = A;
\]

whence the mapping \( (R, S) \mapsto \langle (R, S) \rangle \) is "onto." If \( A = 0 \) then each \( f_n \) and each \( g_n \) equal 0, from which it follows that \( R = \lim_{n \to \infty} \partial^0 f_n = 0 \) and \( S = \lim_{n \to \infty} \partial^0 g_n = 0 \). The mapping \( (R, S) \mapsto \langle (R, S) \rangle \) is therefore one-to-one.

2.19. Theorem. The space \( \mathcal{L} \) is sequentially complete.

Proof. Suppose \( \{ \lambda_n \} \) is a Cauchy sequence in \( \mathcal{L} \). By 2.17 there exists a unique \( (R_n, S_n) \) in \( \mathcal{D}_b \times \mathcal{D}_a \) such that \( \langle (R_n, S_n) \rangle = A_n \) and, by assumption, the sequence \( \{ \langle (R_n, S_n) \rangle \} \) converges for all \( w \) in \( \mathcal{W} \) and all \( t \in \Omega \). For any \( \phi \) in \( \mathcal{D}((-\infty, b)) \) there exists \( t \in (0, b) \) such that \( \text{supp} \phi \subset (-\infty, t) \) and therefore \( \phi_t \in \mathcal{W} \). Since \( \langle (R_n(x), \phi(x)) = \langle R_n(x), p(x)(\phi_t)_\tau(x) \rangle = \langle (R_n(x), S_n(x)) \rangle \phi_t(t) \) for all \( p \in [\tau] \), the sequence \( \{ \langle (R_n(x), \phi(x)) \} \) converges for all \( \phi \) in \( \mathcal{D}((-\infty, b)) \). For any \( \phi \) in \( \mathcal{D}((a, \infty)) \) there exists \( t \in (a, 0) \) such that \( \text{supp} \phi \subset [t, \infty) \) and therefore \( \phi_t \in \mathcal{W} \). Since \( \langle S_n(x), \phi(x) \rangle = \langle S_n(x), p(x)(\phi_t)_\tau(x) \rangle = \langle (R_n, S_n) \rangle \phi_t(t) \) the sequence \( \{ \langle S_n(x), \phi(x) \} \) converges for all \( \phi \) in \( \mathcal{D}((a, \infty)) \). We may again use [1, Proposition 2, p. 315] to infer the existence of \( (R, S) \) in \( \mathcal{D}_b \times \mathcal{D}_a \) such that \( R = \lim R_n \) and \( S = \lim S_n \). By 2.16 we then have

\[
\langle (R, S) \rangle = \lim_{n \to \infty} \langle (R_n, S_n) \rangle = \lim_{n \to \infty} A_n;
\]

2.20. Lemma. If \( (r, s) \) and \( (R, S) \) belong to \( \mathcal{D}_b \times \mathcal{D}_a \) then \( \langle (r \cdot R, -s \cdot S) \rangle = \langle (r, s) \rangle \langle (R, S) \rangle \).
Proof. Let \( \{(F_n, b_n, J_n, G_n, a_n, K_n)\} \in \Sigma_{R,S} \) and \( \{(f_n, b_n, j_n, g_n, a_n, k_n)\} \in \Sigma_{r,s} \). By 1.05 and 1.04,
\[
\begin{align*}
    r * R &= \lim_{N \to \infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \partial^{i + j} (f_m \land F_n), \\
    -s * S &= \lim_{N \to \infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \partial^{k + l} (g_m \land G_n).
\end{align*}
\]
Therefore, by 2.16,
\[
\begin{align*}
    \langle (r * R, - s * S) \rangle &= \lim_{N \to \infty} \sum_{m=0}^{N} \sum_{n=0}^{N} \langle \partial^{i + j} (f_m \land F_n), \partial^{k + l} (g_m \land G_n) \rangle \\
    &= \lim_{N \to \infty} \sum_{m=0}^{N} \sum_{n=0}^{N} (D^{i + j} (f_m \land F_n) + D^{k + l} (g_m \land G_n)) \\
    &= \lim_{N \to \infty} \sum_{m=0}^{N} \sum_{n=0}^{N} (D^{i} (f_m) D^{j} (F_n) + D^{k} (g_m) D^{l} (G_n));
\end{align*}
\]
the second equality is from 2.13 and the third equality is from (2.03). Let \( w \in W \) and \( t \in \Omega \). Choose \( N \) sufficiently large so that \( a_{N+1} < t < b_{N+1} \). Suppose first that \( 0 < t < b \). Then
\[
\begin{align*}
    \langle (r * R, - s * S) \rangle w(t) &= \sum_{m=0}^{N} \sum_{n=0}^{N} D^{i} (f_m) D^{j} (F_n) w(t) \\
    &= \sum_{m=0}^{N} D^{i} (f_m) \left( \sum_{n=0}^{N} D^{j} (F_n) w \right)(t) \\
    &= \sum_{m=0}^{\infty} D^{i} (f_m) \left( \sum_{n=0}^{N} D^{j} (F_n) w \right)(t) \\
    &= \langle (r, s) \rangle \left( \sum_{n=0}^{N} D^{j} (F_n) w \right)(t);
\end{align*}
\]
the last equality is from 2.12. Therefore, by 2.15 and the fact that \( \mathcal{G} \) is a commutative algebra, we have
\begin{align*}
\langle (r \ast R, - s \ast S) \rangle w(t) &= \sum_{n=0}^{N} D^{Jn} \langle F_n \rangle \langle ((r, s)) \rangle w(t) \\
&= \sum_{n=0}^{\infty} D^{Jn} \langle F_n \rangle \langle ((r, s)) \rangle w(t) = \langle (R, S) \rangle \langle ((r, s)) \rangle w(t) \\
&= \langle (R, S) \rangle \langle (r, s) \rangle w(t) = \langle (r, s) \rangle \langle (R, S) \rangle w(t).
\end{align*}

And, if \( a < t < 0 \), then

\begin{align*}
\langle (r \ast R, - s \ast S) \rangle w(t) &= \sum_{m=0}^{N} \sum_{n=0}^{N} D^{k} \langle g_m \rangle D^{\infty} \langle G_n \rangle w(t) \\
&= \sum_{m=0}^{N} D^{k} \langle g_m \rangle \left( \sum_{n=0}^{N} D^{\infty} \langle G_n \rangle w(t) \right)(t) = \sum_{n=0}^{\infty} D^{k} \langle g_m \rangle \left( \sum_{n=0}^{N} D^{\infty} \langle G_n \rangle w(t) \right)(t) \\
&= \langle (r, s) \rangle \left( \sum_{n=0}^{N} D^{\infty} \langle G_n \rangle w(t) \right)(t) = \sum_{n=0}^{\infty} D^{\infty} \langle G_n \rangle \langle ((r, s)) \rangle w(t) = \langle (R, S) \rangle \langle ((r, s)) \rangle w(t) = \langle (r, s) \rangle \langle (R, S) \rangle w(t).
\end{align*}

2.21. Definition. For any \( F \) in \( B \) we denote the element \( \langle (F_+, F_-) \rangle \) of \( A \) by \( \langle F \rangle \).

2.22. Theorem. The equation \( \langle \partial^0 f \rangle = \langle f \rangle \) holds for all \( f \) in \( L \).

Proof. Observing that \( \langle \partial^0 f \rangle_+ = \partial^0 f_+ \) and \( \langle \partial^0 f \rangle_- = \partial^0 f_- \) we may combine 2.21 with 2.13 to obtain the theorem.

2.23. Theorem. The mapping \( F \mapsto \langle F \rangle \) is an isomorphism of \( B \) into \( A \) and the equation \( \langle F \wedge G \rangle = \langle F \rangle \langle G \rangle \) holds for all \( F \) and \( G \) in \( B \).

Proof. The first assertion comes from combining 1.16 and 2.18. As for the second, since \( \langle F \wedge G \rangle = \langle F_+ \ast G_+ - F_- \ast G_- \rangle = \langle F_+ \ast G_+ - F_- \ast G_- \rangle \) (see 1.18 and 1.19), we may use 2.20 to obtain \( \langle F \wedge G \rangle = \langle (F_+, F_-) \rangle \langle (G_+, G_-) \rangle = \langle F \rangle \langle G \rangle \).

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