SPACES OF SET-VALUED FUNCTIONS

BY

DAVID N. O'STEEN(1)

ABSTRACT. If $X$ and $Y$ are topological spaces, the set of all continuous functions from $X$ into $CY$, the space of nonempty, compact subsets of $Y$ with the finite topology, contains a copy (with singleton sets substituted for points) of $Y^X$, the continuous point-valued functions from $X$ into $Y$. It is shown that $Y^X$ is homeomorphic to this copy contained in $(CY)^X$ (where all function spaces are assumed to have the compact-open topology) and that, if $X$ or $Y$ is $T_2$, $(CY)^X$ is homeomorphic to a subspace of $(CY)^{CX}$. Further, if $Y$ is $T_2$, then these images of $Y^X$ and $(CY)^X$ are closed in $(CY)^X$ and $(CY)^{CX}$ respectively.

Finally, it is shown that, under certain conditions, some elements of $X^Y$ may be considered as elements of $(CY)^X$ and that the induced 1-1 function between the subspaces is open.

If $X$ and $Y$ are topological spaces then $Y^X$ denotes the space of all continuous functions from $X$ into $Y$. We shall assume throughout that all function spaces have the compact-open topology, which, we recall, is that topology having as a subbasis $\{(A, W)|A \subset X$ is compact, $W \subset Y$ is open$\}$ where $(A, W) = \{f \in Y^X|f(a) \in W$ for each $a \in A\}$. $CY$ will designate the space of nonempty, compact subsets of $Y$ and will be assumed to have the finite topology, which is obtained by taking as a subbasis $\{\overline{U}|U$ is an open subset of $Y\}\cup \{\overline{U}|U$ is an open subset of $Y\}$, where $\overline{U} = \{A \in CY|A \subset U\}$ and $\overline{U} = \{A \in CY|A \cap U \neq \emptyset\}$.(2) A basis for $CY$ is formed by the collection of all subsets of the form $t(U) \cap (\bigcap_{i=1}^{n} U(i))$ where $U$ and each of $V(1), \cdots, V(n)$ are open in $Y$.

$(CY)^X$ contains a copy (with singleton sets substituted for points) of $Y^X$ and we first show that $Y^X$ is homeomorphic to this subspace of $(CY)^X$ and that, if $X$ or $Y$ is $T_2$, $(CY)^X$ is homeomorphic to a subspace of $(CY)^{CX}$. Further, if $Y$ is $T_2$, then these images of $Y^X$ and $(CY)^X$ are closed in $(CY)^X$ and $(CY)^{CX}$ respectively.

Theorem 1. $Y^X$ is homeomorphic to a subspace of $(CY)^X$ and, if $Y$ is $T_2$, this subspace of $(CY)^X$ is closed.

Proof. Define $\phi: Y^X \rightarrow (CY)^X$ by $\phi(f)(x) = f(x)$. Since $\{y||y \in Y\}$ is a...
homeomorphic copy of $Y$ contained in $CY$, the continuity of $f$ certainly implies that of $\phi(f)$. This shows $\phi$ to be well defined and that $\phi$ is 1-1 is clear. Now if $(A, W)$ is a subbasic open set in $Y^X$ then $\phi((A, W)) = (A, \overline{f} W) \cap \phi(Y^X)$ which, since $\phi$ is 1-1, shows $\phi$ to be open. Likewise, if $(A, \mathcal{U})$ is a subbasic open set in $(CY)^X$ containing $\phi(f)$, then, since $A$ is compact and $\{\overline{f} U\mid U \subset Y \text{ is open}\}$ is a basis for the subspace topology of $\{y\mid y \in Y\}$ in $CY$, there exist finitely many open subsets, $U(1), \ldots, U(n)$, of $Y$ such that $\{\phi(f)(a)\mid a \in A\} \subset U(i)$ and $\phi(A) \subset (A, \mathcal{U}) \subset (A, \mathcal{U})$. Thus $f \in (A, \bigcup_{i=1}^{n} U(i))$ and $\phi(A, \bigcup_{i=1}^{n} U(i)) \subset (A, \mathcal{U})$ which implies the continuity of $\phi$.

Suppose that $Y$ is $T_2$ and let $g$ be an element of $(CY)^X - \phi(Y^X)$. Then there is a point $x$ of $X$ and distinct points $y_1$ and $y_2$ of $Y$ such that $\{y_1, y_2\} \subset g(x)$. Choose disjoint open subsets $U(1)$ and $U(2)$ of $Y$ such that $y_1 \in U(1)$ and $y_2 \in U(2)$ and then $g \in \{x\}, \overline{U(1)} \cap \overline{U(2)}$. Since $\{x\}, \overline{U(1)} \cap \overline{U(2)} \cap \phi(Y^X) = \emptyset$, $\phi(Y^X)$ is closed in $(CY)^X$.

**Lemma 1.** If either of $X$ or $Y$ is $T_2$ and if $\mathcal{B}$ is a subbasis for $Y$, then $\{(A, B)\mid A$ is a compact subset of $X$ and $B$ is an element of $\mathcal{B}\}$ is a subbasis for the compact-open topology on $Y^X$.

**Lemma 2.** $(CY)^X$ is $T_2$ if and only if $Y$ is $T_2$ and $(CY)$ is regular (if and only if $Y$ is regular).

For Lemma 1 see [2] and [3], and for Lemma 2 see [5]. (It is not necessary for compact subsets of $Y$ to be closed in order to prove either part of Lemma 2.)

**Corollary.** $(CY)^X$ is $T_2$ if and only if $Y$ is $T_2$ and $(CY)^X$ is regular if and only if $Y$ is regular.

Proof. This follows from Lemma 2 and the fact that $(CY)^X$ is $T_2$ if and only if $CY$ is $T_2$ and $(CY)^X$ is regular if and only if $CY$ is regular.

**Theorem 2.** If $X$ or $Y$ is $T_2$ then $\mathcal{B}[(CY)^X] = \{(A, \overline{f} U)\mid A \subset X$ is compact and $U \subset Y$ is open\} is a subbasis for $(CY)^X$.

Theorem 2 is an immediate consequence of Lemmas 1 and 2. The following example demonstrates that if neither $X$ nor $Y$ is $T_2$ then $\mathcal{B}[(CY)^X]$ need not be a subbasis for $(CY)^X$.

**Example 1.** Let $R$ and $S$ be two disjoint, countably infinite, dense subsets of $(0,1)$.

We define a topology $\mathcal{T}_{[0,1]}$ on $[0,1]$ such that $\mathcal{T}_{[0,1]}(0,1)$ is the usual topology.

(3) For the case that $Y$ is $T_2$, Theorem 2 is shown in [2].
topology and such that \(|1| \subset U \ (|0| \subset U)\), where \(U\) is an element of \(\mathcal{F}_{[0,1]}\), if and only if \(U\) is open in the usual topology for \([0,1]\) and \(R \subset U \ (S \subset U)\). With this topology \([0,1]\) is a compact, \(T_0\) space. Let \((X, \mathcal{F}_X)\) be \([0,1] - (R \cup S)\) with the subspace topology inherited from \(([0,1], \mathcal{F}_{[0,1]}\)). \((X, \mathcal{F}_X)\) is a compact, \(T_1\) space.

Let \(Y' = ([2, 3] \cup \{4\} \cup \{5\})\) and define a topology \(\mathcal{F}_{Y'}\) on \(Y'\) such that \(\mathcal{F}_{Y'}([2, 3])\) is the usual topology and such that \(|3|, |4|, \text{ or } |5|\) is a subset of \(U\), where \(U\) is an element of \(\mathcal{F}_{Y'}\), if and only if \(U\) is open in the usual topology for \(([2, 3] \cup \{4\} \cup \{5\})\) and \(|2| \subset U\). Let \((Y, \mathcal{F}_Y)\) be \(Y' - \{2\}\) with the subspace topology inherited from \((Y', \mathcal{F}_{Y'})\). We note that \((Y', \mathcal{F}_{Y'})\) is a compact, \(T_0\) space and that \((Y, \mathcal{F}_Y)\) is a compact, \(T_1\) space. Further \([2, 3], \{4\} \cup (2, 3], \text{ and } \{5\} \cup (2, 3]\) are compact subsets of \((Y, \mathcal{F}_Y)\).

Let \(f: X \to CY\) be defined by \(f(q) = (2, 3]\) if and only if \(q \in ((0, 1) \cap X)\), \(f(0) = (2, 3] \cup \{4\}, \ f(1) = (2, 3] \cup \{5\}\). It is easily shown that \(f\) is continuous and therefore \(f \in \mathcal{C}(CY)\).

Now \((2, 3], \{2, 3\} \cup \{4\}, \text{ and } (2, 3] \cup \{5\}\) are all open in \((Y, \mathcal{F}_Y)\). Thus \([\{2, 3\} \cup \{4\} \cap L_{(2,3)}] \cup \{2, 3\} \cup \{5\} \cap L_{(2,3)}\} = Z\) is an open subset of \(CX\). Further, it is clear that \(f \in (X, Z)\), open in \((CY)^X\), since \(X\) is compact and \(f(q) \in Z\), for each \(q \in X\).

To show \(B[(CY)^X]\) is not a subbasis for \((CY)^X\) it will be sufficient to show that every finite intersection of elements of \(B[(CY)^X]\) containing \(f\) contains an element of \((CY)^X\) not contained in \((X, Z)\). Further, we should first note that \(g \in (CY)^X\) is contained in \((X, Z)\) only if \(g(q)\) is a subset of \((2, 3] \cup \{4\}\) or \(g(q)\) is a subset of \((2, 3] \cup \{5\}\), for each \(q \in X\).

Let \(B = (\bigcap_{i=1}^n (A_i, \mathcal{F}_W(i))) \cap (\bigcap_{j=1}^m (D_j, \mathcal{F}_V(j)))\) be a finite intersection of elements of \(B[(CY)^X]\) containing \(f\). Without loss of generality, we may assume no \(A_i\) contains both \(0\) and \(1\) else \((A_i, \mathcal{F}_W(i))\) would equal \((CY)^X\).

Assuming no \(A_i\) contains both \(0\) and \(1\), we first claim \(X \not\subset \bigcup_{i=1}^n A_i\). Suppose \(X \subset \bigcup_{i=1}^n A_i\). Then let \(K = \text{the union of all } A_i\) such that \(1 \notin A_i\), \(1 \leq i \leq n\), and let \(L = \text{the union of all } A_i\) such that \(0 \notin A_i\), \(1 \leq i \leq n\). Neither \(K\) nor \(L\) is empty, \(0 \in K, 1 \in L, K \cup L = X\), and both \(K\) and \(L\) are compact. Let \(q\) be an element of \(R\) and let \(U_1, \ldots, U_n, \ldots\) be a nested sequence of open subsets of \(([0, 1], \mathcal{F}_{[0,1]}\)) such that \(\bigcap_{r=1}^\infty U_r = q\). For the sequence \(\{U_r \cap X\}_{r=1}^\infty\) of open subsets of \((X, \mathcal{F}_X)\) it either is, or is not, the case that there exists an \(n\) such that, for each \(m > n\), \(U_m \cap X \subset L\). If there exists such an \(n\), then an open cover of \(L\) having no finite subcover can be constructed and, if there does not exist such an \(n\) one can do the same for \(K\). Thus the assumption that \(X \subset \bigcup_{i=1}^n A_i\) leads to a contradiction, and, therefore, there exists a point \(\tilde{x}\) of \(X\) such that \(\tilde{x} \notin \bigcup_{i=1}^n A_i\).
Define the function \( g: X \to CY \) by \( g(x) = Y \) and \( g(q) = /g(q) \) if \( q \neq \hat{q} \). Since \( X \) is \( T_1 \) and \( / \) is continuous, it follows that \( g \) is continuous. Hence \( g \in (CY)^X \).

From the definition of \( g \), \( g \in \bigcap_{i=1}^n (A_i, \tilde{T}_W(i)), \) for \( \big| A_i \big| = /| A_i|, \) 1 \( \leq i \leq n. \)

For each \( q \in X \), \( g(q) \cap (2, 3) \neq \emptyset \), and, since any open subset of \( Y \) intersects \( (2, 3) \), it follows that \( g \in \bigcap_{j=1}^m (D_j, \tilde{T}_V(j)), \) and therefore \( g \in B \).

Since \( g(x) \) is an element of neither \( \tilde{T}_{(2,3)} \cup \{4\} \) nor \( \tilde{T}_{(2,3)} \cup \{5\} \), \( g \) is evidently not an element of \( (X, Z) \). Thus \( B[(CY)^X] \) is not a subbasis for the compact-open topology on \( (CY)^X \).

Evidently \( Y \) being \( T_1 \) does not imply that \( CY \) is \( T_1 \) for, in the above example, the element \( (2, 3) \) of \( CY \) is not closed.\(^{(4)}\)

To prove the following lemmas, see Theorem 2.5.2 and Corollary 9.6 of \([5]\) respectively. (That compact sets be closed is not necessary in either.)

**Lemma 3.** If \( X \) is any topological space and \( A \) is a compact subset of \( CX \), then \( \bigcup_{A \in A} A \), is a compact subset of \( X \).

**Lemma 4.** If \( f \in (CY)^X \) and \( A \subset X \) is compact, then \( \bigcup_{x \in A} f(x) \) is a compact subset of \( Y \).

**Theorem 3.** \( (CY)^X \) is homeomorphic to a subspace of \( (CY)^{CX} \), with the compact-open topology, provided that either of \( X \) or \( Y \) is \( T_2 \).

**Proof.** Define a function \( \Sigma: (CY)^X \to (CY)^{CX} \) by \( \Sigma(f)(A) = D \) where \( A \in CX \), and \( D \subset CY \) such that \( D = \bigcup_{x \in A} f(x) \). \( D \) is a compact subset of \( Y \) by Lemma 4 and thus, if \( f \in (CY)^X \), then \( \Sigma(f) \) is a function from \( CX \) to \( CY \), and we first show that \( \Sigma(f) \) is actually an element of \( (CY)^{CX} \).\(^{(5)}\)

It suffices to prove that the pre-images of subbasic open sets are open. Let \( \tilde{T}_U \) be a subbasic open set of \( CY \) containing \( \Sigma(f)(A) \) where \( A \subset CX \). Since \( f \) is continuous, \( \{x| f(x) \in \tilde{T}_U\} \equiv Z \) is open in \( X \). Hence \( \tilde{T}_Z \) is open in \( CX \), and, since \( f(x) \in \tilde{T}_U \), for each \( x \in A \), \( A \subset \tilde{T}_Z \). Clearly, if \( B \in \tilde{T}_Z \) then \( f(b) \in \tilde{T}_U \) for each \( b \in B \) and hence \( \Sigma(f)(B) \subset \tilde{T}_U \). Similarly, if \( \Sigma(f)(A) \in \tilde{T}_V \), where \( V \subset Y \) is open, then there is an element \( a \) of \( A \) such that \( f(a) \subset \tilde{T}_V \). Since \( \{x| f(x) \in \tilde{T}_V\} \equiv W \) is open in \( X \), \( \tilde{T}_W \) is an open subset of \( CX \) containing \( A \) and \( \Sigma(f)(\tilde{T}_W) \subset \tilde{T}_V \). Thus \( \Sigma(f) \in (CY)^{CX} \).

Clearly \( \Sigma \) is 1-1.

Now we shall show that \( \Sigma \) is continuous. Since \( X \) or \( Y \) is \( T_2 \), we know that \( CX \) or \( CY \) is \( T_2 \). Thus by Lemma 1, \( \{(\hat{A}, \tilde{T}_U)| \hat{A} \subset CX \) is compact and \( U \subset Y \) open \} \bigcup \{(\hat{A}, \tilde{T}_V)| \hat{A} \subset CX \) is compact and \( U \subset Y \) open \} forms a subbasis for \( (CY)^{CX} \).

\(^{(4)}\) Compare with Theorem 4.9.2 of \([5]\).

\(^{(5)}\) It is shown in Lemma A of \([6]\) that if \( f \) is an element of \( Y^X \) then \( \Sigma \circ \phi(f) \), where \( \phi \) is as in Theorem 1, is an element of \( (CY)^{CX} \).
Let \((\mathcal{A}, \mathcal{T}_U)\) and \((\mathcal{B}, \mathcal{T}_V)\) be subbasis elements of \((\mathcal{C}Y)^{\mathcal{X}}\). If \(f \in (\mathcal{C}Y)^{\mathcal{X}}\) such that \(\Sigma(f) \subseteq (\mathcal{A}, \mathcal{T}_U)\), then, by Lemma 3, \(\bigcup_{\mathcal{A} \subseteq \mathcal{A}^*} \mathcal{A} = \mathcal{A}^*\) is a compact subset of \(\mathcal{X}\). Further, if \(g \in (\mathcal{C}Y)^{\mathcal{X}}, \Sigma(g) \subseteq (\mathcal{A}, \mathcal{T}_U)\) if and only if \(g(x) \in \mathcal{T}_U\) for each \(x \in \mathcal{A}^*\). But this is true if and only if \(g \in (\mathcal{A}^*, \mathcal{T}_U)\). Hence \(f \in (\mathcal{A}^*, \mathcal{T}_U), open in (\mathcal{C}Y)^{\mathcal{X}}, and \Sigma(\mathcal{A}^*, \mathcal{T}_U) \subseteq (\mathcal{A}, \mathcal{T}_U)\).

Now \(\Sigma(f) \subseteq (\mathcal{B}, \mathcal{T}_V)\) implies that if \(D \in \mathcal{D}\) then there exists some \(x \in \mathcal{D}\) such that \(f(x) \in \mathcal{T}_V\). By Lemma 3, \(\bigcup_{D \in \mathcal{D}} D = \mathcal{D}^*\) is a compact subset of \(\mathcal{X}\) and \([x]/(x) \in \mathcal{T}_V\) is an open subset of \(\mathcal{X}\). Further, \(\mathcal{D}^* \cap \mathcal{W} \neq \emptyset\) since \(D \in \mathcal{D}\) implies that \(\Sigma(f)(D) \subseteq \mathcal{T}_V\) which implies that there exists some \(x \in \mathcal{D}\) such that \(f(x) \subseteq \mathcal{T}_V\). But this says that \(x \in \mathcal{W}\), and, since \(x \in \mathcal{D}^*\), we have \(x \in (\mathcal{D}^* \cap \mathcal{W})\).

Hence for each \(D \in \mathcal{D}\), there exists some \(x \in \mathcal{D}\) such that \(x \in (\mathcal{D}^* \cap \mathcal{W})\). Since \(\mathcal{D}^*\) is compact, it follows that, if \(\mathcal{X}\) is \(T_2\), \(\mathcal{D}^*\), as a subspace of \(\mathcal{X}\), is regular.

Therefore, since \(\mathcal{D}^* \cap \mathcal{W}\) is open in the subspace topology of \(\mathcal{D}^*\), there exists for each \(x \in (\mathcal{D}^* \cap \mathcal{W})\), a subset \(O(x)\) of \(\mathcal{X}\), open in the subspace topology of \(\mathcal{D}^*\), such that \(x \in O(x) \subseteq (\mathcal{D}^* \cap \mathcal{W})\). Note that the closure of \(O(x)\), in \(\mathcal{X}\), is contained in \(\mathcal{D}^*\) since \(\mathcal{D}^*\) being compact and \(\mathcal{X}\) being \(T_2\) implies that \(\mathcal{D}^*\) is closed.

If \(Y\) is \(T_2\) then \(\mathcal{C}Y\) is \(T_2\). Further, for \(x \in (\mathcal{D}^* \cap \mathcal{W})\), if \(q\) is an element of \([\mathcal{D}^* \cap \mathcal{W}] \setminus (\mathcal{D}^* \cap \mathcal{W})\) then \(f(x) \neq f(q)\), since \(f(x) \in \mathcal{T}_V\) and \(f(q) \notin \mathcal{T}_V\). Thus, since \(\mathcal{C}Y\) is \(T_2\) and \(f\) is continuous, there are disjoint sets, \(M(x,q)\) and \(N(x,q)\), open in the subspace topology for \(\mathcal{D}^*\) and containing \(x\) and \(q\) respectively, such that \(M(x,q) \subseteq (\mathcal{D}^* \cap \mathcal{W})\).

Since \(\mathcal{D}^*\) is compact and \([\mathcal{D}^* \cap \mathcal{W}] \setminus (\mathcal{D}^* \cap \mathcal{W})\) is closed, \(\mathcal{D}^*\) is closed in \(\mathcal{D}^*\), which implies that there exist \(N(q_1), \ldots, N(q_k)\) such that \(\bigcup_{i=1}^k N(q_i) \subseteq [\mathcal{D}^* \setminus \mathcal{A}] \). Hence \(x \in \bigcap_{i=1}^k M(x, q_i) = O(x) \subseteq (O(x) \cap \mathcal{D}^*) \subseteq (\mathcal{D}^* \cap \mathcal{W})\).

Thus, whether \(X\) or \(Y\) is \(T_2\) we have, for each \(x \in (\mathcal{D}^* \cap \mathcal{W})\), a subset \(O(x)\), open in the subspace topology for \(\mathcal{D}^*\) such that \(O(x) \subseteq (O(x) \cap \mathcal{D}^*) \subseteq (\mathcal{D}^* \cap \mathcal{W})\).

Now if \(D \in \mathcal{D}\), \(f(D) \subseteq \mathcal{T}_V\), which implies that \(D \cap (\mathcal{D}^* \cap \mathcal{W}) \neq \emptyset\) and thus \(D \cap O(x) \neq \emptyset\) for some \(x \in (\mathcal{D}^* \cap \mathcal{W})\). Since each such \(x \in (\mathcal{D}^* \cap \mathcal{W})\) is covered by some \(O(x)\), \(\bigcup \{O(x) \mid x \in (\mathcal{D}^* \cap \mathcal{W})\}\) is an open cover of \(\mathcal{D}\) and because \(\mathcal{D}\) is compact there exist \(x(1), \ldots, x(r)\) elements of \(\mathcal{D}^* \cap \mathcal{W}\), such that \(\mathcal{D} \subseteq \bigcup_{l=1}^r O'(x(l))\). Thus, if \(D \in \mathcal{D}\), \(D \cap O'(x(l)) \neq \emptyset\) for some \(l\) such that \(1 \leq l \leq r\). Since \(D \subseteq \mathcal{D}^*\), this implies \(D \cap [O'(x(l))] \cap \mathcal{D}^* \neq \emptyset\).

Thus each \(D\) contains a point of \(O(x(l))\) for some \(l\) such that \(1 \leq l \leq r\) where \(D \in \mathcal{D}\). Now \(E = \bigcup_{l=1}^r (O(x(l)) \cap \mathcal{D}^*)\) is closed in the subspace \(\mathcal{D}^*\), and...
hence compact, and is also contained in $\mathcal{W}$. Thus, for each $x \in E$, $f(x) \in L_V$, and further, if $D \in \mathcal{D}$, there exists some $x \in D$ such that $x \in E$. Thus $f \in (E, L_V)$, and, if $g \in (E, L_V)$, then for each $D \in \mathcal{D}$, there exists an element $x \in D$ such that $x \in E$ and hence $g(x) \in L_V$. This implies that $\Sigma(g) \in (\mathcal{D}, L_V)$. Hence $f \in (E, L_V)$ and $\Sigma((E, L_V)) \subseteq (\mathcal{D}, L_V)$. This shows $\Sigma$ to be continuous.

Let $(A, \mathcal{W})$ be a subbasic open set of $(CY)^X$, and then $\mathcal{A} = \{\{x\} \in CX | x$ is a point of $A\}$ is a compact subset of $CX$. If $f \in (A, \mathcal{W})$, then from the definition of $\Sigma$, $\Sigma(f) \in (\mathcal{A}, \mathcal{W})$. Further, it is clear that if $\Sigma(f) \in (\mathcal{A}, \mathcal{W})$ then $f(x) \in \mathcal{W}$, for each $\{x\} \in \mathcal{A}$, which implies that $f \in (A, \mathcal{W})$. We have shown $\Sigma((A, \mathcal{W})) = (\mathcal{A}, \mathcal{W}) \cap \Sigma((CY)^X)$ which is open in $(CY)^{CX} \cap \Sigma((CY)^X)$, and, since $\Sigma$ is 1-1, this shows $\Sigma$ to be open.

Let $\overline{C}$ denote the image of $\Sigma$. An element of $(CY)^{CX}$ will be called consistent if and only if it is an element of $\overline{C}$. An element of $(CY)^{CX} - \overline{C}$ will be called inconsistent.

**Corollary.** An element $f$ of $(CY)^{CX}$ is consistent if and only if, for each $A \in CX$, $f(A) = D$, where $D = \bigcup_{x \in A} f(\{x\})$.

Easy examples, where $X$ and $Y$ are finite discrete spaces, show that $(CY)^{CX} - \overline{C}$ may be nonempty. We may also observe that $\overline{C}$ is not necessarily closed by letting $X$ be a two point discrete space and $Y$ the Sierpinski space.(6) However, if $Y$ is $T_2$ we have the following theorem.

**Theorem 4.** If $Y$ is $T_2$ then $\overline{C}$ is a closed subset of $(CY)^{CX}$.

**Proof.** Suppose $f$ is an inconsistent element of $(CY)^{CX}$. Then there is an element $A$ of $CX$ such that $f(A) \neq \bigcup_{x \in A} f(\{x\})$.

Assume there is a point $p$ of $Y$ such that $p \in f(A)$ but $p \notin \bigcup_{x \in A} f(\{x\})$. Now $\{(x) | x \in A\} \equiv \mathcal{A}$ is a compact subset of $CX$ and this implies that $f(\mathcal{A}) = \{f(\{x\}) | x \in A\}$ is a compact subset of $CY$. Hence, by Lemma 3, $\bigcup_{x \in A} f(\{x\})$ is a compact subset of $Y$. Therefore, since $Y$ is $T_2$, there exist disjoint open sets, $U$ and $V$, of $Y$ such that $p \in U$ and $\bigcup_{x \in A} f(\{x\}) \subseteq V$. Then $f$ is an element of $\{(A), L_U) \cap (\mathcal{A}, \overline{L}_V)$, which is an open subset of $(CY)^{CX}$ and clearly contains no element of $\overline{C}$.

Now assume there is a point $p$ of $Y$ such that $p \in \bigcup_{x \in A} f(\{x\})$ but $p \notin f(A)$. Then there is an element $x$ of $X$ such that $x \in A$ and $p \notin f(\{x\})$. Since $f(A)$ is compact and $Y$ is $T_2$, there exist disjoint open sets, $U$ and $V$, of $Y$ such that $f(A) \subseteq U$ and $p \notin V$. Therefore, $f$ is an element of $\{(A), L_U) \cap (\{x\}, L_V)$, an open subset of $(CY)^{CX}$ which contains no element of $\overline{C}$. Thus $(CY)^{CX} - \overline{C}$ is open which completes the proof.

(6) By the "Sierpinski space" we mean a topological space consisting of two points, $x_1$ and $x_2$, with the totality of open sets being $\{x_1, x_2\}, \{x_1\}$, and $\emptyset$. 

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If \( f \in X^Y \) then \( f^{-1} \) is a set-valued function from the image of \( f \) to \( Y \).

Theorem 5 shows how, under certain conditions, some elements of \( X^Y \) may be considered as elements of \((CY)^X\). However, the function between the two subspaces is not necessarily a homeomorphism.

Let \( D_X^Y \equiv \{ f \in X^Y \mid f \text{ is open, closed, and onto} \} \) and \( D_X^Y \equiv \{ f \in (CY)^X \mid f \text{ is disjointly } 1-1, \bigcup_{x \in M} f(x) \text{ is open (closed) whenever } M \subseteq X \text{ is open (closed), and } \bigcup_{x \in X} f(x) = Y \}. \) (By "disjointly 1-1" we simply mean that if \( x_1 \) and \( x_2 \) are distinct elements of \( X \) then \( f(x_1) \cap f(x_2) = \emptyset \).)

**Theorem 5.** If \( Y \) is compact, \( X \) or \( Y \) is \( T_2 \), and \( D_X^Y \neq \emptyset \), then \( \psi \), defined by \( \psi(f)(x) = f^{-1}(x) \), is a 1-1, open function from \( D_X^Y \) onto \( D_Y^X \).

**Proof.** If \( f \) is in \( D_X^Y \), it is easily verified that \( \psi(f) \in D_Y^X \).

Let \( b \in D_Y^X \) and define \( f : Y \to X \) by \( f(y) = x \) if and only if \( y \in b(x) \). Then \( f \in D_X^Y \) and \( \psi(f) = b \). Clearly \( \psi \) is 1-1.

We have only to show that \( \psi \) is open: Let \( (A, W) \) be a subbasis element of \( X^Y \) and let \( f \in (D_X^Y \cap (A, W)) \). Now since \( W \) is open in \( X \), and \( f : Y \to X \) is continuous and onto, it follows that \( X \) must be compact and that \( X - W \) must be compact. Hence \( (X - W, \tau_{X - W}) \) is open in \( (CY)^X \). Assume \( X \) is \( T_2 \). If \( x' \in (X - W) \) and \( x \in f(A) \) there exist disjoint open sets \( U(x) \) and \( V(x, x') \) containing \( x \) and \( x' \) respectively and, since \( \psi(f) = f^{-1} \) is an element of \( D_Y^X \), it follows that \( \bigcup_{p \in U(x)} \psi(f)(p) \) and \( \bigcup_{p \in V(x, x')} \psi(f)(p) \) are disjoint open sets, containing \( \psi(f)(x) \) and \( \psi(f)(x') \), respectively. By repeating this process for all \( x \in f(A) \), we observe that \( \{ \bigcup_{p \in U(x)} \psi(f)(p) \mid x \in f(A) \} \) is an open cover of \( A \) and since \( A \) is compact there exist \( x_1, \ldots, x_m \) elements of \( f(A) \) such that

\[
\bigcup_{j=1}^m \left( \bigcup_{p \in V(x_j, x')} \psi(f)(p) \right) \supset \psi(f)(x')
\]

and

\[
\left[ \bigcup_{j=1}^m \left( \bigcup_{p \in U(x_j)} \psi(f)(p) \right) \right] \cap \left[ \bigcap_{j=1}^m \left( \bigcup_{p \in V(x_j', x')} \psi(f)(p) \right) \right] = \emptyset.
\]

This implies that no point of \( \psi(f)(x') \) is in \( \overline{A} \), for each \( x' \in (X - W) \). Therefore, \( \psi(f) \in (X - W, \tau_{X - W}) \) if \( X \) is \( T_2 \).

Now assume \( Y \) is \( T_2 \). Then \( A = \overline{A} \) since \( A \) is compact, and thus \( (X - W, \tau_{X - W}) = (X - W, \tau_{X - W}) \). But \( x \in (X - W) \) implies that \( x \notin f(A) \), which implies

\( \psi(f) \) is essentially shown in [5] that \( \psi(f) \) is continuous if and only if \( f \) is open and closed (where the notation \( f^{-1} \) is used instead of our \( \psi(f) \)).
\[
\psi(f)(x) \cap A = f^{-1}(x) \cap A = \emptyset. \text{ Hence } \psi(f) \in (X - W, \bar{Y} - A). \text{ Therefore, if either of } X \text{ or } Y \text{ is } T_2, \psi(f) \in (X - W, \bar{Y} - A).
\]

Let \( g \in ((X - W, \bar{Y} - A) \cap \mathcal{D}_Y^X). \) We know \( \psi^{-1}(g) \) exists and is exactly one element of \( D_Y^X. \) Let \( y \in A. \) Since \( g(x) \cap A = \emptyset, \) for each \( x \in (X - W) \), it follows that \( \psi^{-1}(g)(y) \in W. \) Thus \( \psi^{-1}(g) \in (A, W) \) which implies that \( g \in \psi((A, W) \cap D_Y^X). \) This shows \( ((X - W, \bar{Y} - A) \cap \mathcal{D}_Y^X) \subset \psi((A, W) \cap D_Y^X) \) which, since \( \psi \) is 1-1, completes the proof that \( \psi \) is an open function.

The following example shows that \( \psi: D_Y^X \rightarrow \mathcal{D}_Y^X \) need not be continuous.

Example 2. Let \( X \) be the topological space consisting of the set of all real numbers of the form \( 1/i, \) \( i = 1, 2, \ldots, \) together with \( 0, \) with the subspace topology inherited from the reals. Let \( Y \) be the topological space consisting of the set of all real numbers of the form \( 1/i, \) \( i = 1, 2, \ldots, \) or of the form \( 2 - 1/i, \) \( i = 1, 2, \ldots, \) together with \( 0 \) and \( 2 \) with the subspace topology inherited from the reals. We note that \( X \) and \( Y \) are both compact, \( T_2 \) spaces.

Define \( f: Y \rightarrow X \) by \( f(y) = y, \) \( 0 < y < 1, \) and \( f(y) = 2 - y, \) \( 1 < y < 2. \) Observe that \( f \) is continuous, open, closed, and onto. Thus \( f \in D_Y^X. \) Now \( \psi(f) \in ((0, 1] \cap \mathcal{D}_Y^X) = U, \) open in \( \mathcal{D}_Y^X. \) We will show that given any basic open set \( V, \) of \( D_Y^X, \) containing \( f, \) there exists an element of \( (\mathcal{D}_Y^X - U) \) contained in \( \psi(V). \)

Let \( V = (\bigcap_{i=1}^n (A_i, W_i)) \cap D_Y^X \) be a basic open set of \( D_Y^X \) containing \( f. \)

Let \( M = \bigcup_{i \in A} A_i, \) \( 1 \leq i \leq n, 2 \in A \setminus 1, \)

Let \( K = \bigcup_{i \in A} A_i. \) Assume \( K \neq \emptyset. \) Then \( K \) is a compact subset of \( [0, 2] \cap Y \) and hence there is an element \( k \) of \( (1, 2) \cap Y \) such that \( k' \in K \) then \( k > k'. \)

If \( K = \emptyset, \) choose \( k \) to be \( 1 + \frac{1}{2}. \)

Suppose that \( M \neq \emptyset. \) Then if \( A_i \in M, \) there is a half-open interval \( Z_i \) such that \( 0 \in Z_i \) and \( Z_i \cap X \subset W_i. \) Let \( Z = (\bigcap_{i \in M} Z_i) \cap X \) and choose an element \( z \) of \( Z \) such that \( z \in ((0, 1) \cap X). \) If \( M = \emptyset, \) choose \( z \) to be \( \frac{1}{2}. \)

Let \( l \) be the greater of \( k \) and \( 2 - z \) and define \( g: Y \rightarrow X \) by \( g(y) = f(l), \) \( y \leq l, \) and \( g(y) = f(y), \) \( l < y. \) Clearly \( g \) is continuous, open, closed, and onto, and hence an element of \( D_Y^X. \) Also, \( g \in (\bigcap_{i=1}^n (A_i, W_i) \cap D_Y^X) \) since \( g|\bigcup_{A_i \in M} = f |\bigcup_{A_i \in A} A_i, \)

\( g(y) = f(y), \) for each \( y \in ([0, l] \cap Y), \) and \( g(y) \in \bigcap_{A_i \in M} W_i, \) for each \( y \in ([l, 2] \cap Y). \) This implies that \( \psi(g) \notin ((0, 1] \cap \mathcal{D}_Y^X) \) and \( \psi(g) \in (\mathcal{D}_Y^X - U), \) and it follows that \( \psi: D_Y^X \rightarrow \mathcal{D}_Y^X \) is not continuous.

I would like to thank Dr. A. R. Vobach for his valuable suggestions during the development of this topic. I would like also to thank the referee for his aid in simplifying the proofs and notation.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77004

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30601

*Current address*: 216 N. Elam Ave., Greensboro, North Carolina 27403