TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 171, September 1972

# APPROXIMATION ON DISKS(1)

BY

KENNETH JOHN PRESKENIS

ABSTRACT. Let $D$ be a closed disk in the complex plane, $f$ a complex valued continuous function on $D$ and $R_f(D)$ = the uniform closure on $D$ of rational functions in $z$ and $f$ which are finite. Among other results we obtain the following. **Theorem.** *If $f$ is of class $C^1$ in a neighborhood of $D$ and $|f_{\bar z}| > |f_z|$ everywhere (i.e., $f$ is an orientation reversing immersion of $D$ in the plane), then $R_f(D) = C(D)$.* **Theorem.** *Let $f$ be a polynomial in $z$ and $\bar z$. If for each $a$ in $D$, $f - \sum (j!)^{-1} D^j f(a)(z-a)^j = (\bar z - \bar a)^k g$ with $|g_{\bar z}| > |g_z|$ at the zeros of $g$ in $D$ where $Df = f_z$, then $R_f(D) = C(D)$.* **Corollary.** *Let $f$ be a polynomial in $z$ and $\bar z$ and let $|f_{z\bar z}(0)| < |f_{\bar z\bar z}(0)|/2$. Then there exists an $r > 0$ such that, for $D = (|z| \le r)$, $R_f(D) = C(D)$.* The proofs of the theorems use measures and the conditions involved in the theorems are independent of each other. Concerning the corollary, results of E. Bishop and G. Stolzenberg show that if $f_{\bar z}(0) = 0$ and $|f_{\bar z\bar z}(0)| < |f_{z\bar z}(0)|$, then there exists no $r$ such that $R_f(D) = C(D)$ where $D = (|z| \le r)$.

Let $F = (f_1, \cdots, f_n)$ be a map on $B$ = unit polydisk in $\mathbf{C}^n$ with values in $\mathbf{C}^n$, $P_F$ = uniform closure on $B$ of polynomials in $z_1, \cdots, z_n, f_1, \cdots, f_n$. **Theorem.** *If $F$ is of class $C^1$ in a neighborhood of $B$, $F_{\bar z}$ is invertible and if for each $a = (a_1, \cdots, a_n)$ in $B$, there exist complex constants $\{c_j\}, \{d_{ij}\}$, $i, j = 1, \cdots, n$, such that $\sum c_j(z_j - a_j)(f_j(z) - f_j(a)) + \sum d_{ij}(z_i - a_i)(z_j - a_j)$ has positive real part for all $z \ne a$, then $\{(\zeta, F(\zeta)) : \zeta \in B\}$ is a polynomially convex set.* **Corollary.** *If $F = (f, g)$ where $f(z, w) = \bar z + cz\bar z + d\bar z^2 + q\bar z w$, $g(z, w) = \bar w + sw\bar w + t\bar w^2 + p\bar w z$ and the coefficients satisfy $|\bar c + d| + |d| + |q| < 1$ and $|\bar s + t| + |t| + |p| < 1$, then $P_F = C(B)$.* **Corollary.** *If $F(z) = \bar z + R(z)$ where $R = (R_1, \cdots, R_n)$ is of class $C^2$ and satisfies the Lipschitz condition $|R(\zeta) - R(\eta)| \le k|\zeta - \eta|$ with $k < 1$, then $P_F = C(B)$.* This last corollary is a result of Hörmander and Wermer. The proof of the theorem uses methods from several complex variables.

**Introduction.** In this study we are concerned with uniform approximation on closed disks in the complex plane and on closed polydisks in complex $n$-space. Our work is principally based on two papers of John Wermer [18] and [19] in this area and our results are related to those in his papers.

The basic problem under consideration may be stated simply. We denote by $\mathbf{C}$ the complex plane and by $z$ the identity function of $\mathbf{C}$ to $\mathbf{C}$ or its restrictions.

Presented in part to the Society, April 9, 1971 under the title *A theorem on polynomials in $z$ and $\bar z$*; received by the editors August 3, 1971.

*AMS* 1969 *subject classifications*. Primary 4130, 3270; Secondary 3070.

(1) This work was part of the author's Doctoral Thesis written at Brown University in 1971. Work was supported by National Science Foundation grant GP-28574.

Let $D$ be any disk in $\mathbf{C}$ and let $C(D)$ be the set of all continuous complex valued functions on $D$. For $f$ in $C(D)$ we denote by $P_f(D)$ the uniform closure on $D$ of polynomials in $z$ and $f$. What conditions on $f$ yield $P_f(D) = C(D)$?

In dealing with the question, it is usually convenient to consider $D$ to be the closed unit disk and in this case we simply write $P_f$ for $P_f(D)$. A familiar theorem of Weierstrass states that if $f = \bar{z}$, then $P_f = C(D)$. Other special cases worth mentioning are when $f$ is the zero function and when $f = |z|^2$. In the first case, $P_f$ is the uniform closure on $D$ of all polynomials in $z$ and it is well known that what is obtained here is all continuous functions on $D$ which are analytic in the interior of $D$. Concerning the case $f = |z|^2$, we see that $f$ is constant on any circle in $D$ with center at the origin, and so any function $g$ in $P_f$ is a uniform limit of polynomials in $z$ on each such circle. In this event, we know that $g$ must agree on each circle with a unique function which is continuous in the closed disk determined by the circle and analytic inside the circle. Hence, in these two cases we have $P_f \neq C(D)$.

Whenever $P_f \neq C(D)$, it is natural to ask how the elements of $P_f$ are characterized. Some interesting results in this area can be found in [11] and [19]. However, in this work we will not be concerned with this aspect of the problem, but will limit ourselves to the task of trying to determine when $P_f = C(D)$.

In § 1, we examine closely the statement and proof of a very interesting result of J. Wermer [18]. The statement of the result is: if $f = \bar{z} + R$ where $|R(s) - R(a)| < |s - a|$ for all $s$, $a$ in $D$ with $s \neq a$, then $P_f = C(D)$. In order to put this fact in its proper setting we observe that there exist functions $f$ arbitrarily close to $\bar{z}$ in the uniform norm on $D$ for which $P_f \neq C(D)$ (approximate $\bar{z}$ by functions vanishing in a neighborhood of zero). And also we observe that a consequence of Wermer's result is that if a function $f$ is "close to" $\bar{z}$ in a class $C^1$ sense, then we have $P_f = C(D)$. We also point out that in perturbing $\bar{z}$ as Wermer did, two things which remained unchanged are the facts that the map is one-to-one and that (for a smooth function $f$) $|f_{\bar{z}}| > |f_z|$ everywhere (see definitions in § 1). This last property is equivalent to saying that $f$, regarded as a map from $E_2$ to $E_2$, has negative Jacobian determinant everywhere. A reasonable conjecture is that if $f$ is a smooth function in a neighborhood of $D$ such that $|f_{\bar{z}}| > |f_z|$ everywhere, then $P_f = C(D)$. Or perhaps it might be true under the additional hypothesis that $f$ be one-to-one. We have not been able to prove the conjecture but some progress has come from our efforts. In fact, let us denote by $R_f$ the uniform closure on $D$ of rational functions in $z$ and $f$ which are finite. We have proven that if $f$ is of class $C^1$ in a neighborhood of $D$ and if $|f_{\bar{z}}| > |f_z|$ everywhere, then $R_f = C(D)$.

Roughly speaking, this result differs from Wermer's result in that the latter concerns a function which is "near" $\bar{z}$ while our result concerns a function which is "like" $\bar{z}$ in the sense that it is an orientation reversing immersion of $D$ in the plane. Also we would like to mention that our result seems to add evidence to the

conjecture since in all known examples where $P_f \neq C(D)$ we also have $R_f \neq C(D)$.

In §2, we focus attention on the special case when $f$ is a polynomial in $z$ and $\bar{z}$. Among other things, we present a sufficient condition for such a function $f$ to satisfy $R_f = C(D)$ (see Theorem 2.10). This sufficient condition is independent of the condition $|f_{\bar{z}}| > |f_z|$ and in some respects is easier to work with. Also we present an interesting consequence, namely, if $|f_{z\bar{z}}(0)| < |f_{\bar{z}\bar{z}}(0)|/2$, then there exists an $r > 0$ such that for $D_0 = \{s : |s| \leq r\}$ we have $R_f(D_0) = C(D_0)$. Here $R_f(D_0)$ denotes the uniform closure on $D_0$ of all rational functions in $z$ and $f$ which are finite. This result can be viewed as a contribution to the local version of the problem with which we are dealing. In order to relate this to other contributions we remark that results of E. Bishop [2] and G. Stolzenberg [16] show that if $f$ is smooth, $f_{\bar{z}}(0) = 0$ and $|f_{\bar{z}\bar{z}}(0)| < |f_{z\bar{z}}(0)|$, then there exists no $r$ such that $R_f(D_0) = C(D_0)$, while J. Wermer [18] has shown that if $f_{\bar{z}}(0) \neq 0$, then there exists a disk $D_0$ about the origin for which $P_f(D_0) = C(D_0)$.

In the final section, we consider some generalizations to higher dimensions of the previous material. We shall denote by $z_1, z_2, \cdots, z_n$ the coordinate functions on $\mathbf{C}^n$ and by $B$ the unit polydisk in $\mathbf{C}^n$, i.e. the set of all points $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbf{C}^n$ satisfying $|\zeta_i| \leq 1$ for each $i$. If $F = (f_1, f_2, \cdots, f_n)$ is a vector valued function on $B$ with each $f_j \in C(B)$, we denote by $P_F$ the uniform closure on $B$ of polynomials in $z_1, \cdots, z_n, f_1, \cdots, f_n$. As in the case $n = 1$, the problem is to determine conditions on $F$ which give $P_F = C(B)$.

It turns out that the measure techniques used in the first two sections are not so productive in this higher dimensional context. However, another way of looking at the problem combines with some methods from several complex variables to allow some progress to be made. More specifically, whenever $F$ is as above we shall denote by $X_F$ the graph of $F$, i.e. the set $\{(\zeta, F(\zeta)) : \zeta \in B\}$ (which is a compact subset of $\mathbf{C}^{2n}$). It is straightforward to verify that $P_F$ is isometrically isomorphic to $P(X_F)$, the set of all continuous functions on $X_F$ which can be uniformly approximated on $X_F$ by polynomials in the coordinate functions on $\mathbf{C}^{2n}$. And in trying to prove $P(X_F) = C(X_F)$ we can call upon the famous Oka-Weil theorem (see Theorem 3.1) as well as the more recent work of R. Nirenberg-R. O. Wells, Jr. [12] and L. Hörmander-J. Wermer [8] (which is based on some deep results of J. J. Kohn [9] giving $L^2$-estimates for the solution of the inhomogeneous Cauchy-Riemann equations). From these several complex variable results, we can conclude that for a smooth function $F$, $P(X_F) = C(X_F)$ providing that $X_F$ is a polynomially convex set (for definition, see § 3) and $X_F$ lies on a real manifold which has no complex tangents (see the definition in §3). The condition of having no complex tangents is precisely the condition that the matrix $F_{\bar{z}}$ (see §3) be invertible everywhere. And this calls attention to the difficult problem of showing $X_F$ is a polynomially convex set directly from local conditions on $F$.

This section deals with this problem. Specifically, we describe a method by which polynomial convexity may be verified in some cases. And in particular we prove the following. Let $F$ be of class $C^1$ in a neighborhood of $B$. Assume for each $a = (a_1, \cdots, a_n)$ in $B$ we have $F_{\bar{z}}(a)$ is invertible and there exist complex constants $\beta_j$ and $\gamma_{ij}$ with $i = 1, \cdots, n$, $j = 1, \cdots, n$, such that

$$\sum_{j=1}^{n} \beta_j(z_j - a_j)(f_j - f_j(a)) + \sum_{i,j=1}^{n} \gamma_{ij}(z_i - a_i)(z_j - a_j)$$

has positive real part in $B - \{a\}$. Then $X_F$ is polynomially convex.

1. **Sufficient conditions for $P_f = C(D)$ and $R_f = C(D)$.** Let $X$ be a compact Hausdorff space. By a measure on $X$ we will mean a complex regular Borel measure. For any measure $\mu$ on $X$, $|\mu|$ will denote the associated positive total variation measure. Also for each $\mu$ we can determine a smallest closed set on which $\mu$ "lives", called the support of $\mu$, and we will denote this set by supp $\mu$. For any measurable function $f$ on $X$ satisfying $\int_X |f|\, d|\mu| < \infty$, the Borel measure $f\mu$ is defined by $(f\mu)(F) = \int_F f\, d\mu$ for every Borel set $F$ in $X$.

Our concern with measures derives from their connection with the Banach space $C(X)$ consisting of all complex valued continuous functions on $X$. Specifically, in the first two sections we use the following argument. Let $E$ be a closed subspace of $C(X)$. It follows from a standard corollary of the Hahn-Banach theorem that $E = C(X)$ if and only if there exists no nonzero continuous linear functional on $C(X)$ which annihilates $E$. And according to the well-known representation theorem of F. Riesz (due in full generality to Kakutani), each continuous linear functional on $C(X)$ is of the form $f \longrightarrow \int f\, d\mu$, where $\mu$ is a (uniquely determined) complex regular Borel measure on $X$. (See [15], for instance.) Thus, to show that $E = C(X)$, it suffices to show that if $\mu$ is a complex Borel measure on $X$ such that $\int f\, d\mu = 0$ for all $f$ in $E$, then $\mu$ is the zero measure. We shall adopt the convention of expressing the condition "$\int f\, d\mu = 0$ for all $f$ in $E$" as simply "$\mu \perp E$" and we shall call $\mu$ an annihilating measure for $E$.

The usefulness of the above argument arises from some important properties of compactly supported measures in $\mathbf{C}$. And before we devote ourselves to the analysis of the main problem, we quote these needed definitions and results.

Lebesgue two-dimensional measure in $\mathbf{C}$ will be denoted by $m$.

**Definition.** Let $K$ be a compact set in $\mathbf{C}$, $\mu$ a measure on $K$. For all $s \in \mathbf{C}$, we put $\tilde{\mu}(s) = \int d|\mu|/|z - s|$.

**Lemma 1.1.** *With $K$, $\mu$ as above, $\tilde{\mu}$ is integrable over any compact set (with respect to $m$); in parti ular, $\tilde{\mu} < \infty$ a.e. ($m$).*

**Definition.** Let $K$, $\mu$ be as above. For each $s \in \mathbf{C}$ such that $\tilde{\mu}(s) < \infty$, we define $\hat{\mu}(s) = \int d\mu/(z - s)$.

Thus $\hat{\mu}$ is defined a.e. ($m$), and integrable over bounded sets.

**Definition.** The differential operators $\partial/\partial z$, $\partial/\partial \bar{z}$ are given by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

We often denote $\partial f/\partial z$ by $f_z$ and $\partial f/\partial \bar{z}$ by $f_{\bar{z}}$.

**Lemma 1.2.** *Let $K$, $\mu$ be as above and let $f$ be a class $C^1$ function having compact support in* $\mathbf{C}$. *Then* $\int_K f\,d\mu = (1/\pi)\int_{\mathbf{C}} f_{\bar{z}}\hat{\mu}\,dm$.

**Corollary 1.3.** *Let $\mu$ be a compactly supported measure on* $\mathbf{C}$. *For any open set $U$ in* $\mathbf{C}$ *such that $\hat{\mu}(s) = 0$ for almost all $s$ in $U$, we have $|\mu|(U) = 0$; in particular, $\mu = 0$ if $\hat{\mu}(s) = 0$ a.e. in* $\mathbf{C}$.

For a proof of Lemmas 1.1 and 1.2 see [3].

We are now prepared to treat our problem. By $D$ we shall continue to understand the unit disk, unless stated otherwise. As indicated before, in order to show that $P_f = C(D)$ ($R_f = C(D)$), it suffices to prove any measure $\mu$ on $D$ satisfying $\mu \perp P_f$ ($\mu \perp R_f$) is the zero measure. A key role in our work is the idea (see Corollary 1.3) of proving a measure $\mu$ is the zero measure by showing $\hat{\mu} = 0$ a.e. This idea seems to have originated from some work of E. Bishop; at least it has been a commonly used tool since his paper [1] on peak points in 1959.

For an arbitrary function $f$ in $C(D)$, it is the case that if $\mu$ is a measure on $D$ such that $\mu \perp P_f$ (or $\mu \perp R_f$), then $\hat{\mu} = 0$ in $\mathbf{C} - D$. In fact, for any $a$, $a \notin D$, we have $1/(z-a) = -\sum_0^\infty z^n/a^{n+1}$, the series converging uniformly on $D$, and since $\int z^n\,d\mu = 0$ for all $n$, we have $\hat{\mu}(a) = 0$. This together with Corollary 1.3 allows us to call upon the following principle.

**Proposition A.** *Let $f$ be fixed in $C(D)$ and let $\mu$ be a measure on $D$ such that $\mu \perp P_f$. Then $\mu = 0$ if and only if $\hat{\mu}(s) = 0$ for almost all $s$ in $D$. The same conclusion holds when $\mu \perp R_f$.*

At this point we would like to present two technical lemmas.

**Lemma 1.4.** *Let $E = \{re^{i\theta} : 0 < r < d,\ -\alpha < \theta < \alpha\}$ where $0 < \alpha \le \pi/2$. Let $0 < \epsilon \le d/2$. Then $1/|s-\epsilon| \le k/|s|$ for all $s \notin E$ where $k = \max\{2, 1/\sin\alpha\}$.*

**Proof.** If $\operatorname{Re} s \le 0$, then $|s-\epsilon| > |s|$; so that $1/|s-\epsilon| < 1/|s|$. If $\operatorname{Re} s > 0$ and $\arg s \notin \langle -\alpha, \alpha\rangle$, then $|s|\sin\alpha \le |s|\,|\sin(\arg s)| = |\operatorname{Im} s| \le |s-\epsilon|$; so that $1/|s-\epsilon| \le 1/|s|\sin\alpha$. Finally, if $\operatorname{Re} s > 0$, $\arg s \in \langle -\alpha, \alpha\rangle$ and $|s| \ge d$, then $|s-\epsilon| \ge |s| - \epsilon \ge |s|/2$; so that $1/|s-\epsilon| \le 2/|s|$.

Throughout this chapter we shall denote by $H$ the half-plane $\{s \in \mathbf{C} : \operatorname{Re} s \ge 0\}$.

**Lemma 1.5.** *Let $K$ be a compact subset of $H$. Then $1/(z+\epsilon)$ is uniformly approximable on $K$ by polynomials, for any $\epsilon > 0$.*

**Proof.** $\{s : \operatorname{Re} s > -\epsilon\} = \bigcup_{M>0}\{s : |s-M| < M+\epsilon\}$ and so there exists an $M > 0$ such that $K \subset \{s : |s-M| < M+\epsilon\}$. Now we have

$$\frac{1}{z+\epsilon} = \sum_{0}^{\infty} (-1)^n \frac{(z-M)^n}{(M+\epsilon)^{n+1}},$$

the series converging uniformly on any compact subset of $\{s : |s - M| < M + \epsilon\}$.

We are now in a position to present the statement and proof of a theorem which we use as a frame of reference throughout this paper.

**Wermer's Theorem 1.6.** *Let $f = \bar{z} + R$ where $R \in C(D)$ satisfies the Lipschitz condition $|R(s) - R(t)| < |s - t|$ for all $s, t \in D$, $s \neq t$. Then $P_f = C(D)$.*

**Proof.** Let $\mu$ be a measure annihilating $P_f$ and take $a \in D$ with $\tilde{\mu}(a) < \infty$. We aim to show $\hat{\mu}(a) = 0$. Consider the function $\lambda = (z - a)(f - f(a))$ and observe that $\lambda \in P_f$. Also since $\lambda = (z - a)(\bar{z} - \bar{a} + R - R(a)) = |z - a|^2 + (z - a)(R - R(a))$ we have that $\operatorname{Re} \lambda > 0$ in $D - \{a\}$ and so $\lambda(D)$ is a compact subset of $H$. Next, consider the functions $p_n$ defined by $p_n = 1/(z + 1/n)$, $n = 1, 2, \cdots$. Quickly we have that $|p_n| \leq 1/|z|$ in $H$, independent of $n$, and also, as $n \to \infty$, $p_n \to 1/z$ everywhere in $H$. By Lemma 1.5, $p_n$ is uniformly approximable on $\lambda(D)$ by polynomials and so $p_n(\lambda) \in P_f$, independent of $n$. Now if we put $g_n = (f - f(a))p_n(\lambda)$, then we have $g_n \in P_f$ and, as $n \to \infty$, $g_n \to (f - f(a))/(f - f(a))(z - a)$ in $D$, so $g_n \to 1/(z - a)$ in $D - \{a\}$ (thus almost everywhere in $D$ with respect to $\mu$). Further, $|g_n| \leq 1/|z - a|$ in $D$, so the dominated convergence theorem applies and we have $\hat{\mu}(a) = \lim \int g_n \, d\mu = 0$. By Proposition A we have $\mu = 0$. The theorem follows.

Extracting the essential ingredients from Wermer's theorem, we offer the following mild generalization.

**Theorem 1.7.** *Let $f \in C(D)$. Suppose that for almost all $a \in D$ there exists $\phi_a \in P_f$ with the following three properties:*

(i) *$(z - a)\phi_a$ maps $D$ into $H$;*

(ii) *$\{s : \phi_a(s) = 0\}$ is at most countable, for every $a$;*

(iii) *$\{a : \phi_a(s) = 0\}$ has Lebesgue measure zero, for every $s$.*

*Then $P_f = C(D)$.*

**Proof.** Let $\mu \perp P_f$ and put $E = \{b : \mu(\{b\}) \neq 0\}$. $E$ is a countable set. For each $b \in E$, put $E_b = \{a : \phi_a(b) = 0\}$ and let $F = \bigcup_{b \in E} E_b$. By hypothesis (iii), each $E_b$ has measure zero, and hence, so does $F$. Thus to prove the theorem it suffices to show that $\hat{\mu}(a) = 0$ whenever $\tilde{\mu}(a) < \infty$, $\phi_a$ exists and $a \notin F$. Let $a$ be such a point and put $\lambda = (z - a)\phi_a$. Also for $n = 1, 2, \cdots$, put $p_n = 1/(z + 1/n)$ and $g_n = \phi_a p_n(\lambda)$. It follows from hypothesis (i) that $g_n \in P_f$ for each $n$. From the definition of $p_n$, it follows that $g_n(s) = 0$ whenever $\phi_a(s) = 0$ and that $|g_n| \leq 1/|z - a|$ everywhere in $D$, both holding independent of $n$. Now if $(s - a)\phi_a(s) \neq 0$, we have $g_n(s) \to 1/(s - a)$. But $\{s : \phi_a(s) = 0\} \cup \{a\}$ is a countable set by (ii), and is disjoint from $E$ by our choice of $a$, and thus is a set of $\mu$-measure zero. Hence, we have $g_n$

$\longrightarrow 1/(z-a)$ a.e. $(\mu)$, and $|g_n| \leq 1/|z-a|$ in $D$, so by dominated convergence we have $\hat{\mu}(a) = \lim \int g_n \, d\mu = 0$. We conclude $\mu = 0$ (Proposition A). The theorem is proved.

We would like to point out that the hypothesis on $R$ in Wermer's theorem cannot be weakened to $|R(s) - R(t)| \leq |s-t|$ without adding some other hypothesis (for instance, consider $R = -\bar{z}$). We also note the following corollary implies Wermer's theorem.

**Corollary 1.8.** *Let $f = \bar{z} + R$, where $R$ satisfies the condition $|R(s) - R(t)| \leq |s-t|$ for every $s, t \in D$, and where $f$ is locally one-to-one. Then $P_f = C(D)$.*

**Proof.** Put $\phi_a = f - f(a)$. Independent of $a \in D$ we have

$$(z-a)\phi_a = (z-a)(\bar{z} - \bar{a} + R - R(a)) = |z-a|^2 + (z-a)(R - R(a))$$

and the condition on $R$ shows hypothesis (i) of Theorem 1.7 is satisfied. Next, if $a \in D$ and $\phi_a(s) = 0$ we have $f(s) = f(a)$, which can only occur for finitely many $s$ since $f$ is locally one-to-one. Hence, hypothesis (ii) is satisfied. For the same reason, (iii) holds, and the conclusion follows.

For smooth functions, the conditions of Theorem 1.7 can be relaxed somewhat.

**Theorem 1.9.** *Let $f$ be of class $C^1$ in a neighborhood of $D$. Suppose that, for almost all $a \in D$, $f_{\bar{z}}(a) \neq 0$, and there exists $\phi_a \in P_f$ satisfying*

(i) *$(z-a)\phi_a$ maps $D$ into $H$;*

(ii) *for each $a$, $\{s : \phi_a(s) = 0\}$ has Lebesgue measure zero.*

*Then $P_f = C(D)$.*

**Proof.** Let $\mu \perp P_f$. Suppose first that $\mu$ is absolutely continuous with respect to two-dimensional Lebesgue measure. Fix $a \in D$ where $\tilde{\mu}(a) < \infty$ and $\phi_a$ exists. Again, put $g_n = \phi_a p_n((z-a)\phi_a)$ where $p_n = 1/(z + 1/n)$. As before, we have $g_n \in P_f$, $|g_n| < 1/|z-a|$ everywhere in $D$, and $g_n(s) \longrightarrow 1/(s-a)$ whenever $(s-a)\phi_a(s) \neq 0$. In view of (ii) and the fact that $\mu$ is absolutely continuous with respect to Lebesgue measure, we have $g_n \longrightarrow 1/(z-a)$ a.e. $(\mu)$, and it follows by dominated convergence that $\hat{\mu}(a) = 0$. Thus $\mu = 0$ by Proposition A. Next, let $\mu$ be an arbitrary measure which annihilates $P_f$. From Lemma 1.2 we know $\int g \, d\mu = (1/\pi) \int g_{\bar{z}} \hat{\mu} \, dm$ for any $C^1$ function $g$. We deduce that for any nonnegative integers $k$, $l$,

$$\int z^k f^l f_{\bar{z}} \hat{\mu} \, dm = \pi \int z^k \frac{f^{l+1}}{l+1} \, d\mu = 0.$$

Thus $f_{\bar{z}} \hat{\mu} m$ is an annihilating measure for $P_f$, absolutely continuous with respect to $m$. By what we proved above, we have $f_{\bar{z}} \hat{\mu} = 0$ a.e. $(m)$. Since $f_{\bar{z}} \neq 0$ a.e., we conclude $\hat{\mu} = 0$ a.e., and hence that $\mu = 0$. The theorem follows.

Wermer [19] has proved that if $f$ is of class $C^1$ in a neighborhood of $D$, $f_{\bar{z}} \neq$

0 a.e., and every homomorphism of the algebra $P_f$ onto $\mathbf{C}$ is given by evaluation at some point of $D$, then $P_f = C(D)$. That some hypothesis in addition to $f_{\bar{z}} \neq 0$ a.e. is necessary is shown by the example $f = |z|^2$. It is not enough to assume that $f_{\bar{z}} \neq 0$ everywhere, as shown by the following example of Wermer.

**Example.** Let $f = (1/3)z^2\bar{z}^3 + (i/2)z\bar{z}^2 - (1/3 + i/2)\bar{z}$. Then $f_{\bar{z}} = |z|^4 + i|z|^2 - 1/3 - i/2$ which vanishes nowhere in $\mathbf{C}$. Also if $|s| = 1$, $f(s) = (1/3)\bar{s} + (i/2)\bar{s} - (1/3 + i/2)\bar{s} = 0$. Hence, if $g \in P_f$, then $g$, restricted to the unit circle, is approximable by polynomials in $z$. It follows that $P_f \neq C(D)$. Adding the hypothesis that $f$ is one-to-one does not help either. In fact, we simply need to consider $z + \epsilon f$ where $f$ is Wermer's function and $\epsilon$ is sufficiently small.

Returning to Wermer's theorem temporarily, we would like to point out that if his function $f$ is of class $C^1$, then it satisfies $|f_{\bar{z}}| > |f_z|$ everywhere. One way to see this is as follows. For any function $g$, we denote by $J_g$ the Jacobian matrix of $g$ and observe that $\det J_g = |g_z|^2 - |g_{\bar{z}}|^2$. Let $f_t = \bar{z} + tR$ for all $t \in [0, 1]$ and consider the map given by $t \longrightarrow \det J_{f_t}$. It is continuous and takes a negative value at $t = 0$, also it is never zero. In fact, for each $t$ the norm of $J_{tR}$, as a linear map, is less than one. And this fact combines with the invertibility of $J_{\bar{z}}$ to yield that $J_{f_t}$ is invertible for all $t$.

From our point of view, it seemed reasonable and desirable to prove that, for a smooth function $f$, the condition $|f_{\bar{z}}| > |f_z|$ is sufficient to give $P_f = C(D)$. Although we have not been able to verify this, we have proven that the condition is sufficient for the intermediary set $R_f$ to coincide with $C(D)$. This is what we do next, but first we remark that the rather abstract results in Theorems 1.7, 1.10 and 1.11 yield additional concrete ones in §2.

**Theorem 1.10.** *Let $f \in C(D)$, and suppose that for almost all $a \in D$ there exists $\phi_a \in R_f$ such that*

(i) *$(z - a)\phi_a$ maps $D$ into the complement of an open circular sector $\{re^{i\theta}: 0 < r < d, -\alpha < \theta < \alpha\}$ where $0 < \alpha < \pi/2$;*

(ii) *for each $a$, $\{s : \phi_a(s) = 0\}$ is at most countable;*

(iii) *for every $s$, $\{a : \phi_a(s) = 0\}$ has Lebesgue measure zero.*

*Then $R_f = C(D)$.*

**Proof.** Let $\mu \perp R_f$ and put $E = \{b : \mu(\{b\}) \neq 0\}$. For each $b \in E$, put $E_b = \{a : \phi_a(b) = 0\}$ and let $F = \bigcup_{b \in E} E_b$. For each $b \in E$, $mE_b = 0$ by (iii) and so $mF = 0$. Let $a \in D$ be such that $\tilde{\mu}(a) < \infty$, $\phi_a$ exists, and $a \notin F$. We aim to prove $\hat{\mu}(a) = 0$. Put $\lambda = (z - a)\phi_a$ and for $n = 2, 3, \cdots$ consider the functions $\gamma_n = 1/(z - d/n)$. Quickly we have that, as $n \longrightarrow \infty$, $\gamma_n \longrightarrow 1/z$ everywhere on the compact set $\lambda(D)$, and by Lemma 1.4, $1/|s - 1/n| \leq k/|s|$ for all $s$ outside the circular sector in (i) where $k = \max\{2, 1/\sin\alpha\}$ (this holding independent of $n$). Now put $g_n = \phi_a\gamma_n(\lambda)$. Then $g_n \in R_f$ and $|g_n| \leq k/|z - a|$ in $D$, independent of $n$. Also as $n$

$\to \infty$, $g_n \to 1/(z-a)$ in the set $\{s : \phi_a(s) \neq 0\} - \{a\}$. The set $\{s : (s-a)\phi_a(s) = 0\}$ is a countable set by (ii) and does not meet $E$ by our choice of $a$. Thus it has $|\mu|$ measure zero, and we have $g_n \to 1/(z-a)$ a.e. $(\mu)$. By dominated convergence, we conclude that $\hat{\mu}(a) = \lim \int g_n \, d\mu = 0$. Since this holds for every $a$ not in one of the null sets $F$, $\{a : \tilde{\mu}(a) = \infty\}$, $\{a : \phi_a$ not exist$\}$, we conclude $\mu = 0$ (Proposition A). The theorem follows.

**Theorem 1.11.** *Let $f \in C(D)$ where $f^{-1}(f(a))$ is finite for each $a \in D$. Assume for almost all $a \in D$, there exist a neighborhood $N(a)$, a complex constant $b_a \neq 0$, and a function $S_a$ such that for any $s \in N(a)$ we have $f(s) = b_a \bar{s} + S_a(s)$ where $|S_a(s) - S_a(a)| \leq |b_a|\,|s-a|$. Then $R_f = C(D)$.*

**Proof.** Fix $a \in D$ having $N(a)$, $b_a$ and $S_a$ as in the hypothesis. Let $f^{-1}(f(a)) = \{a = a_1, a_2, \cdots, a_n\}$ and put $A_i = \prod_{j \neq i} b_{a_j}$, $B_i = \prod_{j \neq i} (a_i - a_j)^{-1}$ for each $i = 1, 2, \cdots, n$. Take $p$ to be the unique polynomial of degree $n-1$ satisfying $p(a_i) = \log B_i A_i$. We claim that the function

$$\phi_a = (-1) \prod_{j=1}^{n} b_{a_j}^{-1} \exp(p) \prod_{j=2}^{n} (z - a_j)(f - f(a))$$

satisfies (i), (ii) and (iii) of Theorem 1.10. To verify our claim, observe first that (ii) and (iii) follow from the fact that $f^{-1}(f(a))$ is finite. Next, to see that (i) is true, consider the behavior of $(z-a)\phi_a$ in $N(a_i)$.

$$\begin{aligned}(z - a_i)(f - f(a)) &= (z - a_i)(b_{a_i}(\bar{z} - \bar{a}_i) + S_{a_i} - S_{a_i}(a_i)) \\ &= b_{a_i}|z - a_i|^2 + (z - a_i)(S_{a_i} - S_{a_i}(a_i)),\end{aligned}$$

so $b_{a_i}^{-1}(z - a_i)(f - f(a))$ maps $N(a_i)$ into $H$. Also, if $s \in N(a_i)$ and $b_{a_i}^{-1}(s - a_i)(f(s) - f(a)) = 0$, we have $f(s) = f(a)$. Thus for all $s \in N(a_i)$ (shrinking $N(a_i)$ if necessary) with $s \neq a_i$, $-b_{a_i}^{-1}(s - a_i)(f(s) - f(a))$ has negative real part. Also in $N(a_i)$ we may write $\exp(p) = B_i A_i (1 + r_i)$ where $r_i(a_i) = 0$ and $\prod_{j \neq i}^{n} (z - a_j) = (1/B_i)(1 + \eta_i)$ where $\eta_i(a_i) = 0$. Hence, by shrinking $N(a_i)$ if necessary, we have that in $N(a_i)$, $(z-a)\phi_a = (1 + \gamma_i)b_i$ where $|\gamma_i| < 1/\sqrt{2}$ throughout $N(a_i)$ and $\operatorname{Re} b_i < 0$ in $N(a_i) - \{a_i\}$. This implies that $(z-a)\phi_a$ maps $N(a_i)$ outside the open sector $-\pi/4 < \arg w < \pi/4$. Going through this procedure for each $i$, we have that $(z-a)\phi_a$ maps $\bigcup_{j=1}^{n} N(a_j)$ outside the above sector. Finally, $(z-a)\phi_a$ has no zeros on the compact set $D - \bigcup_{j=1}^{n} N(a_j)$ and so there exists a $d > 0$ with $|(z-a)\phi_a| \geq d$ on this set. This completes the verification that $\phi_a$ satisfies (i) of Theorem 1.10. The theorem now follows.

**Corollary 1.12.** *Let $f$ be of class $C^1$ in a neighborhood of $D$ and assume $|f_{\bar{z}}| > |f_z|$ everywhere. Then $R_f = C(D)$.*

**Proof.** The function $f$ is locally one-to-one, so $f^{-1}(f(a))$ is finite for all $a \in D$. By Taylor's formula at a point $a$, $f = f(a) + f_z(a)(z-a) + f_{\bar{z}}(a)(\bar{z}-\bar{a}) + R_a$ where $\lim_{s\to a} |R_a(s)|/|s-a| = 0$. For each $a \in D$, put $b_a = f_{\bar{z}}(a)$, $S_a = f(a) - f_{\bar{z}}(a)\bar{a} + f_z(a)(z-a) + R_a$ and choose $N(a)$ so that $|R_a| \le (|f_{\bar{z}}(a)| - |f_z(a)|)|z-a|$ throughout $N(a)$. The conditions of Theorem 1.11 are then satisfied.

In fact, we can say more. All that is needed to satisfy the hypotheses of Theorem 1.11 is that, for each $a \in D$, $f^{-1}(f(a))$ be finite and for almost all $a \in D$, we have $f_{\bar{z}}(a) \neq 0$ and the existence of an $N(a)$ in which $f = f(a) + f_z(a)(z-a) + f_{\bar{z}}(a)(\bar{z}-\bar{a}) + R_a$ where $|f_z(a)(z-a) + R_a| \le |f_{\bar{z}}(a)|\,|z-a|$.

2. **The special case: $f$ a polynomial in $z$ and $\bar{z}$.** In this section our aim is to apply the results of §1 and some additional information to arrive at some facts concerning $P_f$ and $R_f$ in the special case when $f$ is a polynomial in $z$ and $\bar{z}$. First we supply the additional information which will be needed.

A result which will be useful to us is the following theorem of S. N. Mergelyan.

**Theorem 2.1.** *Let $D$ be a closed disk in $\mathbf{C}$ and let $f$ be a real-valued function in $C(D)$. If for each $a$ in $D$, $f^{-1}(f(a))$ has no interior and does not separate $\mathbf{C}$, then $P_f(D) = C(D)$.*

As Mergelyan observed in [11], this theorem is a consequence of the following theorem of Lavrentiev.

**Lavrentiev's theorem 2.2.** *Let $X$ be a compact subset of $\mathbf{C}$ and let $P(X)$ denote the uniform closure on $X$ of all polynomials in $z$. If $X$ has empty interior and connected complement, then $P(X) = C(X)$.*

However, in proving Theorem 2.1, Mergelyan gives an argument which seems to have a gap. We present here another proof using the methods of functional analysis. This proof is essentially a restatement of some of the arguments used in the proof by I. Glicksberg of Bishop's General Stone-Weierstrass Theorem [3]. Another proof may be given by directly using Bishop's theorem.

Before supplying the proof, we would like to mention that Lavrentiev's theorem is a special case of the more celebrated theorem of Mergelyan, which says that if $X$ is compact with connected complement then $P(X)$ is precisely the set of all continuous functions on $X$ which are holomorphic in the interior of $X$. This latter theorem may be found in [11], or a more accessible exposition in [15]. (Also see [14].) A functional-analytic proof of Lavrentiev's theorem may be found in [3].

We now turn to the proof of Theorem 2.1. The abstract part of our argument is contained in the following lemma.

**Lemma 2.3.** *Let $E$ be a Banach space and $E'$ its dual space. Let $F$ be a subset of $E$ and let $F^{\perp} = \{\mu \text{ in } E' : \mu(f) = 0 \text{ for all } f \text{ in } F\}$. Put $K = \{\mu \text{ in } F^{\perp} : \|\mu\| \le 1\}$. Then $K$ has an extreme point, i.e., there exists $\mu$ in $K$ such that if $\mu = t\mu_1 +$*

$(1-t)\mu_2$, *with* $0 < t < 1$ *and* $\mu_1, \mu_2$ *in* $K$, *then* $\mu = \mu_1 = \mu_2$.

**Proof.** $K$ is a weak-star closed subset of the closed unit ball of $E'$, which is compact by the Banach-Alaoglu theorem, so $K$ is weak-star compact. Clearly, $K$ is convex, so the Kreĭn-Milman theorem [10] applies to yield the existence of the desired extreme point.

The proof of Theorem 2.1 will readily follow, once we have verified the following lemma which will allow us to make use of the fact that $P_f(D)$ is a subalgebra of $C(D)$.

**Lemma 2.4 (deBranges [4]).** *Let* $A$ *be a subalgebra of* $C(D)$ *containing the constant functions and containing a real-valued function* $f$. *Let* $\mu$ *be an extreme point of the closed unit ball of* $A^{\perp}$. *Then the support of* $\mu$ *is contained in a level set of* $f$.

**Proof.** Since $A$ is a linear space containing constants, there is no loss of generality in assuming $0 < f < 1$. Since $A$ is an algebra, $f\mu$ and $(1-f)\mu$ belong to $A^{\perp}$. If $\mu = 0$, there is nothing to prove. If $\mu \neq 0$, we may write

$$\mu = \|f\mu\|(f\mu/\|f\mu\|) + \|(1-f)\mu\|((1-f)\mu/\|(1-f)\mu\|),$$

and we observe that

$$\|f\mu\| + \|(1-f)\mu\| = \int f\,d|\mu| + \int (1-f)\,d|\mu| = \int d|\mu| = 1.$$

Since $\mu$ is an extreme point of the closed unit ball of $A^{\perp}$, it follows that $f\mu = \|f\mu\|\mu$, so $f = \|f\mu\|$ a.e. $(\mu)$. It now follows from the continuity of $f$ that $f = \|f\mu\|$ everywhere on the support of $\mu$, which was to be proved.

**Proof of Theorem 2.1.** Let $\mu$ be an extreme point of the closed unit ball of $P_f(D)^{\perp}$. Such a $\mu$ exists by Lemma 2.3 and, in view of Lemma 2.4, the supp $\mu$ is contained in a level set. Denote this level set by $X$. The hypothesis of Theorem 2.1 allows us to apply Lavrentiev's theorem to $X$, and thus, $P(X) = C(X)$. Further, it is evident that $\mu$ must annihilate $P(X)$; it follows that $\mu = 0$. Now if $P_f(D)^{\perp}$ contained any nonzero measure $\mu$, we could write $0 = \frac{1}{2}(\mu/\|\mu\|) + \frac{1}{2}(-\mu/\|\mu\|)$, and 0 would not be extreme in the unit ball of $P_f(D)^{\perp}$. We conclude $P_f(D)^{\perp} = \{0\}$, so $P_f(D) = C(D)$.

**Corollary 2.5.** *Let* $D$ *be a closed disk in* $\mathbf{C}$ *and let* $f \in C(D)$. *Suppose that* $\bar{f}$ *is a nonconstant analytic function in the interior of* $D$. *Then* $P_f(D) = C(D)$.

**Proof.** Choose $c > \|f\|$, and put $g = (f+c)(\bar{f}+\bar{c})$. Since $\bar{f}+\bar{c}$ is uniformly approximable on $D$ by polynomials in $z$, we have $g \in P_f(D)$, and so $P_g(D) \subset P_f(D)$. Let $X$ be a level set of $g$, say $X = g^{-1}(a^2)$ where $a > 0$. Certainly $X$ has no interior. If $X$ has disconnected complement, then some complementary component $G$ of $X$ is contained in the interior of $D$. By the maximum principle, $|\bar{f}+\bar{c}| \leq a$ in $G$; also $(\bar{f}+\bar{c})^{-1}$ is analytic in $G$ and the same reason gives $|\bar{f}+\bar{c}| \geq a$ in $G$. Thus, $f+c$

is constant in $G$ which is impossible. We conclude that $g$ satisfies the hypotheses of Theorem 2.1, so $P_g(D) = C(D)$, and $P_f(D) = C(D)$.

The next idea we present is a trivial one, but it is useful so often that we formalize it as a lemma.

**Lemma 2.6.** *Let $f \in C(D)$. Assume $\alpha$ is a nonzero complex constant and assume $q$ is a fixed polynomial in $z$. If $g = \alpha f + q$, then $P_f = P_g$ and $R_f = R_g$.*

**Proof.** The set of polynomials in $z$ and $g$ coincides with the set of polynomials in $z$ and $f$. Also the set of rational functions in $z$ and $g$ which are finite coincides with the set of rational functions in $z$ and $f$ which are finite.

Finally, we quote a result which comes from the work of G. Stolzenberg.

**Theorem 2.7.** *Let $D$ be any closed disk in $\mathbf{C}$ and let $f \in C(D)$. If $f$ agrees with an analytic function on a curve which separates $\mathbf{C}$, then $R_f(D) \neq C(D)$.*

The proof may be found in [16].

Throughout the rest of this section, we denote by $\mathcal{P}$ the collection of all polynomials in $z$ and $\bar{z}$. For our initial consideration we restrict our attention to the case when $f \in \mathcal{P}$ and the degree of $f$ is two. In this case we can find necessary and sufficient conditions for $P_f = C(D)$ or $R_f = C(D)$.

In view of Lemma 2.6, there is no loss of generality in assuming that $f = b\bar{z} + cz\bar{z} + d\bar{z}^2$, with $b$, $c$, $d$ not all $= 0$. If $c = 0$, then Corollary 2.5 tells us that $P_f = C(D)$. For the case $c \neq 0$ we follow an idea of E. Bishop. Let $\theta = \arg(d/c)$ and consider the real-valued function $h$ given by $h(s) = c^{-1}f(e^{i\theta/2}s) + |d/c|s^2 + (\bar{b}/\bar{c})e^{i\theta/2}s$. While we cannot say $P_h = P_f$, it is clear that $P_h = C(D)$ if and only if $P_f = C(D)$. Writing $h$ in the form $2\operatorname{Re}(\alpha z) + z\bar{z} + 2\operatorname{Re}(\beta z^2)$, where $\alpha = (\bar{b}/\bar{c})e^{i\theta/2}$, $\beta = |d/c|$, we see that the level sets of $h$ in $\mathbf{C}$ are conic sections; thus they separate the plane if and only if they are ellipses, i.e. if and only if $\beta < 1/2$. Rewriting $h$ in the real form $h = (x-\gamma)^2/A^2 + (y-\delta)^2/B^2 + \epsilon$ when $\beta < 1/2$, we can recognize that the level sets of $h$ are concentric ellipses with center $z_0 = \gamma + i\delta$, where $z_0$ is the unique point in $\mathbf{C}$ where the gradient of $h$ is zero (i.e., $z_0$ is the unique zero of $h_{\bar{z}}$). Therefore, the level sets of $h$ in $D$ separate the plane if and only if $|d/c| < 1/2$ and $|z_0| < 1$. Now a quick calculation gives $z_0 = -(\operatorname{Re}\alpha)/(1+2\beta) + i[(\operatorname{Im}\alpha)/(1-2\beta)]$; the corresponding expression in terms of $b$, $c$, $d$ is too ugly to write down. Also it is clear that the zero of $h_{\bar{z}}$ lies on the same circle with center at the origin as the zero of $f_{\bar{z}}$. Therefore we have proved the following result.

**Proposition 2.8.** *Let $f \in \mathcal{P}$ be of degree two. Then $P_f \neq C(D)$ if and only if $|f_{\bar{z}\bar{z}}| < |f_{z\bar{z}}|$ and $f_{\bar{z}}$ has a zero in the interior of $D$.*

In view of Theorem 2.7, it is the case here that $P_f = C(D)$ if and only if $R_f = C(D)$. And so, the second degree case is settled.

We shall now examine the situation of an arbitrary function in $\mathcal{P}$. For our purposes, we may consider a function $f$ in $\mathcal{P}$ to have the form (see Lemma 2.6)

$$(1)\qquad f = b\bar{z} + \left(\sum_{k=1}^{n_1} b_{k,1}z^k\right)\bar{z} + \left(\sum_{k=0}^{n_2} b_{k,2}z^k\right)\bar{z}^2 + \cdots + \left(\sum_{k=0}^{n_r} b_{k,r}z^k\right)\bar{z}^r.$$

In looking for conditions which give $P_f = C(D)$, we can be successful if $b \neq 0$. For consider the function defined for each $a$ in $D$ by

$$(2)\qquad \phi_a = f - \sum_{j=0}^{\infty} (j!)^{-1}D^j f(a)(z-a)^j$$

where $D^j f(a) = \partial^j f(a)/\partial z^j$. We claim that one can place appropriate conditions on the coefficients of $f$ which will insure that $\lambda = (z-a)\phi_a$, or more precisely $\phi_a$, satisfies the hypothesis of Theorem 1.7 and so will yield $P_f = C(D)$.

To see what the appropriate conditions on the coefficients of $f$ would be, we use the fact that if $g = z^k\bar{z}^n$ then

$$g - \sum_{0}^{\infty} (j!)^{-1}D^j g(a)(z-a)^j = z^k(\bar{z}^n - \bar{a}^n).$$

This fact readily yields the following fact: if $f$ and $\phi_a$ are as in (1) and (2), then

$$(3)\qquad \phi_a = b(\bar{z} - \bar{a}) + \left(\sum_{k=1}^{n_1} b_{k,1}z^k\right)(\bar{z} - \bar{a}) + \cdots + \left(\sum_{k=0}^{n_r} b_{k,r}z^k\right)(\bar{z}^r - \bar{a}^r).$$

And this yields the following

**Proposition 2.9.** *Let $f$ be as in* (1) *and $\phi_a$ be as in* (2). *If $b > 0$ and*

$$\sum_{k=1}^{n_1} |b_{k,1}| + 2\sum_{k=0}^{n_2} |b_{k,2}| + \cdots + r\sum_{k=0}^{n_r} |b_{k,r}| < b,$$

*then $P_f = C(D)$.*

**Proof.** Considering $\phi_a$ in the form (3), we see that the conditions placed on the coefficients give $\operatorname{Re}((z-a)\phi_a) > 0$ for all points in $D - \{a\}$. That $\phi_a$ satisfies the hypothesis of Theorem 1.7 is easy to check. It follows that $P_f = C(D)$.

Actually, for any specific $f$ in $\mathcal{P}$ we can usually do better than this rough estimate on the coefficients. For example, let $f = \bar{z} + cz\bar{z} + d\bar{z}^2$. Then $\operatorname{Re}((z-a)\phi_a) = |z-a|^2(1 + \operatorname{Re}(cz + d\bar{z} + d\bar{a}))$. And independent of the value of $a$ chosen in $D$, we have $\operatorname{Re}((z-a)\phi_a) \geq |z-a|^2(1 - |\bar{c} + d| - |d|)$. So if $|\bar{c} + d| + |d| < 1$, then $\operatorname{Re}((z-a)\phi_a) > 0$ in $D - \{a\}$ (independent of $a$). We conclude $P_f = C(D)$. In general, however, it is rather messy to write down the conditions on the

coefficients which are better than the $L^1$-estimates given above.

In keeping with our intention of using Wermer's theorem (of §1) as a touchstone, we offer the following two observations, which show that the information obtained above gives a strengthening of Wermer's theorem in certain cases.

**Example.** Let $f = \bar{z} + (1/4)z\bar{z} + (1/3)\bar{z}^2$. Then $|1/4 + 1/3| + |1/3| < 1$, while if $R = (1/4)z\bar{z} + (1/3)\bar{z}^2$, we have $|R(1) - R(3/4)| > 1/4 = |1 - 3/4|$.

**Proposition 2.10.** *Let $R = cz\bar{z} + d\bar{z}^2$ where $c \neq 0$. Assume that for all $s$, $t$ in $D$ with $s \neq t$ we have $|R(s) - R(t)| < |s - t|$. Then $|\bar{c} + d| + |d| < 1$.*

**Proof.** Fix $s$, $|s| = 1$, and put $t = rs$ where $r < 1$. Then $|R(s) - R(t)| = |(c + d\bar{s}^2)(1 - r^2)| < 1 - r$ and so $|c + d\bar{s}^2| < 1/(1 + r)$ (independent of $r < 1$). Thus $|c + d\bar{s}^2| \leq 1/2$. Next, choosing $s$ so that $c$ and $d\bar{s}^2$ have the same argument, we get $|c| + |d| \leq 1/2$. From this, we have $|\bar{c} + d| + |d| \leq 1$. Now if $|\bar{c} + d| + |d| = 1$, we have necessarily that $|d| = |\bar{c} + d| = 1/2$ and $|\bar{c} + d| \leq |c| + |d| \leq 1/2$, so $c = 0$ which is impossible. The conclusion follows.

The main feature in what has been said about functions in $\mathcal{P}$ is the consideration of the function $\phi_a$ in (2).

We have done more with this consideration. We intend to present another condition which yields that $R_f = C(D)$ for a function $f$ in $\mathcal{P}$. This condition is independent of the negative determinant condition of §1. First we supply a necessary lemma.

**Lemma 2.11.** *For any $f \in \mathcal{P}$ and any $b \in \mathbf{C}$,*

$$\left\{a \in \mathbf{C} : f(b) - \sum_{j=0}^{\infty} (j!)^{-1} D^j f(a)(b - a)^j = 0\right\}$$

*is a finite set.*

**Proof.** If the nonanalytic terms of $f$ form an expression of the type in (1), then the function $f - \Sigma\, (j!)^{-1} D^j f(a)(z - a)^j$ has the form (3). Thus, for $b$ fixed, the function of $a$ defined by $f(b) - \Sigma_{j=0}^{\infty} (j!)^{-1} D^j f(a)(b - a)^j$ is a polynomial in $\bar{a}$ and so has a finite number of zeros.

In order to cut down some notation, we shall write $Tf_a$ for the function $f - \Sigma_{j=0}^{\infty} (j!)^{-1} D^j f(a)(z - a)^j$ where $f \in \mathcal{P}$.

**Theorem 2.12.** *Let $f \in \mathcal{P}$ and assume that, for each $a$ in $D$, $Tf_a = (\bar{z} - \bar{a})^k \psi$ where $|\psi_z(b)| < |\psi_{\bar{z}}(b)|$ whenever $\psi(b) = 0$. Then $R_f = C(D)$.*

**Proof.** We aim to show that for each $a$ in $D$, there exists $\phi_a \in P_f$ which satisfies the conditions (i), (ii) and (iii) of Theorem 1.10. The function $\phi_a$ will have a slightly different form depending on whether $\psi(a) = 0$ or $\psi(a) \neq 0$.

*Case* 1. Assume $\psi(a) \neq 0$. Our hypothesis shows $\psi$ has a finite set of zeros in $D$, say $\{a_1, a_2, \cdots, a_n\}$. Put $A_i = \Pi_{j \neq i}\, \psi_{\bar{z}}(a_j)$, $B_i = \Pi_{j \neq i}\, (a_i - a_j)^{-1}$ for each

$i = 1, \cdots, n$, and $A = \prod_{j=1}^{n} (\psi_{\bar{z}}(a_j)(a - a_j)^{-1})$. Take $p$ to be the unique polynomial of degree $n$ satisfying $p(a_i) = \log(A_i B_i \psi(a))$ for each $i$ and $p(a) = \log A$. Define $\phi_a$ by

$$\phi_a = (-1)\psi(a)^{-1} \prod_{j=1}^{n} \psi_{\bar{z}}(a_j)^{-1} \exp(p)(z - a)^{k-1} \prod_{j=1}^{n} (z - a_j) T f_a.$$

Observe $\phi_a \in P_f$. Next, we point out that $\phi_a$ satisfies (ii) of Theorem 1.10 since $\{s : \phi_a(s) = 0\} = \{a, a_1, \cdots, a_n\}$. As in the proof of Theorem 1.11, we will show (i) is also satisfied by showing the function $\lambda = (z - a)\phi_a$ maps small neighborhoods $N(a), N(a_1), \cdots, N(a_n)$ outside the open sector in the $w$-plane $\{-\pi/4 < \arg w < \pi/4\}$. The compactness of $D - (\bigcup_{j=1}^{n} N(a_j) \cup N(a))$ then yields a positive constant $d$ such that $|\lambda| \geq d > 0$ on this set. Now consider $\lambda$ in a neighborhood of $a$. The function $h = (-1)\psi(a)^{-1}(z - a)^k T f_a$ has the form $-|z - a|^{2k}(1 + r)$ where $r(a) = 0$ and so $\operatorname{Re} h(s) < 0$ for all $s \neq a$ in some small neighborhood $N(a)$. Also we may assume that, in $N(a)$, $\exp(p) = A(1 + \sigma)$ where $\sigma(a) = 0$ and $\prod_{j=1}^{n} (z - a_j) = \prod_{j=1}^{n} (a - a_j)(1 + \eta)$ where $\eta(a) = 0$. By shrinking $N(a)$, if necessary, we have $\lambda = (1 + \gamma)h$ where $|\gamma| < 1/\sqrt{2}$ throughout $N(a)$ and $\operatorname{Re} h < 0$ in $N(a) - \{a\}$. This implies $\lambda$ maps $N(a)$ outside the sector above. Next, consider $\lambda$ in a neighborhood of $a_i$. The function $h_i = (-1)\psi_{\bar{z}}(a_i)^{-1}(z - a)^k (z - a_i) T f_a$ has the form

$$-|z - a|^{2k} \left( |z - a_i|^2 + \frac{\psi_z(a_i)}{\psi_{\bar{z}}(a_i)} (z - a_i)^2 + \frac{(z - a_i)}{\psi_{\bar{z}}(a_i)} R_{a_i} \right).$$

Since $|\psi_z(a_i)| < |\psi_{\bar{z}}(a_i)|$ and $\lim(|R_{a_i}|/|z - a_i|) = 0$, for a sufficiently small neighborhood $N(a_i)$, $\operatorname{Re} h_i < 0$ in $N(a_i) - \{a_i\}$. Also, we may assume that in $N(a_i)$, $\exp(p) = \psi(a) A_i B_i (1 + \sigma_i)$ where $\sigma_i(a_i) = 0$ and $\prod_{j \neq i} (z - a_j) = (1/B_i)(1 + \eta_i)$ where $\eta_i(a_i) = 0$. By shrinking $N(a_i)$, if necessary, we have $\lambda = (1 + \gamma_i)h_i$, where $|\gamma_i| < 1/\sqrt{2}$ throughout $N(a_i)$ and $\operatorname{Re} h_i < 0$ in $N(a_i) - \{a_i\}$. This implies $\lambda$ maps $N(a_i)$ outside the sector mentioned above. Thus, we have shown that the function $\phi_a$ (which is in $P_f$) satisfies (i) and (ii) of Theorem 1.10.

*Case* 2. Assume $\psi(a) = 0$. Our hypothesis shows $\psi$ has a finite number of other zeros, say $a_1, a_2, \cdots, a_n$. Let $A_i$, $B_i$ and $A$ be as given in Case 1 and take $p$ to be the unique polynomial of degree $n$ such that $p(a) = \log A$ and $p(a_i) = \log(\psi_{\bar{z}}(a) A_i B_i / (a_i - a))$. Define $\phi_a$ by

$$\phi_a = (-1)\psi_{\bar{z}}(a)^{-1} \prod_{j=1}^{n} \psi_{\bar{z}}(a_j)^{-1} \exp(p)(z - a)^k \prod_{j=1}^{n} (z - a_j) T f_a.$$

Observe $\phi_a \in P_f$. Next, we point out that $\phi_a$ satisfies (ii) of Theorem 1.10 since $\{s : \phi_a(s) = 0\} = \{a, a_1, \cdots, a_n\}$. Again, to show $\phi_a$ satisfies (i), it suffices to prove the function $\lambda = (z - a)\phi_a$ maps small neighborhoods $N(a), N(a_1), \cdots, N(a_n)$ out-

side the sector $\{-\pi/4 < \arg w < \pi/4\}$. This we now do.

Consider $\lambda$ in a neighborhood of $a$ first. The function

$$h = (-1)\psi_{\bar{z}}(a)^{-1}(z-a)^{k+1}Tf_a$$

has the form $(-1)\psi_{\bar{z}}(a)^{-1}|z-a|^{2k}(z-a)\psi$ and so has negative real part (since $|\psi_z(a)| < |\psi_{\bar{z}}(a)|$) in $N(a) - \{a\}$ for some neighborhood $N(a)$. Also, in $N(a)$, we may assume $\exp(p) = A(1+\sigma)$ where $\sigma(a) = 0$ and $\prod_{j=1}^n (z-a_j) = \prod_{j=1}^n (a-a_j)(1+\eta)$ where $\eta(a) = 0$. Thus, $\lambda = (1+\gamma)h$ where $|\gamma| < 1/\sqrt{2}$ throughout a suitably small neighborhood $N(a)$ and where $\operatorname{Re} h < 0$ in $N(a) - \{a\}$. This implies $\lambda$ maps $N(a)$ outside the sector above. Next, consider $\lambda$ in a neighborhood of $a_i$. The function $h_i = (-1)\psi_{\bar{z}}(a_i)(z-a)^k Tf_a$ has the form $-|z-a|^{2k}[(z-a_i)/\psi_{\bar{z}}(a_i)]\psi$ and so has negative real part (except at $a_i$) in a sufficiently small neighborhood of $a_i$ (since $|\psi_z(a_i)| < |\psi_{\bar{z}}(a_i)|$). Also in some $N(a_i)$, we may assume $\exp(p) = [\psi_{\bar{z}}(a)A_iB_i/(a_i-a)](1+\sigma_i)$ and $\prod_{j\neq i}(z-a_j) = (1/B_i)(1+\eta_i)$ where $\sigma_i(a_i) = \eta_i(a_i) = 0$, and $(z-a) = (a_i-a)(1+(z-a_i)/(a-a_i))$. Thus, $\lambda = (1+\gamma_i)h_i$ where $|\gamma_i| < 1/\sqrt{2}$ throughout some $N(a_i)$ and where $\operatorname{Re} h_i < 0$ in $N(a_i) - \{a_i\}$. This implies $\lambda$ maps $N(a_i)$ outside the sector above. And we have $\phi_a$ (besides being in $P_f$) satisfies (i) and (ii). Finally, even though the $\phi_a$ are constructed differently in Case 1 and Case 2, we have that $\phi_a(s) = 0$ if and only if $Tf_a(s) = 0$. Or for every $s$, $\{a : \phi_a(s) = 0\} = \{a : Tf_a(s) = 0\}$ and this set is finite by Lemma 2.11. We conclude $\phi_a$ satisfies (iii) of Theorem 1.10 and this theorem is proved.

It is not hard to find examples which show the condition $|f_{\bar{z}}| > |f_z|$ of §1 is independent of the hypothesis of Theorem 2.12. In fact, $f = \bar{z} + (1/4)z\bar{z} + (1/3)\bar{z}^2$ satisfies the latter, but $|f_z(-1)| > |f_{\bar{z}}(-1)|$, while $f = \bar{z} + (1/4)z\bar{z}$ does not satisfy the hypothesis of Theorem 2.12 and yet $|f_z| < |f_{\bar{z}}|$ everywhere in $D$. One advantage of the new condition is that we are dealing with a polynomial in $z$ and $\bar{z}$ of smaller degree than the original function $f$.

Another application of what we have done is the following "either/or" result for functions in $\mathcal{P}$.

**Theorem 2.13.** *Let $f \in \mathcal{P}$. Assume for each $a$ in $D$ that either $|f_{\bar{z}}| > |f_z|$ at the points of $f^{-1}(f(a))$ or $Tf_a = (\bar{z} - \bar{a})^k\psi$ where $|\psi_{\bar{z}}| > |\psi_z|$ at the zeros of $\psi$. Then $R_f = C(D)$.*

**Proof.** Again we show for each $a$ in $D$, there exists $\phi_a \in P_f$ which satisfies the conditions (i), (ii) and (iii) of Theorem 1.10. In the event that $a$ has $|f_{\bar{z}}| > |f_z|$ at the points of $f^{-1}(f(a))$, we must have $f^{-1}(f(a))$ is a finite set, say $\{a = a_1, \cdots, a_n\}$. And we take

$$\phi_a = (-1)\prod_{j=1}^n f_{\bar{z}}(a_j)^{-1}\exp(p)\prod_{j=2}^n (z-a_j)(f-f(a)) \tag{4}$$

where $p$ is a suitably chosen polynomial of degree $n-1$. (See Theorem 1.11 and Corollary 1.12.) If $a$ has $Tf_a = (\bar{z} - \bar{a})^k \psi$ where $|\psi_{\bar{z}}| > |\psi_z|$ at the zeros of $\psi$, we must have a finite set of zeros for $\psi$, say $\{a_1, \cdots, a_n\}$. And we take

$$\phi_a = (-1)\beta \prod_{j=1}^{n} \psi_{\bar{z}}(a_j)^{-1} \exp(p)(z-a)^m \prod_{j=1}^{n} (z - a_j) Tf_a \tag{5}$$

where $\beta$ is a suitably chosen complex constant (nonzero), $p$ is a suitably chosen polynomial of degree $n-1$, and $m$ is one of $k$ or $k-1$. (See Theorem 2.12.)

So for each $a$ in $D$, no matter which of the two situations exist at $a$, $\phi_a \in P_f$ and $\{s : \phi_a(s) = 0\}$ is finite. Also from our previous work (see the proofs of Theorem 1.11, Corollary 1.12 and Theorem 2.12), $\phi_a$ satisfies (i) of Theorem 1.10. Finally, we claim that, for every $s$ in $D$, $\{a : \phi_a(s) = 0\}$ is finite. In fact, let $s \in D$. When $\phi_a(s) = 0$ with $\phi_a$ as in (4) we must have $f(s) = f(a)$. And the fact that $f^{-1}(f(a))$ is finite shows only finitely many $a$ can have $\phi_a(s) = 0$. When $\phi_a(s) = 0$ with $\phi_a$ as in (5), we must have $Tf_a(s) = 0$ and Lemma 2.11 shows this can happen only a finite number of times. The theorem follows.

Our final result in this chapter is related to a different line of thought; dealing with a local version of our main problem. We address ourselves to the following question: For $f$ in $\mathcal{P}$ does there exist an $r > 0$ such that if $D_0 = \{z : |z| \le r\}$, then $R_f(D_0) = C(D_0)$? The following theorem gives an affirmative answer when $f_{\bar{z}}(0) \neq 0$, which is exactly the case when $f$ is in $\mathcal{P}$ and has the form (1) with $b \neq 0$.

**Theorem 2.14 (Wermer).** *If $f$ is a class $C^2$ function in a neighborhood of 0 satisfying $f_{\bar{z}}(0) \neq 0$, then there exists a closed disk $D_0$ about 0 for which $P_f(D_0) = C(D_0)$.*

Another contribution to the question under consideration comes from the work of E. Bishop [2].

**Theorem 2.15.** *Let $f \in \mathcal{P}$. Assume $f_{\bar{z}}(0) = 0$ and $|f_{z\bar{z}}(0)| > |f_{\bar{z}\bar{z}}(0)|$. Then there is no $r > 0$ such that, if $D_0 = \{s : |s| \le r\}$, $R_f(D_0) = C(D_0)$.*

The result we offer here is essentially a corollary to the proof of Theorem 2.12.

**Theorem 2.16.** *Let $f \in \mathcal{P}$ and assume $f$ satisfies $|f_{z\bar{z}}(0)| < |f_{\bar{z}\bar{z}}(0)|/2$. Then there exists an $r > 0$ such that $R_f(D_0) = C(D_0)$ where $D_0 = \{s : |s| \le r\}$.*

**Proof.** We will show there exists a disk $D_0$ which satisfies: for each $a$ in $D_0$, $Tf_a = (\bar{z} - \bar{a})\psi$ where $|\psi_z(s)| < |\psi_{\bar{z}}(s)|$, independent of $s$ in $D_0$. The theorem will follow from the proof of Theorem 2.12.

By our hypothesis on $f$ we may choose $d > 0$ such that for any $a$ with $|a| \le d$ we have $0 < m \le \frac{1}{2}|f_{\bar{z}\bar{z}}(a)| - |f_{z\bar{z}}(a)|$ for some constant $m$. Also we must have $Tf_a = (\bar{z} - \bar{a})(f_{\bar{z}}(a) + f_{z\bar{z}}(a)(z-a) + \frac{1}{2}f_{\bar{z}\bar{z}}(a) + S)$ for each $a$ with $|a| \le d$. And the function $S$ satisfies the condition that there exists a constant $M$, independent

of $a$, such that $|S_z(t)| \leq M|t-a|$ and $|S_{\bar{z}}(t)| \leq M|t-a|$ for all $t$ with $|t| \leq d$. Choose $r < d$ so that whenever $|t| \leq r$ and $|a| \leq r$ we have $|t-a| < m/4M$. Put $D_0 = \{|z| \leq r\}$. For each $a$ in $D_0$, $Tf_a = (\bar{z} - \bar{a})\psi$ where, for any $s$ in $D_0$, $|\psi_{\bar{z}}(s)| - |\psi_z(s)| = |\frac{1}{2}f_{\bar{z}\bar{z}}(a) + S_{\bar{z}}(s)| - |f_{z\bar{z}}(a) + S_z(s)| \geq m/2 > 0$. The theorem is proved.

3. **Generalizations to higher dimensions.** In this section, we deal with the higher dimensional version of our problem and we use the methods of several complex variables. Let us begin by quoting some definitions and results.

Let $X$ be a compact set in $\mathbf{C}^n$ and let $z_1, \cdots, z_n$ denote the coordinate functions on $\mathbf{C}^n$. We shall denote by $P(X)$ the set of functions in $C(X)$ which can be uniformly approximable by polynomials in $z_1, \cdots, z_n$ and we shall be interested in the case when $P(X) = C(X)$.

As an aid in studying this case, we introduce the following

**Definition.** Let $X$ be as above. The polynomial convex hull of $X$, denoted by $\hat{X}$, is the set of all points $\zeta$ in $\mathbf{C}^n$ satisfying $|p(\zeta)| \leq \max_X |p|$ for every polynomial $p$.

Evidently, $\hat{X}$ is a compact subset of $\mathbf{C}^n$ and we have $X \subset \hat{X}$. We call $X$ polynomially convex if $X = \hat{X}$. Let us denote by $A(X)$ the set of functions in $C(X)$ which can be uniformly approximated by functions analytic in a neighborhood of $X$. A relevant result here is the well-known theorem of K. Oka and A. Weil.

**Theorem 3.1 (Oka-Weil).** *Let $X$ be a compact set in $\mathbf{C}^n$ which is polynomially convex. Then $P(X) = A(X)$.*

Oka's proof may be found in [13]; an easier proof may be found in either [7] or [6].

If we replace $X$ by a smooth real submanifold of an open set in $\mathbf{C}^n$, there is some pertinent information about when $A(X) = C(X)$. First we make the following definition.

**Definition.** Let $M$ be a smooth real submanifold of $\mathbf{C}^n$ having dimension $k$ and, for $x \in M$, let $T_x$ denote the tangent space at $x$ (regarded as a real-linear subspace of $\mathbf{C}^n$). A complex tangent to $M$ at $x$ is a two-dimensional real-linear subspace in $T_x$ invariant under multiplication by $i$ (i.e., a complex line in $T_x$).

It has recently been proved (see [8] and [12]) that if $M$ is as in the definition and if $M$ has no complex tangents, then $A(X) = C(X)$ for any compact set $X$ in $M$. Thus, if $X$ is a compact subset of such an $M$ which is polynomially convex, then $P(X) = C(X)$. This information comes from the efforts of several people. For $k = 2$, the main contributions were from J. Wermer [19] and M. Freeman [5]; for $k > 2$, the work of R. Nirenberg-R. O. Wells, Jr. [12], and L. Hörmander-J. Wermer [8] yields the key results.

Let us now put the above facts into our context. If $B$ is the unit polydisk in

$\mathbf{C}^n$ and if $F = (f_1, \cdots, f_n)$ where each $f_j \in C(B)$, we seek conditions on $F$ which yield $P_F = C(B)$ where $P_F$ is the uniform closure on $B$ of polynomials in $z_1, \cdots, z_n, f_1, \cdots, f_n$. As mentioned in the introduction, if we put $X_F = \{(\zeta, F(\zeta)) : \zeta \in B\}$, then it is the case that $P_F = C(B)$ when and only when $P(X_F) = C(X_F)$. And calling on the results quoted above we can say the following statements hold. When $X_F$ is polynomially convex, $P(X_F) = A(X_F)$ (Oka-Weil Theorem) and when $X_F$ is a subset of a smooth real submanifold of $\mathbf{C}^{2n}$ which has no complex tangents, $A(X_F) = C(X_F)$ (work of Nirenberg-Wells and Hörmander-Wermer).

Finally, the condition of the graph of $F$ having no complex tangents can be neatly characterized by a local condition on $F$.

**Definition.** The differential operators $\partial/\partial z_j$, $\partial/\partial \bar{z}_j$, $j = 1, \cdots, n$, are given by

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right), \qquad \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right).$$

We write $F_z$ for the $n$-by-$n$ matrix whose $(j, k)$th entry is $\partial f_j/\partial z_k$ and write $F_{\bar{z}}$ for the similar matrix involving $\partial f_j/\partial \bar{z}_k$.

**Theorem 3.2.** *Let $F = (f_1, \cdots, f_n)$ be a smooth function defined in a neighborhood $N$ of $0$ in $\mathbf{C}^n$ and let $M = \{(\zeta, F(\zeta)) : \zeta \in N\}$. Then $M$ has no complex tangent at $(0, F(0))$ if and only if the matrix $F_{\bar{z}}$ is invertible at $0$.*

**Proof.** If $\Phi(\zeta) = (\zeta, F(\zeta))$ for $\zeta \in N$, then the tangent space to $M$ at $\Phi(0)$ is the set of points $d\Phi(\zeta)$ where $\zeta \in \mathbf{C}^n$. Thus, $M$ has a complex tangent there if and only if there exist $\xi, \eta \in \mathbf{C}^n$ (different from 0) such that $d\Phi(\eta) = id\Phi(\xi)$. Or what is the same, $(\eta, F_z(0)\eta + F_{\bar{z}}(0)\bar{\eta}) = i(\xi, F_z(0)\xi + F_{\bar{z}}(0)\bar{\xi})$. So $\eta = i\xi$ and $F_z(0)\eta + F_{\bar{z}}(0)\bar{\eta} = i(F_z(0)\xi + F_{\bar{z}}(0)\bar{\xi})$. These two identities are equivalent to $F_{\bar{z}}(0)\bar{\xi} = 0$. The theorem follows.

Let us briefly state the consequence of what has been said so far. Assume $F = (f_1, \cdots, f_n)$ is a smooth function defined in a neighborhood $N$ of $B$ in $\mathbf{C}^n$. Let $M$ denote the submanifold of an open subset of $\mathbf{C}^{2n}$ (of dimension $2n$) consisting of all points $(\zeta, F(\zeta))$ where $\zeta \in N$. Finally, assume $F_{\bar{z}}$ is always invertible and the compact subset $X_F$ of $M$ is a polynomially convex set. Then $P_F = C(B)$.

This points up the problem of showing $X_F$ is a polynomially convex set directly from local conditions on $F$. This is a difficult problem and has only been solved in special cases. For instance, we have the following higher dimensional version of Wermer's theorem (of §1).

**Theorem (Hörmander-Wermer) 3.3.** *Let $R = (R_1, \cdots, R_n)$ be a smooth function defined in a neighborhood $N$ of $B$ in $\mathbf{C}^n$. Assume there is a constant $k < 1$ such that $|R(\zeta) - R(\zeta')| \le k|\zeta - \zeta'|$ for all $\zeta, \zeta' \in N$. If $f_j = \bar{z}_j + R_j$ and $F = (f_1, \cdots, f_n)$, then $P_F = C(B)$.*

The proof may be found in [8] and it consists in proving $F_{\bar{z}}$ is always invertible and $X_F$ is polynomially convex. Our aim here is to introduce a method for verifying the polynomial convexity of certain $X_F$. We refer to this method as the "sweeping technique". And we start by offering another proof (one which we believe is very instructive) that $\hat{X}_F = X_F$ for the special case above (whether $R$ is a smooth function or not).

**Lemma 3.4.** *Let $R = (R_1, \cdots, R_n)$ be a continuous function in $\mathbf{C}^n$ where, for all $\zeta, \zeta'$ in $\mathbf{C}^n$, $|R(\zeta) - R(\zeta')| \leq k|\zeta - \zeta'|$, $k$ being a fixed number with $0 < k < 1$. Then the map given by $\zeta \longrightarrow \bar{\zeta} + R(\zeta)$ is a homeomorphism from $\mathbf{C}^n$ to $\mathbf{C}^n$.*

**Proof.** First, $\bar{\zeta} + R(\zeta) = \bar{\eta} + R(\eta)$ if and only if $\bar{\zeta} - \bar{\eta} = R(\eta) - R(\zeta)$. And the hypothesis on $R$ implies that the map is one-to-one. Next, fix $\tau \in \mathbf{C}^n$, then the map assumes the value $\tau$ if and only if $\bar{\tau} - \bar{R}$ has a fixed point. Let $G = \bar{\tau} - \bar{R}$ and observe $G$ satisfies the same Lipschitz condition that $R$ does. Put $\zeta_0 = 0$, $\zeta_1 = G(\zeta_0), \cdots, \zeta_{j+1} = G(\zeta_j)$. We have, for each $j$, $|\zeta_{j+1} - \zeta_j| \leq k|\zeta_j - \zeta_{j-1}|$ and so $|\zeta_{j+1} - \zeta_j| \leq k^j|\zeta_1|$. We conclude there exists a $\sigma$ in $\mathbf{C}^n$ such that $\zeta_j \longrightarrow \sigma$ as $j \longrightarrow \infty$. The definition of $\zeta_j$ and the continuity of $G$ now combine to yield $G(\sigma) = \sigma$. Since $\tau$ was arbitrary the result follows.

**Lemma 3.5.** *Let $R$ be as in Lemma 3.4 and let $F(\zeta) = \bar{\zeta} + R(\zeta)$. Then the map given by $(\zeta, \eta) \longrightarrow (i\zeta + \eta, i(F(\zeta) - F(0)) + F(\eta))$ is a homeomorphism from $\mathbf{C}^{2n}$ to $\mathbf{C}^{2n}$.*

**Proof.** First we show the map is onto. Let $(\zeta_0, \eta_0) \in \mathbf{C}^{2n}$. Observe that the map assumes the value $(\zeta_0, \eta_0)$ if and only if $\eta_0 = i(F(\zeta) - F(0)) + F(\eta)$ for some $(\zeta, \eta)$ on the hyperplane $\Pi$ consisting of those $(\zeta, \eta)$ satisfying $\zeta_0 = i\zeta + \eta$. But for the function $F$ in question this is equivalent to saying $\eta_0 = \bar{\zeta}_0 + 2i\bar{\zeta} + i(R(\zeta) - R(0)) + R(\zeta_0 - i\zeta)$ for some $\zeta \in \mathbf{C}^n$. And since $|R(\zeta) + R(\zeta_0 - i\zeta) - R(\eta) - R(\zeta_0 - i\eta)| \leq 2k|\zeta - \eta|$ independent of $\zeta, \eta$ chosen in $\mathbf{C}^n$, this fact follows from Lemma 3.4. In order to show the map is one-to-one, observe that $(i\zeta' + \eta', i(F(\zeta') - F(0)) + F(\eta')) = (i\zeta'' + \eta'', i(F(\zeta'') - F(0)) + F(\eta''))$ if and only if $(\zeta', \eta')$, $(\zeta'', \eta'')$ lie on the same hyperplane, say $\Pi$, as given above where $\zeta_0 = i\zeta' + \eta' = i\zeta'' + \eta''$, and they are mapped into the same element of $\mathbf{C}^n$ by the function of $\zeta$ and $\eta$ given by $i(F(\zeta) - F(0)) + F(\eta)$. But this last function, restricted to $\Pi$, is one-to-one as was seen above. Thus, $(\zeta', \eta') = (\zeta'', \eta'')$. The conclusion follows.

**Theorem 3.6.** *Let $R = (R_1, \cdots, R_n)$ be a continuous function on $B$ and satisfy the Lipschitz condition given above. If $F = (\bar{z}_1 + R_1, \cdots, \bar{z}_n + R_n)$, then $\hat{X}_F = X_F$.*

**Proof.** A theorem of F. Valentine [17] allows $R$ to be extended to all of $\mathbf{C}^n$

preserving the Lipschitz condition. By the previous lemma, the map given by $(\zeta, \eta) \to (i\zeta + \eta, i(F(\zeta) - F(0)) + F(\eta))$ is a homeomorphism from $\mathbf{C}^{2n}$ to $\mathbf{C}^{2n}$. Evidently, it suffices to prove $\hat{X}_F \cap \{(\alpha, \beta) : \alpha, \beta \in \mathbf{C}^n, \beta \neq F(\alpha)\} = \emptyset$. Fix $(\alpha, \beta)$ satisfying $\beta \neq F(\alpha)$. For each $c = (c_1, \cdots, c_n)$, let $Q_c$ be the polynomial given by $Q_c = -\sum_{j=1}^n (z_j - c_j)(z_{j+n} - \bar{c}_j - R_j(c))$. The Lipschitz condition combines with the Cauchy-Schwarz inequality to give $\operatorname{Re} Q_c < 0$ on $X_F - \{(c, F(c))\}$. Now pick the unique $a, b \in \mathbf{C}^n$ such that $(\alpha, \beta) = (ia + b, i(F(a) - F(0)) + F(b))$ and consider $Q_b$. $Q_b(\alpha, \beta) = -Q_0(a, F(a))$ as can be easily checked and by our choice of $(\alpha, \beta)$ we have $a \neq 0$. Thus $\operatorname{Re} Q_b(\alpha, \beta) > 0$. Putting $H = \exp Q_b$, we have $|H(\alpha, \beta)| > 1 \geq \max_{X_F} |H|$ and so $(\alpha, \beta) \notin \hat{X}_F$. The theorem now follows.

Actually, for $R$ of class $C^1$ we can dispense with the extension theorem of Valentine in proving Theorem 3.6. In that case, our "sweeping technique" allows us to prove a more general result. Before giving this result, we state a useful theorem.

**Theorem 3.7.** *Let $F = (f_1, \cdots, f_n)$ where each $f_j \in C(B)$. Assume that, for each $x \in X_F$, there exists a neighborhood $N(x)$ in $\mathbf{C}^{2n}$ such that $N(x) \cap \hat{X}_F = N(x) \cap X_F$. Then $\hat{X}_F = X_F$.*

**Proof.** Assume $\hat{X}_F - X_F \neq \emptyset$. By hypothesis there exists a function $H$ which is analytic on $\hat{X}_F$, equals zero on $X_F$ and equals one on $\hat{X}_F - X_F$. By Oka-Weil theorem, $H$ can be approximated on $\hat{X}_F$ to within $\frac{1}{4}$ by a polynomial $p$ which is impossible.

**Theorem 3.8.** *Let $F$ be of class $C^1$ in a neighborhood of $B$. Assume $F = (f_1, \cdots, f_n)$ and satisfies the following two conditions for each $a = (a_1, \cdots, a_n) \in B$:*

(i) *$F_{\bar{z}}(a)$ is invertible,*

(ii) *there exist complex constants $\beta_j, \gamma_{ij}$, $i, j = 1, \cdots, n$, such that $\sum_{j=1}^n \beta_j(z_j - a_j)(f_j - f_j(a)) + \sum_{i,j=1}^n \gamma_{ij}(z_i - a_i)(z_j - a_j)$ has positive real part on $B - \{a\}$.*

*Then $X_F$ is polynomially convex.*

**Proof.** It suffices to prove that every point on $X_F$ has a neighborhood $N$ in $\mathbf{C}^{2n}$ such that $\hat{X}_F \cap N = X_F \cap N$. Fix $x = (a, F(a))$ on $X_F$. Consider the function given by $(\zeta, \eta) \to (i(\zeta - \eta) + \eta, i(F(\zeta) - F(\eta)) + F(\eta))$ in a neighborhood of $(a, a)$. It has a nonsingular differential at $(a, a)$. In order to see this, observe that this differential is precisely the map given by $(\zeta, \eta) \to (i(\zeta - \eta) + \eta, idF(\zeta - \eta) + dF(\eta))$ where $dF(\alpha_1, \cdots, \alpha_n) = (\beta_1, \cdots, \beta_n)$ with

$$\beta_k = \sum_{j=1}^n \left( \frac{\partial f_k}{\partial z_j}(a)\alpha_j + \frac{\partial f_k}{\partial \bar{z}_j}(a)\bar{\alpha}_j \right).$$

And this map is onto $\mathbf{C}^{2n}$ if and only if the map given by $(\zeta, \eta) \to idF(\zeta - \eta) + dF(\eta)$ takes every hyperplane $\Pi_\xi = \{(\zeta, \eta) : \xi = i(\zeta - \eta) + \eta\}$ onto $\mathbf{C}^n$. But on $\Pi_\xi$ we have $\zeta = (1 + i)\eta - i\xi$, so what must be verified is that the map given by $\eta \to idF(i\eta) + dF(\eta) - idF(-i\xi)$ takes $\mathbf{C}^n$ onto $\mathbf{C}^n$. And this follows from the facts that $idF(i\eta) + dF(\eta) = 2F_{\bar{z}}\bar{\eta}$ and $F_{\bar{z}}$ is invertible. We conclude that there exists a neighborhood $N$ of $x$ such that every point in $N$ has the form $(i(\zeta - \eta) + \eta, i(F(\zeta) - F(\eta)) + F(\eta))$ where $(\zeta, \eta)$ is uniquely determined in a suitable neighborhood of $(a, a)$.

Next, fix $(\alpha, \beta) \in N$ satisfying $\beta \neq F(\alpha)$ and choose $c, b \in \mathbf{C}^n$ such that $(\alpha, \beta) = (i(c - b) + b, i(F(c) - F(b)) + F(b))$. Let $b = (b_1, \cdots, b_n)$ and consider the polynomial $Q_b$ given by

$$Q_b = -\sum_{j=1}^{n} \beta_j(z_j - b_j)(z_{j+n} - f_j(b)) - \sum_{i,j=1}^{n} \gamma_{ij}(z_j - b_j)(z_i - b_i).$$

By (ii), $\operatorname{Re} Q_b < 0$ on $X_F - \{(b, F(b))\}$. It can be easily checked that $Q_b(\alpha, \beta) = -Q_b(c, F(c))$ and, by our choice of $(\alpha, \beta)$, we have $c \neq b$. Thus, $\operatorname{Re} Q_b(\alpha, \beta) > 0$. Putting $H = \exp Q_b$, we have $|H(\alpha, \beta)| > 1 \geq \max_{X_F} |H|$ and so $(\alpha, \beta) \notin \hat{X}_F$. We conclude that $N \cap \hat{X}_F = N \cap X_F$. The theorem follows.

As a final observation we have this corollary.

**Corollary 3.9.** *Let $z$, $w$ be the coordinate functions on $\mathbf{C}^2$. Assume $f = \bar{z} + cz\bar{z} + d\bar{z}^2 + qz\bar{w}$, $g = \bar{w} + sw\bar{w} + t\bar{w}^2 + p\bar{w}z$ where the coefficients satisfy $|\bar{c} + d| + |d| + |q| < 1$ and $|\bar{s} + t| + |t| + |p| < 1$. If $F = (f, g)$, then $\hat{X}_F = X_F$. (In fact, $P(X_F) = C(X_F)$.)*

**Proof.** The inequalities easily yield that both $f_{\bar{z}}$ and $g_{\bar{w}}$ are never zero. Thus,

$$\begin{pmatrix} f_{\bar{z}} & 0 \\ 0 & g_{\bar{w}} \end{pmatrix}$$

is invertible. Further, for each element $a = (\zeta, \eta)$ of the unit polydisk in $\mathbf{C}^2$ we have

$$\begin{aligned}(z - \zeta)(f - f(a) - f_z(a)(z - \zeta) - f_w(a)(w - \eta)) \\ + (w - \eta)(g - g(a) - g_z(a)(z - \zeta) - g_w(a)(w - \eta))\end{aligned}$$

is precisely

$$|z - \zeta|^2(1 + cz + qw + d\bar{z} + d\zeta) + |w - \eta|^2(1 + sw + pz + t\bar{w} + t\bar{\eta})$$

which has positive real part on $B - \{a\}$ by the inequalities.

## REFERENCES

1. E. Bishop, *A minimal boundary for function algebras*, Pacific J. Math. 9 (1959), 629–642. MR 22 #191.

2. ———, *Boundary measure of analytic differentials*, Duke Math. J. 27 (1960), 331–340. MR 22 #9621.

3. A. Browder, *Introduction to function algebras*, Benjamin, New York, 1969. MR 39 #7431.

4. L. De Branges, *The Stone-Weierstrass theorem*, Proc. Amer. Math. Soc. 10 (1959), 822–824. MR 22 #3970.

5. M. Freeman, *Some conditions for uniform approximation on a manifold*, Proc. Internat. Sympos. on Function Algebras (Tulane Univ., 1965), Scott-Foresman, Chicago, Ill., 1966, pp. 42–60. MR 33 #1758.

6. R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N. J., 1965. MR 31 #4927.

7. L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N. J., 1966. MR 34 #2933.

8. L. Hörmander and J. Wermer, *Uniform approximation on compact sets in* $C^n$, Math. Scand. 23 (1968), 5–21. MR 40 #7484.

9. J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds*. I, II, Ann. of Math. (2) 78 (1963), 112–148; ibid. (2) 79 (1964), 450–472. MR 27 #2999; MR 34 #8010.

10. M. Kreĭn and D. Milman, *On extreme points of regular convex sets*, Studia. Math. 9 (1940), 133–138. MR 3, 90.

11. S. N. Mergelian, *Uniform approximations to functions of a complex variable*, Uspehi Mat. Nauk 5 (1952), no. 2 (48), 31–122; English transl., Amer. Math. Soc. Transl. (1) 3 (1962), 294–391. MR 14, 547.

12. R. Nirenberg and R. O. Wells, Jr., *Holomorphic approximation on real submanifolds of a complex manifold*, Bull. Amer. Math. Soc. 73 (1967), 378–381. MR 35 #746.

13. K. Oka, *Domaines convexes par rapport aux fonctions rationelles*, J. Sci. Hiroshima Univ. 6 (1936), 245–255.

14. K. Preskenis, *On a theorem of S. N. Mergelyan*, Master's Thesis, Brown University, Providence, R. I., 1967.

15. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966. MR 35 #1420.

16. G. Stolzenberg, *Polynomially and rationally convex sets*, Acta. Math. 109 (1963), 259–289. MR 26 #3929.

17. F. A. Valentine, *A Lipschitz condition preserving extension for a vector function*, Amer. J. Math. 67 (1945), 83–93. MR 6, 203.

18. J. Wermer, *Approximation on a disk*, Math. Ann. 155 (1964), 331–333. MR 29 #2670.

19. ———, *Polynomially convex disks*, Math. Ann. 158 (1965), 6–10. MR 30 #5158.

DEPARTMENT OF MATHEMATICS, NEWTON COLLEGE, NEWTON, MASSACHUSETTS 02159