STRICTLY IRREDUCIBLE \(*\)-REPRESENTATIONS OF BANACH \(*\)-ALGEBRAS(1)

BY

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ABSTRACT. In this paper strictly irreducible \(*\)-representations of Banach \(*\)-algebras and the positive functionals associated with these representations are studied.

Introduction. Let \( A \) be a Banach \(*\)-algebra, and let \( a \mapsto \pi(a) \) be a representation of \( A \) on a Hilbert space \( H \). A subspace \( K \subset H \) is \( \pi \)-invariant if \( \pi(a)K \subset K \) for every \( a \in A \). The representation \( \pi \) is irreducible if \( \pi \) is nonzero and the only closed \( \pi \)-invariant subspaces of \( H \) are \( H \) and \( \{0\} \). \( \pi \) is strictly irreducible if \( \pi \) is nonzero and the only \( \pi \)-invariant subspaces of \( H \) are \( H \) and \( \{0\} \). In the case where \( A \) is a \( B^* \)-algebra, R. V. Kadison proved the remarkable result that every irreducible \(*\)-representation of \( A \) is strictly irreducible [6, Theorem 1]. Aside from this theorem of Kadison, there are only a few minor isolated results concerning strictly irreducible \(*\)-representations of Banach \(*\)-algebras. In this paper we study strictly irreducible \(*\)-representations and certain positive functionals associated with these representations which we call strictly pure states (a positive functional \( \alpha \) on \( A \) is a strictly pure state if \( \alpha \) is a pure state and the \(*\)-representation of \( A \) determined by \( \alpha \) is strictly irreducible). We give necessary and sufficient conditions that a pure state of \( A \) be strictly pure in \( \S \)2. In \( \S\S \)3 and 4 some of the special properties of strictly pure states and strictly irreducible representations are presented. In \( \S \)5 some examples of Banach \(*\)-algebras with the property that every irreducible \(*\)-representation is strictly irreducible are provided.

1. Notation and preliminaries. Throughout this paper \( A \) denotes a Banach \(*\)-algebra. A linear functional \( \alpha \) on \( A \) is positive if \( \alpha(a^*a) \geq 0 \) for all \( a \in A \). When \( \alpha \) is a positive functional on \( A \), let

\[
M(\alpha) = \sup \left\{ \frac{|\alpha(a)|^2}{\alpha(a^*a)} \right\} (a \in A, \ a(a^*a) \neq 0).
\]

Received by the editors July 12, 1971.

AMS 1970 subject classifications. Primary 46K10, 46H05.

Key words and phrases. Strictly irreducible \(*\)-representations, pure states.

(1) This research was partially supported by NSF grant GP-20226.
The set of all positive functionals \( \alpha \) on \( A \) with the properties \( \alpha(a^*) = \overline{\alpha(a)} \) for all \( a \in A \) and \( M(\alpha) < +\infty \), we denote by \( \mathcal{P} \). \( \mathcal{P}_1 \) is the set of all \( \alpha \in \mathcal{P} \) with \( M(\alpha) \leq 1 \). Let \( A_h \) be the real linear space of hermitian elements of \( A \). \( \mathcal{P}_1 \) is a convex subset of \( A_h^* \), the dual space of \( A_h \), and \( \mathcal{P}_1 \) is compact in the weak *-topology on \( A_h^* \) (see [4, Theorem (21.34), p. 328]). The extreme points of \( \mathcal{P}_1 \) are called pure states. For \( \alpha \in \mathcal{P} \), the left kernel of \( \alpha \), denoted \( K_\alpha \), is the set of all \( a \in A \) such that \( \alpha(a^*a) = 0 \). \( K_\alpha \) is a closed left ideal of \( A \). The quotient space \( A - K_\alpha \) is a pre-Hilbert space in the inner-product \( (a + K_\alpha, b + K_\alpha) = \alpha(b^*a) \). Let \( H_\alpha \) denote the Hilbert space which is the completion of this pre-Hilbert space. A *-representation \( \alpha \to \pi_\alpha(a) \) of \( A \) on \( H_\alpha \) is constructed by first defining \( \pi_\alpha(a)(b + K_\alpha) = ab + K_\alpha \) for \( b + K_\alpha \in A - K_\alpha \). Then \( \pi_\alpha(a) \) is a bounded operator on \( A - K_\alpha \) which extends uniquely to a bounded operator on \( H_\alpha \) (also denoted by \( \pi_\alpha(a) \)). For details of this construction see the proof of Theorem (21.24) in [4]. It is a well-known theorem that \( \alpha \in \mathcal{P} \) is a pure state if and only if \( M(\alpha) = 1 \) and the *-representation \( \pi_\alpha \) is irreducible on \( H_\alpha \) [4, Theorem (21.34), p. 328]. We define \( \alpha \in \mathcal{P} \) to be a strictly pure state of \( A \) if \( \alpha \) is a pure state of \( A \) and \( a \to \pi_\alpha(a) \) is strictly irreducible on \( H_\alpha \).

When \( X \) is a normed linear space with norm \( \| \cdot \| \) and \( Y \) is a closed subspace of \( X \), then the quotient norm \( \| \cdot \|_q \) on the quotient space \( X - Y \) is defined as usual by

\[
\|x + y\|_q = \inf \{ \|x - y\| : y \in Y \}.
\]

\( H \) is always a Hilbert space and \( \mathcal{B}(H) \) is the algebra of all bounded operators on \( H \).

2. Necessary and sufficient conditions for a pure state to be strictly pure.

When \( \alpha \in \mathcal{P} \), the quotient space \( A - K_\alpha \) is an inner product space with inner product defined by \( (a + K_\alpha, b + K_\alpha) = \alpha(b^*a) \). Let \( |a + K_\alpha|^2 = (a + K_\alpha, a + K_\alpha)^{1/2} = \alpha(a^*a)^{1/2} \). We prove that a pure state \( \alpha \) of \( A \) is strictly pure if and only if \( A - K_\alpha \) is complete in the norm \( |a + K_\alpha|^2 \).

**Theorem 2.1.** Assume that \( A \) is a Banach *-algebra and that \( \alpha \) is a pure state of \( A \). Then \( \alpha \) is a strictly pure state of \( A \) if and only if \( A - K_\alpha \) is complete in the norm \( |a + K_\alpha|^2 = \alpha(a^*a)^{1/2} \). Also when \( \alpha \) is a strictly pure state of \( A \), then \( K_\alpha \) is a modular maximal left ideal of \( A \).

**Proof.** Assume first that \( \alpha \) is a strictly pure state of \( A \). By the construction of \( H_\alpha \), \( A - K_\alpha \) is an invariant subspace for \( \pi_\alpha(a) \) whenever \( a \in A \). Then \( H_\alpha = A - K_\alpha \), so that \( A - K_\alpha \) is complete in the norm \( |a + K_\alpha|^2 \).

Conversely assume that \( A - K_\alpha \) is complete in this norm. We prove first that the two norms \( | \cdot |_2 \) and \( \| \cdot \|_q \) are equivalent on \( A - K_\alpha \). By the Closed Graph Theorem it suffices to prove that \( \| \cdot \|_q \) dominates \( | \cdot |_2 \). This is exactly the same
as proving that the identity map \( a + K_a \to a + K_a \) is a continuous map from
\((A - K_a, \| \cdot \|)\) onto \((A - K_a, \| \cdot \|)\). Again using the Closed Graph Theorem, it
suffices to show that this map is closed. Therefore assume that \{a_n\} \subset A, a \in A,
\|a_n + K_a\| \to 0, and \(|a_n - a + K_a|^{1/2} \to 0\). Then there exists a sequence \{k_n\} \subset
K_a such that \|a_n + k_n\| \to 0. Therefore \|a^*a_n + a^*k_n\| \to 0, and this implies
\(\alpha(a^*a_n) = \alpha(a^*a_n + a^*k_n) \to 0\). But also \(\alpha(a^*(a_n - a)) = \|((a_n - a) + K_a) a + a\| \leq |(a_n - a) + K_a| a + K_a|^{1/2} \to 0\). Therefore \(\alpha(a^*a) = 0\), so that \(a + K_a = 0\).

Now define a functional \(\overline{\alpha}\) on the Hilbert space \(H_a = A - K_a\) by
\(\overline{\alpha}(a + K_a) = \alpha(a)\). Since \(K_a\) is contained in the null space of \(\alpha\), \(\overline{\alpha}\) is well defined. Also,
\[\|\overline{\alpha}\|^2 = \sup \left\{ \frac{|\alpha(a)|^2}{\alpha(a^*a)} \mid a \in A, \alpha(a^*a) \neq 0 \right\} = M(\alpha) = 1.\]

Since \(A - K_a\) is a Hilbert space, there exists \(v \in A\) such that \(\overline{\alpha}(a + K_a) = (a + K_a, v + K_a) = \alpha(v^*a)\) for all \(a \in A\). Therefore \(\alpha(a) = \alpha(v^*a)\) for all \(a \in A\).

Given any \(a \in A\),
\[\alpha((a(1 - v))^*a(1 - v)) = \alpha(a^*a(1 - v)) - \alpha(v^*a^*a(1 - v)) = 0.\]

Therefore \(A(1 - v) \subset K_a\) so that \(K_a\) is a modular left ideal. Let \(K\) be a maximal
left ideal of \(A\) such that \(K_a \subset K\). Set \(M = \{b + K_a \mid b \in K\}\). \(M\) is a proper \(\pi_a\)-invariant
subspace of \(H_a = A - K_a\). Furthermore \(M\) is \(\| \cdot \|_q\)-closed. Therefore by the
result of the previous paragraph, \(M\) is \(\| \cdot \|_2\)-closed. It follows that \(K_a = K\). Then
since \(K_a\) is a maximal modular left ideal of \(A\), \(\pi_a(A)\) acts strictly irreducibly on
\(H_a = A - K_a\).

Every Banach \(*\)-algebra \(A\) has an algebra pseudonorm called the Gelfand-
Naimark pseudonorm. We denote this pseudonorm by \(|a|, a \in A\). This pseudonorm
has the properties:

1. \(|a^*a| = |a|^2\) for all \(a \in A\).
2. \(|\alpha(a)| \leq M(\alpha)|a|\) whenever \(\alpha \in \mathcal{P}, a \in A\).
3. The \(*\)-radical of \(A\) is the set of all \(a \in A\) such that \(|a| = 0\). See [8, p.
   226] for the details of these results. We prove next that a pure state \(\alpha\) of \(A\) is
   strictly pure if and only if \(|a + K_a|_q = \inf \{|a + k| \mid k \in K_a\}\) is a complete norm on
   \(A - K_a\).

**Theorem 2.2.** Let \(| \cdot |\) denote the Gelfand-Naimark pseudonorm on \(A\). Then
a pure state \(\alpha\) of \(A\) is strictly pure if and only if \(|a + K_a|_q\) is a complete norm
on \(A - K_a\).

**Proof.** For convenience we assume in the proof that \(A\) is reduced (i.e. the
\(*\)-radical of \(A\) is 0). This assumption can be made with no loss of generality.
In this case \(| \cdot |\) is a norm on \(A\) with the \(B^*\)-property by (1) and (3) above. Let
\(B\) denote the \(B^*\)-algebra which is the completion of \(A\) in the norm \(| \cdot |\). Let \(\alpha\)
be a pure state of \( A \). By (2) above \( \alpha \) is \( |\cdot| \)-continuous. Therefore \( \alpha \) has a unique extension \( \tilde{\alpha} \) to \( B \). It is easy to verify that \( \tilde{\alpha} \) is a pure state of \( B \).

Now assume that \( \alpha \) is a strictly pure state of \( A \). Let \( \text{cl}(K_a) \) denote the \( |\cdot| \)-closure of \( K_a \) in \( A \). If \( \text{cl}(K_a) \neq K_{\tilde{\alpha}} \), then by [8, Theorem (4.9.8), p. 251] there exists a pure state \( \tilde{\beta} \) of \( B \) with \( \text{cl}(K_a) \subseteq K_{\tilde{\beta}} \) and \( \tilde{\alpha} \neq \tilde{\beta} \). Let \( \tilde{\beta} \) be the restriction of \( \tilde{\beta} \) to \( A \). \( K_a \subseteq K_{\tilde{\beta}} \) and therefore \( K_a = K_{\tilde{\beta}} \). By a result we prove in the next section, Theorem 3.2, it follows that \( \alpha = \beta \). But then \( \tilde{\alpha} = \tilde{\beta} \), a contradiction. Therefore \( \text{cl}(K_a) = K_{\tilde{\alpha}} \). By Kadison's theorem \( \tilde{\alpha} \) is a strictly pure state of \( B \). Then as noted in Theorem 2.1 there exists \( M > 0 \) such that

\[
M \tilde{\alpha}(b^*b)^{\frac{1}{2}} \geq |b + K_{\tilde{\alpha}}|^q \quad \text{for all } b \in B.
\]

Also using (2) above we have, for \( a \in A \), \( k \in K_a \),

\[
|a + K_a|^2 = \alpha(\alpha + k)^{\frac{1}{2}} \leq |(a + k)^*(a + k)|^{\frac{1}{2}} = |a + k|.
\]

Therefore \( |a + K_a|^2 \leq |a + K_a|^q \). Then for all \( a \in A \),

\[
M|a + K_a|^2 \geq M \tilde{\alpha}(a^*a)^{\frac{1}{2}} \geq |a + K_{\tilde{\alpha}}|^q = |a + K_a|^q \geq |a + K_a|^2.
\]

The norm \( |a + K_a|^2 \) is complete on \( A - K_a \) by Theorem 2.1. Therefore \( |a + K_a|^q \) is a complete norm on \( A - K_a \).

Conversely assume that \( |a + K_a|^q \) is a complete norm on \( A - K_a \). Given \( b \in K_a \), choose \( \{b_n\} \subseteq A \) such that \( |b_n - b| \rightarrow 0 \). Then \( |(b_n - b_m) + K_a|^q \rightarrow 0 \) as \( n, m \rightarrow +\infty \). Therefore there exists \( a \in A \) such that \( |(b_n - a) + K_a|^q \rightarrow 0 \). Choose \( \{k_n\} \subseteq K_a \) such that \( |b_n - a + k_n| \rightarrow 0 \). Then \( |b - a + k_n| \rightarrow 0 \), so that \( b - a \in \text{cl}(K_a) \). It follows that \( a^*b - a^*a \in \text{cl}(K_a) \), and therefore that \( \tilde{\alpha}(a^*b - a^*a) = 0 \). But \( \tilde{\alpha}(a^*b) = 0 \), since \( b \in K_{\tilde{\alpha}} \). Therefore \( a^*a = 0 \), so that \( a \in K_a \). Therefore \( b \in \text{cl}(K_a) \). We have now shown that \( K_a = \text{cl}(K_a) \). We have \( |a + K_a|^q \geq |a + K_a|^2 \) for all \( a \in A \) just as before. By Kadison's theorem \( \tilde{\alpha} \) is a strictly pure state of \( B \). Then by Theorem 2.1 there exists \( M > 0 \) such that \( |b + K_{\tilde{\alpha}}|^q \geq M|b + K_{\tilde{\alpha}}|^q \) for all \( b \in B \). Therefore for all \( a \in A \),

\[
|a + K_a|^q \geq |a + K_a|^2 = |a + K_{\tilde{\alpha}}|^2 \geq m|a + K_{\tilde{\alpha}}|^q = m|a + K_a|^q.
\]

It follows that \( |a + K_a|^2 \) is a complete norm on \( A - K_a \), and therefore \( \alpha \) is strictly pure by Theorem 2.1.

3. Results concerning strictly pure states and strictly irreducible representations. The relationship between a pure state and its left kernel has never been fully explored in a general Banach \(*\)-algebra. In fact to our knowledge none of the following questions have been answered when \( A \) is a Banach algebra with hermitian involution.

**Question 1.** If \( \alpha \) is a pure state of \( A \), is \( K_a \) a maximal left ideal of \( A \)?

**Question 2.** If \( \alpha \) and \( \beta \) are pure states of \( A \) and \( K_a = K_b \), does \( \alpha = \beta \)?
Question 3. If \( \alpha \in \mathcal{P} \), \( M(\alpha) = 1 \), and \( K_\alpha \) is a maximal left ideal of \( A \), is \( \alpha \) a pure state of \( A \)?

We add to this list another closely related question.

Question 4. If \( a \rightarrow \pi(a) \) and \( a \rightarrow \gamma(a) \) are two algebraically equivalent irreducible \(*\)-representations of \( A \) on respective Hilbert spaces, are \( \pi \) and \( \gamma \) necessarily unitarily equivalent?

The answer to all these questions is affirmative when \( A \) is a \( B^* \)-algebra. In this section we deal with special cases of these questions. To begin with, Theorem 2.1 states that when \( \alpha \) is a strictly pure state of \( A \), then \( K_\alpha \) is a modular maximal left ideal of \( A \). This answers Question 1 in the case when \( \alpha \) is strictly pure.

Next we prove a result which easily settles Question 2 if \( \alpha \) or \( \beta \) is strictly pure. Kadison proves in [6] that when \( \alpha \) is a pure state of a \( B^* \)-algebra, then \( \mathfrak{N}(\alpha) = K_\alpha + K_\alpha^* \) where \( \mathfrak{N}(\alpha) \) is the null space of \( \alpha \). We have the following generalization.

**Proposition 3.1.** If \( \alpha \) is a strictly pure state of \( A \), then \( \mathfrak{N}(\alpha) = K_\alpha + K_\alpha^* \)

**Proof.** Since \( M(\alpha) = 1 \), then \( |\alpha(a)|^2 \leq \alpha(a^*a) \) for all \( a \in A \). Therefore \( K_\alpha \subseteq \mathfrak{N}(\alpha) \), and it follows that \( K_\alpha + K_\alpha^* \subseteq \mathfrak{N}(\alpha) \). Now we prove the reverse inclusion.

By Theorem 2.1, \( K_\alpha \) is a modular left ideal of \( A \). Therefore there exists \( u \in A \) such that \( A(1 - u) \subseteq K_\alpha \). When \( a \in \mathfrak{N}(\alpha) \), then \( a^* \in \mathfrak{N}(\alpha) \), and \( (u + K_\alpha, a + K_\alpha) = \alpha(a^*u) = \alpha(a^*u - a^*) = 0 \). Thus \( u + K_\alpha \) is orthogonal to \( a + K_\alpha \) in \( A - K_\alpha = \mathfrak{H}_\alpha \). \( \pi_\alpha(A) \) is a \(*\)-subalgebra of \( \mathcal{B}(\mathfrak{H}_\alpha) \) which acts strictly irreducibly on \( \mathfrak{H}_\alpha \). Let \( B \) be the closure of \( \pi_\alpha(A) \) in the operator norm. By the transitivity theorem [3, Théorème (2.8.3)] there exists \( T \in B \), \( T = T^* \), such that \( T(u + K_\alpha) = 0 \) and \( T(a + K_\alpha) = a + K_\alpha \). Then there exists \( \{v_n\} \subseteq A \) such that \( v_n = v_n^* \) for all \( n \) and \( |\pi_\alpha(v_n) - T| \to 0 \) where \( |.| \) denotes the operator norm. Therefore \( |v_n - u + K_\alpha|_2 \to 0 \) and \( |v_n a - a + K_\alpha|_2 \to 0 \). Also \( v_n = v_n(1 - u) + v_n u \) and \( v_n(1 - u) \in K_\alpha \) for all \( n \). Then \( v_n + K_\alpha \to 0 \), and finally \( a^* v_n + K_\alpha \to 0 \). From the proof of Theorem 2.1 it follows that \( \|a^* v_n + K_\alpha\|_q \to 0 \) and \( \|(v_n a - a) + K_\alpha\|_q \to 0 \).

Assume for the moment the \( * \) is continuous on \( A \). There exists \( \{k_n\}, \{j_n\} \subseteq K_\alpha \) such that \( a^* v_n - k_n \to 0 \) and \( (a - v_n a) - j_n \to 0 \). Then \( \|a - (j_n + k_n^*)\| \leq \|v_n a - k_n^*\| + \|(a - v_n a) - j_n\| \to 0 \). Therefore in this case \( \mathfrak{N}(\alpha) = K_\alpha + K_\alpha^* \).

In the general case, let \( P_\alpha \) be the kernel of the representation \( \pi_\alpha \). \( A/P_\alpha \) is a semisimple Banach \(*\)-algebra. Note that \( P_\alpha \subseteq K_\alpha \cap K_\alpha^* \). Define \( \alpha_0 \) on \( a + P_\alpha \in A/P_\alpha \) by \( \alpha_0(a + P_\alpha) = \alpha(a) \). Then \( \alpha_0 \) is a strictly pure state of \( A/P_\alpha \). By Johnson's theorem [5, Theorem 2] the involution on \( A/P_\alpha \) is continuous. Therefore \( \mathfrak{N}(\alpha_0) = K_{\alpha_0} + K_{\alpha_0}^* \) by our previous argument. Then when \( a \in \mathfrak{N}(\alpha) \), there exists \( \{k_n\}, \{j_n\} \subseteq K_\alpha \) such that \( \|(a - (k_n + j_n^*)) + P_\alpha\|_q \to 0 \). Then there exists
Let $\alpha$ be a strictly pure state of $A$. Assume that $\beta \in \mathcal{P}$, $M(\beta) = 1$, and $K_\alpha = K_\beta$. Then $\alpha = \beta$.

**Proof.** $K_\beta + K_\beta^* \subset \mathcal{H}(\beta)$. Therefore

$$\mathcal{H}(\alpha) = \frac{K_\alpha + K_\alpha^*}{K_\beta + K_\beta^*} \subset \mathcal{H}(\beta).$$

It follows that there is a scalar $\lambda > 0$ such that $\alpha = \lambda \beta$. Then $1 = M(\alpha) = \lambda M(\beta) = \lambda$.

The next theorem answers Question 4 in a special case.

**Theorem 3.3.** Assume that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and $a \to \pi(a)$ and $a \to \gamma(a)$ are strictly irreducible *-representations of $A$ on $\mathcal{H}$ and $\mathcal{K}$ respectively. Then if $\pi$ and $\gamma$ are algebraically equivalent, then $\pi$ and $\gamma$ are unitarily equivalent.

**Proof.** By hypothesis there exists a linear operator $V$ which maps $\mathcal{K}$ in a one-to-one manner onto $\mathcal{H}$ with the property that $V^{-1} \pi(a) V = \gamma(a)$ for all $a \in A$. Take $\xi \in \mathcal{K}$ with $\| \xi \| = 1$, and set $\alpha(a) = (\gamma(a) \xi, \xi)$ for $a \in A$. By [8, Lemma (4.5.8), p. 217] the representation $\gamma$ is unitarily equivalent to $\pi_\alpha$ on $\mathcal{H}_\alpha$. Also $M(\alpha) = \| \xi \|^2 = 1$ by [4, Theorem (21.25), p. 323]. Then $\alpha$ is a strictly pure state of $A$ by [4, Theorem (21.34), p. 328]. Now set $\eta = V(\xi)/\| V(\xi) \|$. Define $\beta(a) = (\pi(a) \eta, \eta)$ for all $a \in A$. By the same argument as just applied to $\alpha$, $\beta$ is a strictly pure state of $A$, and $\pi_\beta$ is unitarily equivalent to $\pi$. Then

$$a \in K_\alpha \iff \gamma(a) \xi = 0 \iff \gamma(\alpha) (V^{-1}(\eta)) = 0$$

$$\iff V^{-1}(\pi(a)(\eta)) = 0 \iff \pi(a) \eta = 0 \iff a \in K_\beta.$$
4. Irreducible representations which are similar to \(*\)-representations. Let $a \rightarrow \pi(a)$ be a strictly irreducible representation (but not necessarily a \(*\)-representation) of $A$ on a Hilbert space $\mathcal{H}$. If $\xi \in \mathcal{H}$, $\xi \neq 0$, then a straightforward algebraic argument proves that $K_\xi = \{a \in A | \pi(a)\xi = 0\}$ is a modular maximal left ideal of $A$. We show in the next theorem that when $K_\xi$ is the left kernel of a strictly pure state $\alpha$ of $A$, then $\pi$ is similar to a \(*\)-representation of $A$ on $\mathcal{H}$.

Theorem 4.1. Let $a \rightarrow \pi(a)$ and $K_\xi$ be as above. Assume that $\alpha$ is a strictly pure state of $A$ with $K_\alpha = K_\xi$. Then there exists a \(*\)-representation $a \rightarrow \rho(a)$ of $A$ on $\mathcal{H}$ and a positive operator $V \in \mathcal{B}(\mathcal{H})$ such that, for all $a \in A$,

$$\pi(a) = V^{-1}\rho(a)V.$$  

Proof. Since $\pi$ is strictly irreducible, $a \rightarrow \pi(a)\xi$ is a linear map from $A$ onto $\mathcal{H}$. We define a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{H} \times \mathcal{H}$ by

$$\langle \pi(a)\xi, \pi(b)\xi \rangle = \alpha(b^*a),$$

$a, b \in A$. Whenever $c \in K_\alpha$ and $d \in A$, then $\alpha(d^*c) = 0$. This implies that $\langle \cdot, \cdot \rangle$ is well defined.

Next we prove that $\langle \cdot, \cdot \rangle$ is a bounded form. By a theorem of B. E. Johnson [5, Theorem 1, p. 537] $\pi$ is a continuous map of $A$ into $\mathcal{B}(\mathcal{H})$. If $k \in K_\xi$, $a \in A$,

$$||\pi(a)\xi|| = ||\pi(a + k)\xi|| \leq ||\pi|| ||\xi|| ||a + k||.$$  

Therefore for any $a \in A$,

(1)  $$||\pi(a)\xi|| \leq ||\pi|| ||\xi|| ||a + K_\xi||_q.$$  

Then by the Closed Graph Theorem there exists $N > 0$ such that, for all $a \in A$,

(2)  $$||a + K_\xi||_q \leq N ||\pi(a)\xi||.$$  

As shown in the proof of Theorem 2.1, the norms $||\cdot||_2$ and $||\cdot||_q$ are equivalent on $A - K_\alpha$. In particular there exists $J > 0$ such that $||a + K_\alpha||_2 \leq J ||a + K_\alpha||_q$ for all $a \in A$. Therefore for all $a, b \in A$,

(3)  $$||\alpha(b^*a)|| = ||(a + K_\alpha, b + K_\alpha)|| \leq J^2 ||a + K_\alpha||_q ||b + K_\alpha||_q.$$  

Then combining (2) and (3) we have

$$||\pi(a)\xi, \pi(b)\xi|| = ||\alpha(b^*a)|| \leq J^2 ||a + K_\alpha||_q ||b + K_\alpha||_q \leq J^2 N^2 ||\pi(a)\xi|| ||\pi(b)\xi||.$$  

This proves that $\langle \cdot, \cdot \rangle$ is bounded on $\mathcal{H} \times \mathcal{H}$.

The form $\langle \cdot, \cdot \rangle$ is a symmetric, positive definite, bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$. Therefore there exists an operator $U \in \mathcal{B}(\mathcal{H})$ such that $U = U^*$, $U \geq 0$, and $\langle \phi, \psi \rangle = \langle U\phi, \psi \rangle$, when $\phi, \psi \in \mathcal{H}$.

By (3), for all $a \in A$,

$$||a + K_\alpha||_2^2 = \alpha(a^*a) \leq J^2 (||a + K_\alpha||_q)^2.$$
By the proof of Theorem 2.1 there exists $P > 0$ such that, for all $a \in A$,
\[ \|a + K_a\|_q \leq P \|a + K_a\|_2. \]

Given $a \in A$, set $\psi = \pi(a)\xi$. Then by (1),
\[ \|\psi\| = \|\pi(a)\xi\| \leq \|\pi\| \|\xi\| \|a + K_a\|_q. \]

Set $M = \|\pi\| \|\xi\| P$. Then
\[ \|\psi\|^2 \leq M^2 (\|a + K_a\|_2)^2 = M^2 \alpha(a^*a) = M^2 [\psi, \psi]. \]

Therefore
\[ \|\psi\|^2 \leq M^2 [\psi, \psi] = M^2 (U\psi, \psi) \leq M^2 \|U\psi\| \|\psi\|. \]

Finally $\|\psi\| \leq M^2 \|U\psi\|$, and this proves that $U^{-1} \in \mathcal{B}(\mathcal{H})$.

Now set $V = U^{1/2}$. Then $[\phi, \psi] = (V\phi, V\psi)$ for all $\phi, \psi \in \mathcal{H}$. Let $\rho(a) = V\pi(a)V^{-1}, a \in A$. Given $\psi_1, \psi_2 \in \mathcal{H}$, there exists $\phi_1, \phi_2 \in \mathcal{H}$ and $a_1, a_2 \in A$ such that
\[ \psi_i = V\phi_i \quad \text{and} \quad \phi_i = \pi(a_i)\xi, \quad i = 1, 2. \]

Then
\[ (\rho(a)\psi_1, \psi_2) = (V\pi(a)V^{-1}V\phi_1, V\phi_2) \]
\[ = [\pi(a)\phi_1, \phi_2] = [\pi(a)\pi(a_1)\xi, \pi(a_2)\xi] \]
\[ = \alpha(a^*_a a_{11}) = \alpha((a^*_a a_{22})^* a_{11}) \]
\[ = [\pi(a_1)\xi, \pi(a^*)\pi(a_2)\xi] = [\phi_1, \pi(a^*)\phi_2] \]
\[ = (V\phi_1, V\pi(a^*)\phi_2) = (V\phi_1, V\pi(a^*)V^{-1}V\phi_2) = (\psi_1, \rho(a^*)\psi_2). \]

Therefore $\rho(a^*) = \rho(a)^*$ for all $a \in A$ which completes the proof of the theorem.

Corollary 4.2. Assume that every modular maximal left ideal of $A$ is the left kernel of a strictly pure state of $A$. Let $a \rightarrow \pi(a)$ be a strictly irreducible representation of $A$ on a Hilbert space $\mathcal{H}$. Then there exists a $*$-representation $a \rightarrow \rho(a)$ of $A$ on $\mathcal{H}$ and a positive operator $V \in \mathcal{B}(\mathcal{H})$ such that, for all $a \in A$,
\[ \pi(a) = V^{-1}\rho(a)V. \]

5. Some examples. When $A$ is $B^*$-algebra, then $A$ has the following two properties:

(I) Every pure state of $A$ is strictly pure.

(II) Every modular maximal left ideal of $A$ is the left kernel of a strictly pure state of $A$.

Also when $G$ is a compact topological group and $1 \leq p < +\infty$, then
$A = L^p(G)$ (or $C(G)$, the continuous functions on $G$) has properties (I) and (II).

Here the multiplication is, as usual, convolution. All the irreducible $\ast$-representations of $A$ in this case are finite dimensional. In this section we present two examples of algebras which have properties (I) and (II), but which are not in general $B^\ast$-algebras, and which need not in general have any finite dimensional $\ast$-representations.

Example 5.1. Let $A$ be a Banach algebra which is also a dense $\ast$-ideal in a $B^\ast$-algebra $B$. Any full Hilbert algebra is a particular example of such a Banach algebra; see [1].

Assume that $a \rightarrow \pi(a)$ is an irreducible $\ast$-representation of $A$ on a Hilbert space $H$. Then as shown in [1, Proposition 4.1] $\pi$ extends uniquely to a $\ast$-representation $b \rightarrow \pi(b)$ of $B$ on $H$. Therefore by Kadison's theorem $\pi(B)$ acts strictly irreducibly on $H$. Since $A$ is a dense ideal of $B$, $\pi(A) = \pi(A)$ is a non-zero ideal in $\pi(B)$. Given $\xi \in H$, $\pi(A)\xi$ is a $\pi(B)$-invariant subspace of $H$. Therefore $\pi(A)\xi = H$, so that $a \rightarrow \pi(a)$ is strictly irreducible on $H$. It follows that $A$ has property (I).

Now assume that $M$ is a modular maximal left ideal of $A$. Then there exists $u \in A$ such that $A(1 - u) C M$. Let $N = \{b \in B\} bu \in M\}$. $N$ is a left ideal of $B$ and $M = N \cap A$. Furthermore if $b \in B$, $b(1 - u)u = bu(1 - u) \in M$ since $bu \in A$. Therefore $N$ is a proper modular left ideal of $B$. By [8, Theorem (4.9.8), p. 251] there exists a pure state of $B$ with $N \subset K_{\alpha}$. Then $M = K_{\alpha} \cap A$. It follows that $\alpha$, the restriction of $\alpha$ to $A$, is a strictly pure state of $A$ with $K_{\alpha} = M$. We have shown that $A$ has property (II).

Example 5.2. Assume that $\Omega$ is a compact Hausdorff space and $B$ is a $B^\ast$-algebra with identity $e$. Let $C(\Omega, B)$ be the algebra of all continuous $B$-valued functions on $\Omega$. $C(\Omega, B)$ is a $B^\ast$-algebra with identity. Assume that $A$ is a Banach algebra which is a $\ast$-subalgebra of $C(\Omega, B)$ containing the identity. We also assume that $A$ has the properties:

1. Given $\omega \in \Omega$ and $b \in B$, there exists $f \in A$ such that $f(\omega) = b$.
2. $f \in A$ is left invertible in $A$ if and only if $f(\omega)$ is invertible in $B$ for all $\omega \in \Omega$.

We mention a specific example of such an algebra $A$: Let $\Omega$ be the interval $[0, 2\pi]$ with 0 and $2\pi$ identified and with the usual topology. Let $B$ be any $B^\ast$-algebra with identity. We define $A$ to be the algebra of all functions of the form

$$f(t) = \sum_{n = -\infty}^{+\infty} a_n e^{int}$$

where $t \in \Omega$ and $\{a_n\}$ is any sequence in $B$ such that $\sum_{n = -\infty}^{+\infty} \|a_n\| < +\infty$. When $f(t) = \sum_{n = -\infty}^{+\infty} a_n e^{int}$, let $\|f\| = \sum_{n = -\infty}^{+\infty} \|a_n\|$. The algebra $A$ is discussed by...
Bochner and Phillips in [2]. That $A$ has property (2) above is the assertion of [2, Theorem 1, p. 409]. The rest of the required properties of $A$ are easily verified.

Now assume that $A$ is any Banach *-subalgebra of $C(\Omega, B)$ which contains the identity and satisfies (1) and (2). When $\omega \in \Omega$ and $N$ is a maximal left ideal of $B$, we define

$$K(\omega, N) = \{f \in A \mid f(\omega) \in N\}.$$ 

It is not difficult to see that $K(\omega, N)$ is a maximal left ideal of $A$. We prove the converse of this. Assume that $M$ is a maximal left ideal of $A$. For any $\omega \in \Omega$, $M(\omega) = \{f(\omega) \mid f \in M\}$ is a left ideal of $B$. Suppose that $M(\omega) = B$ for all $\omega \in \Omega$. Then for each $\omega \in \Omega$, we can choose a function $g_\omega \in M$ such that $g_\omega(\omega) = e$.

Therefore there exists an open set $U_\omega$ in $\Omega$ such that $\omega \in U_\omega$ and $g_\omega(y)$ is invertible in $B$ for all $y \in U_\omega$. Then $(g^*_\omega g_\omega)(y)$ is invertible in $B$ for all $y \in U_\omega$.

Choose a finite cover $U_{\omega_1}, \ldots, U_{\omega_n}$ for $\Omega$. Set $f = \sum_{k=1}^n g^*_\omega_k g_\omega_k \in M$. When $b_k \in B$, $b_k \geq 0$, $1 \leq k \leq n$, and $b_j$ is invertible for some $j$, then $h_1 + \cdots + h_n$ is invertible (this is easy to verify when the $b_j$ are positive operators on a Hilbert space, since the lower bound of the numerical range of the sum $h_1 + \cdots + h_n$ is greater or equal to the lower bound of the numerical range of $b_j$). But then for all $y \in \Omega$, $f(y)$ is invertible in $B$. By (2), $f$ is then invertible in $A$, which contradicts the fact that $f \in M$. It follows that for some $\omega \in \Omega$, $M(\omega)$ is a proper left ideal of $B$. Then there exists a maximal left ideal of $B$ such that $M(\omega) \subseteq N$.

Therefore $M \subseteq K(\omega, N)$, so that $M = K(\omega, N)$ by the assumption that $M$ is maximal.

Given $M$ a maximal left ideal of $A$, then as we have shown above $M = K(\omega, N)$ for some $\omega \in \Omega$ and some maximal left ideal $N$ of $B$. Choose $\beta$ a pure state of $B$ such that $K_\beta = N$. Define $\alpha$ on $A$ by $\alpha(f) = \beta(f(\omega))$, $f \in A$. Then $K_\alpha = K(\omega, N) = M$. It is easy to verify that the norm $\|f + K_\alpha\|_2 = \alpha(f^*f)^{1/2}$ is a complete norm on $A - K_\alpha$. Therefore $\alpha$ is a strictly pure state of $A$. This proves that $A$ has property (II).

Now assume that $\alpha$ is a pure state of $A$. Then by [3, Lemma 2.10.1, p. 50] $\alpha$ has an extension to a pure state $\beta$ of $C(\Omega, B)$. By [7, Corollary, p. 337] there exists a point $\omega \in \Omega$ and a maximal left ideal $N$ of $B$ such that $K_\beta = \{f \in C(\Omega, B) \mid f(\omega) \in N\}$. Therefore $K_\alpha = K_\beta \cap A = K(\omega, N)$. It follows that $\alpha$ is a strictly pure state of $A$ by Proposition 3.4. Therefore $A$ has property (I).

REFERENCES


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