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ANALYTIC CAPACITY AND APPROXIMATION PROBLEMS(1)

BY

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ABSTRACT. We consider some problems concerning analytic capacity as a set function, which are relevant to approximation problems for analytic functions on plane sets. In particular we consider the question of semiadditivity of capacity. We obtain positive results in some special cases and give applications to approximation theory. In general we establish some equivalences among various versions of the semiadditivity question and certain questions in approximation theory.

Introduction. Recently Vituškin [14] has developed a technique for tackling problems of (qualitative) rational approximation on compact plane sets using the concept of analytic capacity. Essentially this reduces problems in rational approximation to problems on analytic capacity, and therefore it becomes important to study the properties of analytic capacity as a set function. The purpose of this paper is to study analytic capacity in this light; in particular we concentrate on the problem of semiadditivity, a solution of which would have considerable significance for approximation theory. We do not come near to finding a solution but we obtain some partial results and several equivalent versions of the general problems.

In the first two sections we set up Vituškin's approximation scheme, as amended by Gamelin and Garnett [8] to treat pointwise bounded as well as uniform approximation. In §3 we develop a modified approximation scheme which is required later. §4 considers "negligible" sets—sets which in a certain sense can be neglected as far as approximation theory is concerned. A major problem is to determine which sets are negligible. In §5 we establish the equivalence of several versions of semiadditivity for the analytic capacity γ, and in §§6 and 7 we treat some important special cases with applications to bounded approximation and Dirichlet algebras. In §8 we establish the equivalence of the semiadditivity of the continuous analytic capacity α with a certain conjecture on uniform approximation, and prove this conjecture in a special case. Finally in §9 we consider approximation of functions satisfying Lipschitz and related conditions.

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Notation. If K is a compact plane set then $C(K)$ denotes the algebra of all continuous complex-valued functions on K, $A(K)$ denotes the subalgebra of all functions in $C(K)$ which are analytic on the interior K° of K, and $R(K)$ denotes all functions on K which are uniform limits on K of rational functions with poles outside K.

If U is an open subset of the extended plane S^2 we denote by $A(U)$ the algebra of all continuous functions on its closure $\overline{U}$ which are analytic on U, and by $H^\infty(U)$ the algebra of all bounded analytic functions on U. If V is a subset of the boundary ∂U of U then $H_V^\infty(U)$ denotes all bounded continuous functions on $U \cup V$ which are analytic on U.

The symbol $\|f\|$ always means the supremum of $|f|$ over the domain of definition of f.

1. Definitions and basic properties of analytic capacity. For detailed accounts see [14], [17], and Chapter 8 of [7].

Let $S \subseteq \mathbf{C}$. We define the analytic capacity $\gamma(S)$ of S by $\gamma(S) = \sup\{|f'(\infty)|: f \in B(S, 1)\}$ where $B(S, M) = \{f: f$ is a bounded analytic function on $S^2 \setminus K$ where K is a compact subset of S, $\|f\| \leq M$, $f(\infty) = 0\}$.

Here $f'(\infty)$ is the coefficient a_1 in the Laurent expansion $f(z) = a_1 z^{-1} + a_2 z^{-2} + \cdots$.

The continuous analytic capacity $\alpha(S)$ is defined by $\alpha(S) = \sup\{|f'(\infty)|: f \in C(S, 1)\}$, where $C(S, M) = \{f: f$ is continuous on S^2, analytic outside a compact subset of S, $\|f\| \leq M$, $f(\infty) = 0\}$.

We list some of the basic properties below.

$\alpha(S) \leq \gamma(S) \leq \operatorname{diam}(S)$ for any set S.

$\gamma(K) \geq \frac{1}{4} \operatorname{diam}(K)$ if K is compact and connected.

$\alpha(U) = \gamma(U)$ if U is open.

$\gamma(K) = \inf\{\gamma(U): U$ open, $K \subseteq U\}$ if K is compact.

See [7, Chapter 8, §§1, 2, 3].

We define $\gamma^*(S) = \inf\{\gamma(U): U$ open, $U \supseteq S\}$. Then $\gamma(K) = \gamma^*(K)$ for compact K. It is not clear whether this extends, say, to borel sets. We shall prove in §5 that if S is locally compact then $\gamma^*(S) \leq C\gamma(S)$ where C is an absolute constant.

We shall be concerned mainly with the following conjecture of Vituškin:

Conjecture. There exists an absolute constant M such that

$$\gamma(S_1 \cup S_2) \subseteq M(\gamma(S_1) + \gamma(S_2)), \qquad \alpha(S_1 \cup S_2) \subseteq M(\alpha(S_1) + \alpha(S_2)),$$

for reasonable sets S_1 and S_2.

It is not clear what "reasonable" should mean here; we shall be mainly concerned with the case where S_1 is compact (of course one may always assume $S_1 \cup S_2$ is compact).

2. **The T_ϕ operator and the approximation scheme.** Here we outline Vituškin's construction, following Chapter 8 of [7]. If f is a bounded borel function on $\mathbf{C}$, and ϕ is a continuously differentiable function with compact support, we define

$$(T_\phi f)(\zeta) = \phi(\zeta)f(\zeta) + \frac{1}{\pi} \int \frac{f(z)}{z-\zeta} \frac{\partial\phi}{\partial\bar{z}}\, dm(z)$$

where m is plane Lebesgue measure.

Then (see [7, Chapter 8, Lemma 7.1]) $T_\phi f$ is a bounded borel function on $\mathbf{C}$, analytic outside supp ϕ, analytic on any open set where f is, continuous at any point where f is, and vanishes at ∞. Moreover, $f - T_\phi f$ is analytic on any open set where $\phi = 1$. If f is analytic on a neighborhood of supp ϕ then $T_\phi f = 0$. If f is analytic outside a compact set K and zero at ∞, and $\phi = 1$ on a neighborhood of K, then $T_\phi f = f$.

Finally, we have the following estimate for $T_\phi f$:

$$\|T_\phi f\| \leq 2 \operatorname{diam}(\operatorname{supp}\phi)\, \|\partial\phi/\partial\bar{z}\| \sup\{|f(z) - f(\zeta)|: z, \zeta \in \operatorname{supp}\phi\}.$$

Now fix $\delta > 0$, let $\{z_k\}$ be an enumeration of the points of $\mathbf{C}$ of the form $(m\delta/2, n\delta/2)$ where m and n are integers. Let $\Delta_k = \Delta(z_k, \delta)$, the open disc of center z_k and radius δ. Let $\{\phi_k\}$ be a continuously differentiable partition of 1 subordinate to $\{\Delta_k\}$; i.e. $0 \leq \phi_k \leq 1$, $\Sigma_k \phi_k(z) = 1$ for all $z \in \mathbf{C}$, each ϕ_k is continuously differentiable with support in Δ_k. We can choose $\{\phi_k\}$ so that $\|\operatorname{grad}\phi_k\| \leq A_1/\delta$ where A_1 is an absolute constant.

Let f be a bounded borel function on $\mathbf{C}$, analytic off a compact set. Let $f_k = T_{\phi_k} f$. Then $f_k = 0$ for all but finitely many k, and $f = f(\infty) + \Sigma_k f_k$.

Moreover, f_k is analytic on any open set where f is, and also outside supp ϕ_k, and f_k is continuous where f is. Finally, $\|f_k\| \leq A_2\|f\|$ where A_2 is an absolute constant.

We now give a general lemma which relates approximation questions to analytic capacity. Let A_0 denote the algebra of all bounded borel functions on $\mathbf{C}$ which are analytic outside some compact set. Let A be a subalgebra of A_0. For any set $S \subseteq \mathbf{C}$ we define

$$\gamma_A(S) = \sup\{|f'(\infty)|: f \in A, \text{ and is analytic outside a compact subset of } S, \text{ with } \|f\| \leq 1\}.$$

We say a subalgebra A of A_0 is T_ϕ-*invariant* if whenever $f \in A$ and ϕ is a continuously differentiable function with compact support then $T_\phi f \in A$. Observe that the properties of the T_ϕ operator ensure that any algebra whose definition merely specifies analyticity or continuity of its members at certain points is automatically T_ϕ-invariant.

Now suppose A is a T_ϕ-invariant subalgebra of A_0.

Then if $f \in A$ and ϕ is C' with support in a disc $\Delta = \Delta(z_0, r)$, we have

$$\left|\frac{1}{\pi}\int \bar{\partial}\phi(\zeta)f(\zeta)\,dm(\zeta)\right| = |(T_\phi f)'(\infty)|$$
$$\leq M\gamma_A(\Delta)\delta\|\bar{\partial}\phi\| \sup\{|f(z_1) - f(z_2)|: z_1, z_2 \in \Delta\}.$$

Conversely, we have

Lemma 2.1. *Let $f \in A_0$ and suppose there exist $m, r, \delta_0 > 0$ such that whenever $z \in \mathbf{C}$, $0 < \delta < \delta_0$, and ϕ is C^1 with support in $\Delta = \Delta(z, \delta)$, we have*

$$\left|\frac{1}{\pi}\int \bar{\partial}\phi(\zeta)f(\zeta)\,dm(\zeta)\right| \leq m\gamma_A(\Delta(z, r\delta))\delta\,\|\bar{\partial}\phi\| \sup\{|f(z_1) - f(z_2)|: z_1, z_2 \in \Delta\}.$$

Then we can find a sequence $\{g_n\}$ in A with $\|g_n\| \leq A(m, r)\|f\|$ such that $g_n \to f$ uniformly on any compact set where f is continuous.

Before proving this we make some remarks on the second Laurent coefficient. If $f \in A_0$ and $z_0 \in \mathbf{C}$ we have the Laurent expansion

$$f(z) = f(\infty) + \frac{a_1}{z - z_0} + \frac{a_2}{(z - z_0)^2} + \cdots \qquad (|z| \text{ large}).$$

Then $a_1 = f'(\infty)$ and we define $\beta(f, z_0) = a_2$. Observe that if $a_1 = 0$ then a_2 is independent of z_0; in this case we may write $\beta(f)$. If $g \in A_0$ and $f \in T_\phi g$ then

$$\beta(f, z_0) = \frac{1}{\pi}\int \frac{\partial\phi}{\partial\bar{\zeta}} f(\zeta)(\zeta - z_0)\,dm(\zeta).$$

Proof of Lemma 2.1. We may assume $r \geq 3$. Fix $\delta > 0$ and construct the approximation scheme $\{\Delta_k\}$, $\{\phi_k\}$ for this δ. We consider only those k for which $f_k = T_{\phi_k} f$ does not vanish identically (there are only finitely many); we have $f = f(\infty) + \Sigma_k f_k$.

We use $C_1, C_2, \ldots$ to denote constants depending only on m and r. We claim that for each k we can find $g_k \in A$, with the same first two Laurent coefficients as f_k, analytic outside $\tilde{\Delta}_k = \Delta(z_k, r\delta)$, with $\|g_k\| \leq C_1\rho$ where $\rho = \sup_{z_1, z_2 \in \tilde{\Delta}_k} |f(z_1) - f(z_2)|$.

To do this we fix k and put $\gamma = \min(\delta, \gamma_A(\tilde{\Delta}))$. Construct the approximation scheme $\{\Delta_r^*\}$, $\{\phi_r^*\}$ corresponding to γ; we consider only those r for which Δ_r^* meets $\tilde{\Delta}_r$. Put $\psi_r^* = \phi_r^*\phi_k$ and $f_r^* = T_{\psi_r} f_r$ so that $\phi_k = \Sigma_r \psi_r^*$ and $f_k = \Sigma_r f_r^*$. Let $\tilde{\Delta}_r^*$ be the disc with the same center as Δ_r^* and radius $r\gamma$.

We have $|f_r^{*\prime}(\infty)| < C_2\rho\gamma_A(\Delta_r^*)$, by hypothesis, so that we can find $g_r \in A$, analytic outside $\tilde{\Delta}_r^*$ with $g_r(\infty) = 0$, $g_r'(\infty) = f_r^{*\prime}(\infty)$, and $\|g_r\| \leq C_2\rho$. Put $g_k^* = \Sigma_r g_r$. We observe that the proof of Vituškin's lemma [7, Chapter 8, Theorem 2.7] works equally well with γ_A in place of γ, if one interprets "admissible function for E" as "function in A, bounded by 1, zero at infinity, analytic outside E".

Applying it to the subsets $\tilde{\Delta}_r^*$ of $\tilde{\Delta}_k$ we have $\|g_k^*\| \le C_3\rho$. Also we observe $g_k^{*\prime}(\infty) = f_k'(\infty)$. We wish to modify g_k^* so that the second coefficients are matched as well.

Observing that $(g_r - f_r^*)'(\infty) = 0$, we have

$$|\beta(f_k - g_k^*)| \le \sum_r |\beta(f_r^* - g_r)| \le \sum_r |\beta(f_r^*, z_r^*)| + |\beta(g_r, z_r^*)|.$$

Now

$$|\beta(f_r^*, z_r^*)| \le \frac{1}{\pi}\left| \int \bar{\partial}\psi_r(\zeta) f(\zeta)(\zeta - z_r^*)\, dm(\zeta)\right| \le C_3\gamma\rho\gamma_A(\tilde{\Delta}_r^*),$$

and a similar estimate holds for g_r since $g_r = T_{\theta_r} g_r$ where θ_r has support in $\Delta(z_r^*, 2r\gamma)$ and is 1 in $\tilde{\Delta}_r^*$. Hence

$$|\beta(f_k - g_k^*)| \le C_4\rho\gamma \sum_r \gamma_A(\tilde{\Delta}_r^*)$$

$$\le C_5\rho\gamma^2 \quad \text{by Vituškin's lemma again.}$$

Choose $h \in A$, analytic outside $\tilde{\Delta}_k$, with $h'(\infty) = \gamma$ and $\|h\| < 2$. Put

$$g_k = g_k^* + \beta(f_k - g_k^*)(h/\gamma)^2.$$

Then g_k clearly has the asserted properties.

Finally we put $g = \Sigma_k g_k + f(\infty) \in A$. Then for $z \in \mathbf{C}$,

$$|f(z) - g(z)| \le \sum_k |f_k(z) - g_k(z)|$$

$$\le C_1 \sum_k \sup_{z_1 z_2 \in \tilde{\Delta}_k} |f(z_1) - f(z_2)| \min\left(1, \frac{r^3\delta^3}{|z - z_k|^3}\right).$$

First, this shows that $\|g\| \le A(m, r)\|f\|$. Second, if f is continuous at each point of the compact set K, and $\epsilon > 0$ is given, we can find a neighborhood U of K at each point of which the oscillation of f is less than ϵ. Choosing δ small enough we make the contribution to the above sum from those Δ_k contained in U less than $\epsilon A(m, r)$, and that for the other discs less than ϵ for $z \in K$. Hence if we take a sequence $\delta_n \to 0$ the corresponding sequence of g's proves the theorem.

Notes. (1) If U is an open set and f is locally in A on U, in the sense that $T_\phi f \in A$ whenever supp $\phi \subseteq U$, then we can take $g_k = f_k$ whenever $\Delta_k \subseteq U$. Then we obtain uniform convergence on compact subsets of U. If moreover, f is continuous at each point of $\mathbf{C}\setminus U$, then we obtain uniform convergence on $\mathbf{C}$.

(2) It is not hard to show that we can always ensure convergence almost everywhere (Lebesgue measure) but we shall not use this fact.

Lemma 2.1 has the following immediate consequence:

Corollary 2.2. *Suppose A_1 and A_2 are T_ϕ-stable subalgebras of A_0 and suppose that there exist $m, r, \delta_0 > 0$ such that $\gamma_{A_1}(\Delta(z, \delta)) \leq m\gamma_{A_2}(\Delta(z, r\delta))$ for all $z \in \mathbf{C}$ and $0 < \delta < \delta_0$. Let $f \in A_1$.*

Then we can find $\{f_n\}$ in A_2 with $\|f_n\| \leq A(m, r)\|f\|$, such that $f_n \to f$ uniformly on any compact set where f is continuous. Moreover, if U is any open set where locally $A_1 \subseteq A_2$ (in the sense that $T_\phi g \in A_2$ whenever $g \in A_1$ and supp $\phi \subseteq U$) *the convergence is uniform on compact subsets of U.*

This gives us a general condition for approximation in terms of analytic capacity. In practice, however, it is usually easier to verify the capacity conditions if we allow m, r and δ_0 to vary with z. We next show that in certain cases this weaker condition is sufficient.

Theorem 2.3. *Let A_1 and A_2 be T_ϕ-invariant subalgebras of A_0, and suppose all functions in A_1 are continuous on $\mathbf{C}$. Suppose that for all $z \in \mathbf{C}$ we can find $m, r, \delta_0 > 0$ with $\gamma_{A_1}(\Delta(z, \delta)) \leq m\gamma_{A_2}(\Delta(z, r\delta))$ for all $z \in \mathbf{C}$ and $0 < \delta < \delta_0$. Let $f \in A_1$. Then f is in the uniform closure of A_2.*

Proof. We first prove a lemma:

Lemma. *Let $z_1 \in \mathbf{C}$, $\delta_1 > 0$ and suppose that there exist $m, r, \delta_0 > 0$ such that for all $z \in \Delta(z_1, 2\delta_1)$, $0 < \delta < \delta_0$, $\gamma_{A_1}(\Delta(z, \delta)) \leq m\gamma_{A_2}(\Delta(z, r\delta))$. Then $\gamma_{A_1}(\Delta(z_1, \delta_1)) \leq \gamma_{A_2}(\Delta(z_1, 3\delta_1))$.*

Proof. Let

$$\tilde{A}_1 = \{f \in A_1 : f \text{ is analytic outside } \Delta(z_1, \delta_1)\},$$
$$\tilde{A}_2 = \{f \in A_2 : f \text{ is analytic outside } \Delta(z_1, 3\delta_1)\}.$$

Then for $z \in \mathbf{C}$, $0 < \delta < \min(\delta_0, \delta_1)$, we have

$$\gamma_{\tilde{A}_1}(\Delta(z, \delta)) \leq m\,\gamma_{\tilde{A}_2}(\Delta(z, r\delta)).$$

Thus by Corollary 2.2, $\tilde{A}_2$ is uniformly dense in $\tilde{A}_1$, which implies the lemma.

We can now prove the theorem. Assuming the conclusion is false, by Corollary 2.2 we can find $z_1 \in \mathbf{C}$ and $\delta_1 < \frac{1}{2}$ such that

$$\gamma_{A_1}(\Delta(z_1, \delta_1)) > \gamma_{A_2}(\Delta(z_1, 3\delta_1)).$$

Using the lemma we can construct inductively sequences $\{z_n\}$ in $\mathbf{C}$, $\{\delta_n\} > 0$ with $\delta_n < 2^{-n}$, $z_{n+1} \in \Delta(z_n, 2\delta_n)$, $\delta_{n+1} < \frac{1}{2}\delta_n$, and

$$\gamma_{A_1}(\Delta(z_n, \delta_n)) > n\gamma_{A_2}(\Delta(z_n, 3n\delta_n)).$$

Then $|z_n - z_{n+1}| < 2\delta_n$ so $z_n \to z \in \mathbf{C}$ with $|z - z_n| < 4\delta_n$. Thus

$$\gamma_{A_1}(\Delta(z, 5\delta_n)) \geq \gamma_{A_1}(\Delta(z_n, \delta_n))$$
$$> n\gamma_{A_2}(\Delta(z_n, 3n\delta_n)) \geq n\gamma_{A_2}(\Delta(z, (3n-4)\delta_n)) \quad \text{for all } n,$$

which contradicts the hypothesis of the theorem.

The only place in the above proof where the continuity of the functions in A_1 was used was to show in the lemma that $\tilde{A}_2$ is uniformly dense in $\tilde{A}_1$, so that $\gamma_{A_1}(\Delta(z_1, \delta_1)) \leq \gamma_{A_2}(\Delta(z_1, 3\delta_1))$. It would suffice to know that for $f \in \tilde{A}_1$ we can find $\{f_n\}$ in A_2 with $\|f_n\| \leq \|f\|$ and $f_n'(\infty) \to f'(\infty)$. For certain A_2 we can do this without the continuity assumption on A_1. Let U be an open subset of S^2 containing ∞ and let $A(U)$ denote the algebra of continuous functions on S^2 which are analytic on U. The proof of the following lemma is a simplified version of that in [1].

Lemma 2.4. *Let A be a T_ϕ-invariant subalgebra of A_0 and suppose there exist $m, r, \delta_0 > 0$ such that for all $z \in \mathbf{C}$, $0 < \delta < \delta_0$, $\gamma_A(\Delta(z, \delta)) \leq m\gamma_{A(U)}(\Delta(z, r\delta))$. Let $f \in A$. Then we can find $\{f_n\}$, $f_n \in A(U)$, $\|f_n\| \leq \|f\|$, with $f_n \to f$ uniformly on compact subsets of U.*

Proof. The hypotheses imply that the functions in A are analytic on U. Hence by Corollary 2.2 we can find $M > 0$ such that for each $f \in A$ there is a sequence $\{f_n\}$ with $f_n \in A(U)$, $\|f_n\| \leq M\|f\|$, $f_n \to f$ uniformly on compact subsets of U. We have to prove this with $M = 1$. Assume $f \in A$ with $\|f\| = 1$.

By the Hahn-Banach theorem it suffices to show the following: let K be a compact subset of U and μ a complex borel measure on K such that $|\int h\,d\mu| \leq 1$ for all $h \in A(U)$ with $\|h\| \leq 1$. Then $|\int f\,d\mu| \leq 1$. To prove this we extend the functional $h \to \int h\,d\mu$ on $A(U)$ to a functional on $C(S^2)$ of norm ≤ 1, represented by a borel measure ν, $\|\nu\| \leq 1$. Fix a positive integer k. We have sequences $\{f_n\}$, $\{g_n\}$ in $A(U)$, bounded by M, with $f_n \to f$, $g_n \to f^k$, uniformly on compact subsets of U. Let F, G be weak* cluster points of $\{f_n\}$ and $\{g_n\}$ respectively in $L^\infty(|\sigma|)$. Replacing f_n and g_n by convex combinations we may assume $f_n \to F$ and $g_n \to G$ a.e. $(|\sigma|)$. Let $h_n = f_n^k - g_n \in A(U)$ and $H = F^k - G$ in $L^\infty(|\sigma|)$; then $h_n \to 0$ uniformly on compact subsets of U and $h_n \to H$ a.e. $(|\sigma|)$.

The measure $\tau = H\sigma$ is supported on $S^2 \setminus U$ and for $h \in A(U)$, $\int h\,d\tau = \lim \int hh_n\,d\sigma = 0$. Moreover, if $\zeta \in U \setminus \{\infty\}$ then

$$\int \frac{d\tau(z)}{z-\zeta} = \lim \int \frac{h_n(z) - h_n(\zeta)}{z-\zeta}\,d\sigma(\zeta) = 0,$$

since $h_n(\zeta) \to 0$. The linear span of $A(U)$ and $\{1/(\zeta - z): \zeta \in U \setminus \{\infty\}\}$ is dense in $C(S^2 \setminus U)$ by [1, Lemma 1.1], so $\tau = 0$ and so $H = 0$ in $L^\infty(|\sigma|)$.

Thus $F^k = G$, so $\|F\|_\infty = \|G\|_\infty^{1/k} \leq M^{1/k}$. So $|\int f\,d\mu| = \lim |\int f_n\,d\mu| =$

$\lim |\int f_n \, d\nu| = |\int F \, d\nu| \leq \|F\| \|\nu\| \leq M^{1/k}$. Since k is arbitrary the lemma is proved.

As a consequence of Lemma 2.4 and the preceding remarks, the proof of Theorem 2.3 yields

Theorem 2.5. *Suppose A, $A(U)$ are as before and for each $z \in \mathbf{C}$ there exist m, r, $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, we have*

$$\gamma_A(\Delta(z, \delta)) \leq m \gamma_{A(U)}(\Delta(z, r\delta)).$$

Then for any $f \in A$ we can find a sequence $\{f_n\}$ in $A(U)$ with $\|f_n\| \leq \|f\|$ and $f_n \to f$ uniformly on compact subsets of U.

3. **A modified approximation scheme.** The approximation scheme devised by Vituškin and described in §2 uses coverings by discs of the same radius. For some applications it is useful to have a version in which the discs are allowed to have different radii; in this section we provide the details of the construction. For technical reasons we work with squares instead of discs.

Proposition 3.1. *Let V be an open subset of $\mathbf{C}$ and ρ a positive continuous function on V. We can find a sequence $\{Q_r\}_{r=1}^{\infty}$ of open squares, Q_r having center z_r and side δ_r, whose closures lie in V, such that*

(1) $V = \bigcup_r Q_r$;

(2) *no point of V lies in more than 4 squares Q_r;*

(3) *the square with center z_r and side $\delta_r/3$ meets no Q_s for $s \neq r$;*

(4) *if $z \in Q_r$ then $\delta_r < \rho(z)$;*

(5) *there is a C^1 partition $\{\phi_r\}$ of 1 subordinate to $\{Q_r\}$ with $\|\operatorname{grad} \phi_r\| \leq 100/\delta_r$;*

(6) *for all $z \in \mathbf{C}$, $\Sigma_r \min(1, \delta_r^3/|z - z_r|^3) \leq A$ (absolute constant).*

Proof. For each positive integer k let $\tau_k = \sup\{d(z, \mathbf{C}\setminus V): z \in V, \rho(z) \leq 5 \cdot 2^{-k-2}\}$. Then $\tau_k \to 0$ as $k \to \infty$, so we can choose a sequence $\{\rho_k\}$ of positive numbers such that $\rho_k \to 0$, $\rho_k - \rho_{k+1} > 2^{2-k/2}$, and $\rho_k > \tau_k$ for each k. Let $V_k = \{z \in V: d(z, \mathbf{C}\setminus V) \geq \rho_k\}$. Then V_k is compact and $z \in V_k \Rightarrow \rho(z) > 2^{-k} \cdot 5/4$.

For $k = 1, 2, \cdots$ let $\{\tilde{Q}_{kp}\}_{p=1}^{\infty}$ be an enumeration of the open squares with sides 2^{-k}, parallel to the coordinate axes, and corners at the points whose coordinates are both integral multiples of 2^{-k}. Let Q_{kp} be the square concentric with $\tilde{Q}_{kp}$ obtained by enlarging it by a factor of $5/4$. The covering $\{Q_r\}$ will be a subset of $\{Q_{kp}\}$, defined as follows: let k_0 be the smallest k for which V_k is nonempty. We include all $Q_{k_0 p}$ contained in V_{k_0}. Inductively, for $k > k_0$ we include all Q_{kp} such that Q_{kp} is contained in V_k but $\tilde{Q}_{kp}$ is not contained in some square Q_{ls}, $l < k$, for which Q_{ls} has already been included. We renumber those Q_{kp} selected in this way as $\{Q_r\}_{r=1}^{\infty}$. If $Q_r = Q_{kp}$

put $\tilde{Q}_r = \tilde{Q}_{kp}$ and $\delta_r = 5 \cdot 2^{-k-2}$.

Then it is clear that $\{Q_r\}$ satisfies properties (1)–(4). To prove (5) we choose for each r a continuous differentiable function ψ_r with support contained in Q_r, satisfying $0 \le \psi_r \le 1$, $\psi_r = 1$ on a neighborhood of $\tilde{Q}_r$, and $\|\operatorname{grad} \psi_r\| \le 11/\delta_r$. Let $\psi = \sum_{s=1}^{\infty} \psi_s$ and $\phi_r = \psi_r/\psi$. We have $|\psi| \ge 1$ on V, and for $z \in Q_r$,

$$|\operatorname{grad} \psi(z)| \le \sum_{z \in Q_s} |\operatorname{grad} \psi_s(z)| \le 11 \cdot \sum_{z \in Q_s} \delta_s^{-1} \le 88/\delta_r$$

since at most four Q_s contain z and for each of these $\delta_2 \ge \delta_r/2$. Hence

$$|\operatorname{grad} \phi_r(z)| \le \psi(z)^{-1} |\operatorname{grad} \psi_r(z)| + \psi(z)^{-2} |\operatorname{grad} \psi(z)| \le 99/\delta_r .$$

It is clear that $\{\phi_r\}$ is a partition of unity subordinate to $\{Q_r\}$.

To prove (6) we observe that as in the usual approximation scheme, for each k and each $z \in \mathbf{C}$,

$$\sum_p \min(1, 2^{-3k}/|z - z_{kp}|^3) \le A_1 \min\left(1, 2^{-k} \Big/ d\left(z, \bigcup_p Q_{kp}\right)\right)$$

where the summation extends over those p for which Q_{kp} is included above. If $z \notin V$ then $d(z, \bigcup_p Q_{kp}) \ge \rho_k > 2^{2-k/2}$, so that

$$\sum_r \min(1, \delta_r^3/|z - z_r|^3) \le A_1 \sum_k 2^{k/2-2} \le A_1.$$

On the other hand, if $z \in V$, then $z \in V_l \setminus V_{l-1}$ for some l and then

$$\sum_r \min\left(1, \frac{\delta_r^3}{|z - z_r|^3}\right) \le 3A_1 + A_1\left(\sum_{k=k_0}^{l-2} + \sum_{k=l+2}^{\infty}\right) \frac{2^{-k}}{d(z, \bigcup_p Q_{kp})}$$

$$\le 3A_1 + A_1 \sum_{k=k_0}^{l-2} 2^{k/2-2} + A_1 \sum_{k=l+2}^{\infty} \frac{2^{-k}}{2^{3-k/2} - 2^{-k}} \le 5A_1$$

which completes the proof.

4. **Negligible sets.** Let $E \subseteq S^2$. We say E is α-negligible if whenever $f \in C(S^2)$ is analytic on some open set U, we can find a sequence $\{f_n\}$, $f_n \in C(S^2)$, $f_n \to f$ uniformly on S^2, such that each f_n is analytic on some open set containing $U \cup E$. Similarly we say E is γ-negligible if for some constant $M > 0$, whenever f is a bounded borel function on S^2, analytic on U, we can find $\{f_n\}$, f_n a bounded borel function on S^2, with $\|f_n\| \le M\|f\|$, $f_n \to f$ point-

wise on U, and f_n analytic on an open set containing $U \cup E$.

Vituškin's arguments in [14, Chapter 5, §4], show that $R(K) = A(K)$ provided that for each $z \in \partial K \setminus E$, where E is an α-negligible set, there exist $m, r, \delta_0 > 0$ such that

$$\alpha(\Delta(z, \delta) \setminus K^\circ) \leq m\alpha(\Delta(z, r\delta) \setminus K), \qquad 0 < \delta < \delta_0.$$

Vituškin shows that any Liapunov arc is α-negligible; moreover it is easily seen that a countable union of α-negligible sets is α-negligible. Gamelin and Garnett show in [9, Theorem 7.3], that a set of zero outer length on a Liapunov arc is γ-negligible; moreover, the "reduction of norm" theorem in [9, Theorem 6.7] shows that a countable union of compact γ-negligible sets is γ-negligible. It is not clear whether this remains true if compactness is dropped.

In this section we prove an analogue of Vituškin's theorem for bounded approximation. This is a slight extension of Theorem 8.6 of [9], but the proof is quite different.

Then we turn to the question of determining which sets are negligible, and relate this problem to the semiadditivity problem to be considered in the following section. Finally we weaken the smoothness assumption in Vituskin's result on Liapunov arcs.

Let U be an open subset of S^2; we assume $\infty \in U$. If V is a subset of ∂U we denote by $H_V^\infty(U)$ the set of bounded analytic functions on U which extend continuously to $U \cup V$.

Theorem 4.1. *Let U be as above. Let $E_1, E_2, \cdots$ be γ-negligible subsets of $\mathbf{C}$ such that $E_i \cap \overline{E}_j \subseteq E_j$ for all i, j. Let V be a relatively open subset of ∂U and suppose that for each $z \in \partial U \setminus (V \cup (\bigcup_n E_n))$ we can find $m, r, \delta_0 > 0$ with*

$$(*) \qquad \gamma_V(\Delta(z, \delta) \setminus U) \leq m\alpha(\Delta(z, r\delta) \setminus U), \qquad 0 < \delta < \delta_0.$$

Then $A(U)$ is pointwise boundedly dense in $H_V^\infty(U)$.

Proof. First we prove a lemma:

Lemma. *Let $z_1 \in \mathbf{C}$, $\delta_1 > 0$, let n be a positive integer, and suppose that for each $z \in \Delta(z_1, 2\delta_1) \setminus (V \cup (\bigcup_{i=1}^n E_i))$, we can find $m, r, \delta_0 > 0$ so that $\gamma_V(\Delta(z, \delta) \setminus U) \leq m\alpha(\Delta(z, r\delta) \setminus U)$, $0 < \delta < \delta_0$.*

Then $\gamma_V(\Delta(z_1, \delta_1) \setminus U) \leq \alpha(\overline{\Delta}(z_1, 3\delta_1) \setminus U)$.

Proof of lemma. Let $A_1 = \{f \in A_0 : f$ is continuous on V, analytic in U and outside $\Delta(z_1, \delta)\}$, $A_2 = \{f \in A_0 : f$ is continuous on V, analytic in U and $\bigcup_{i=1}^n E_i$ and outside $\Delta(z_1, \delta)\}$. Let $W = U \cup S^2 \setminus \overline{\Delta}(z_1, 3\delta)$.

If $f \in A_1$ then by γ-negligibility of the E_i we can find a bounded sequence $\{f_n\}$ in A_2, $f_n \to f$ pointwise on U. By Theorem 2.5 $A(W)$ is pointwise boundedly

dense in A_2, hence $A(W)$ is pointwise boundedly dense in A_1, by Lemma 2.4. This proves the lemma.

Proof of theorem. Assuming the conclusion is false, by Theorem 2.5 we can find $z_0 \in \mathbf{C}$, $\delta_0 < 1$, with

$$\gamma_V(\Delta(z_0, \delta_0)\setminus U) > \alpha(\Delta(z_0, 3\delta_0)\setminus U).$$

Using the lemma we construct inductively sequences $\{z_n\}$ in $\mathbf{C}$, $\{\delta_n\} > 0$, with $z_{n+1} \in \Delta(z_n, 2\delta_n)$, $\delta_{n+1} < \frac{1}{2}\delta_n$, $d(z_n, \bigcup_{i=1}^n E_i) > 4\delta_n$, and $\gamma_V(\Delta(z_n, \delta_n)\setminus U) > n\alpha(\Delta(z_n, 3n\delta_n)\setminus U)$.

As before $z_n \to z$ which does not satisfy (*); moreover, $|z - z_n| < 4\delta_n$ so $z \notin \bigcup_{i=1}^\infty E_i$, which is a contradiction.

See §6 for an application of this result.

Next we consider the problem of determining when a set is negligible. It follows from the definitions that if S is α-negligible then $\alpha(S \cup T) = \alpha(T)$ for all sets T. Conversely, if for every open set $U \subseteq S^2$ and $z \in \mathbf{C}$ we can find $m, r, \delta_0 > 0$ with

$$\alpha(\Delta(z, \delta)\setminus U) \leq m\alpha(\Delta(z, r\delta)\setminus(U \cup S))$$

then Theorem 2.3 ensures that S is α-negligible. In particular if $\alpha(S \cup T) \leq M\alpha(T)$ for all sets T, where M is independent of T, then S is α-negligible. Clearly if S is negligible then $\alpha(S) = 0$; a semiadditivity theorem for α would yield a converse, under appropriate conditions on S. More precisely, if we knew that there existed a constant M such that

$$\alpha(S \cup T) \subseteq M(\alpha(S) + \alpha(T))$$

whenever S satisfied a certain condition (of course one may assume $S \cup T$ compact) then any S satisfying that condition with $\alpha(S) = 0$ is α-negligible. In §8 we consider this problem for compact S.

A similar situation obtains for γ-neglibility. A bounded plane set S is γ-negligible if and only if there exists $M > 0$ so that $\gamma(S \cup T) \subseteq M\gamma(T)$ for all plane sets T. It is not clear whether M can always be taken to be 1 (this is so when S is compact). Again if we have a semiadditivity theorem for γ we can deduce S is γ-negligible if $\gamma(S) = 0$. This problem is considered for compact S in §5.

We conclude this section by showing that certain sets lying on smooth arcs are negligible. This sharpens results of Vituškin [14], and Gamelin and Garnett [9].

Definition 4.2. We say that a uniformly continuous function defined on a plane set is *strongly continuous* if its modulus of continuity $\omega(t)$ satisfies $\int_0^1 \omega(t)/t < \infty$. We say that an arc or curve J is *hypo-Liapunov* if it has a tangent at each point and the direction of the tangent (regarded as a point on the unit circle) is strongly continuous as a function on J. The proof of the next

lemma is essentially due to Vituškin in [14, Chapter 1, §6, Theorem 1], with some modifications.

Lemma 4.3. *Let J be a simple closed curve bounding a domain D, and let ϕ map the unit disc Δ conformally onto D. Suppose ϕ satisfies*

(i) *ϕ' extends continuously to $\overline{\Delta}$;*

(ii) *$\arg\phi'$ has a strongly continuous branch on $\overline{\Delta}$;*

(iii) *$\operatorname{Re}\phi' > 0$ on $\overline{\Delta}$.*

Then there exists $M > 0$ such that if E is any compact subset of Δ, then

$$\alpha(E) < M\alpha(\phi(E)) \quad \textit{and} \quad \gamma(E) \leq M\gamma(\phi(E)).$$

Proof. We give the proof for α, that for γ being exactly analogous.

Choose f continuous on $\mathbf{C}^+$, analytic off E, zero at infinity, with $\|f\| \leq 1$ and $|f'(\infty)| > \frac{1}{2}\alpha(E)$. Let $g(z) = f(z) + \overline{f(\bar{z}^{-1})}$ or $i(f(z) - \overline{f(\bar{z}^{-1})})$, whichever makes $(1/2\pi)|\int_\Gamma g(z)dz| \geq \frac{1}{2}\alpha(E)$. Then g is analytic outside $E \cup E^*$, real on Γ, and $\|g\| \leq 2$ (where $E^* = \{z: \bar{z}^{-1} \in E\}$). Let $\psi: \overline{D} \to \overline{\Delta}$ be inverse to ϕ, and let $h = g \circ \psi$. Put

$$F(w) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_J h(u)\psi'(u)(w-u)^{-1}\,du, & w \in \mathbf{C}\setminus\overline{D}, \\ \dfrac{1}{2\pi i}\displaystyle\int_J h(u)\psi'(u)(w-u)^{-1}\,du + h(w)\psi'(w), & w \in D. \end{cases}$$

Then F extends analytically across T to give a continuous function on $\mathbf{C}$, analytic outside $\phi(E)$. For $z \in \Delta$ we have

$$F(\phi(z)) = \frac{g(z)}{\phi'(z)} + \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\phi(z) - \phi(\zeta)}\,d\zeta.$$

We estimate the second term on the right, as follows:

$$\frac{\phi(\zeta) - \phi(z)}{\zeta - z} = \int_0^1 \phi'(t\zeta + (1-t)z)\,dt,$$

and since

$$|\operatorname{Im}[\phi'(t\zeta + (1-t)z)/\phi'(z)]| \leq A_1\omega(|\zeta - z|),$$

where ω is the modulus of continuity of ϕ' on $\overline{\Delta}$ and $A_1, A_2, \cdots$ denote constants depending only on D and ϕ, we have

$$\left|\operatorname{Im}\left\{\frac{\phi(\zeta) - \phi(z)}{(\zeta - z)\phi'(z)}\right\}\right| \leq A_1\omega(|\zeta - z|), \qquad \zeta, z \in \overline{\Delta}.$$

For some $\delta > 0$, $\operatorname{Re}\phi' \geq \delta$ on $\overline{\Delta}$, and so

$$|(\phi(\zeta) - \phi(z))/(\zeta - z)| \geq \delta \quad \text{for } \zeta, z \in \overline{\Delta}.$$

Hence

$$\left|\operatorname{Re}\left\{\phi'(z)\int_\Gamma \frac{g(\zeta)}{\phi(z)-\phi(\zeta)}\,d\zeta\right\}\right|$$

$$= \left|\int_\Gamma g(\zeta)\left\{\operatorname{Re}\frac{1}{z-\zeta}\operatorname{Re}\frac{(\zeta-z)\phi'(z)}{\phi(\zeta)-\phi(z)} + \operatorname{Im}\frac{1}{z-\zeta}\operatorname{Im}\frac{(\zeta-z)\phi'(z)}{\phi(\zeta)-\phi(z)}\right\}d\zeta\right|$$

$$\le A_2\int_\Gamma |g(\zeta)|\frac{\omega(|z-\zeta|)}{|z-\zeta|}|d\zeta| + 2\pi\sup_{\zeta\in\Gamma}\left|\frac{g(\zeta)(\zeta-z)\phi'(z)}{\phi(\zeta)-\phi(z)}\right| \le A_3.$$

Thus $|\operatorname{Re}\{\phi'(z)F(\phi(z))\}| \le A_4$, $z \in \overline{\Delta}$.

The range of $\arg\phi'$ lies in an interval $(-\pi/2+\theta, \pi/2-\theta)$ for some $\theta > 0$; hence $F(\phi(z))$ must lie in the sector

$$\{\zeta : \pi - \theta < \arg(\zeta + A_5) < \pi + \theta\}$$

if A_5 is positive and sufficiently large. Thus if A_6 is large enough, the function $G(w) = 1/(A_6 + F(w))$ will be bounded in modulus by 1 on $\overline{D}$, and so also on $\mathbf{C}$ since $G(\infty) = A_6^{-1}$. Finally

$$F'(\infty) = \frac{1}{2\pi i}\int_J F(w)\,dw = \frac{1}{2\pi i}\int_\Gamma g(z)\,dz$$

so that $|F'(\infty)| \ge \frac{1}{2}\alpha(E)$ and $G'(\infty) = -A_6^{-2}F'(\infty)$, whence $\alpha(\phi(E)) \ge \frac{1}{2}A_6^{-2}\alpha(E)$.

We observe that the constant in the conclusion of Lemma 4.3 depends only on the quantities $\|\phi'\|$, $\inf\arg\phi'$, and $\int_0^1(\omega(t)/t)\,dt$ where ω is the modulus of continuity of $\arg\phi'$.

In order to make use of Lemma 4.3 we have to obtain criteria for the hypotheses to hold. We obtain these by means of some classical results of Warschawski and Courant on conformal mapping, which can be found in Chapter 9 of Tsuji [13].

First we note that if D is a domain bounded by a continuously differentiable simple closed curve J and ϕ maps Δ conformally on D, then $\arg\phi'$ extends continuously on $\overline{\Delta}$ by Theorem IX.6 of [13], and moreover, by equation (2) in the proof of that theorem, $\arg\phi'(e^{i\theta})$ is just the difference between the direction of the tangent to J at $\phi(e^{i\theta})$ and the tangent to Γ at $e^{i\theta}$.

We now consider a special class of domains defined as follows: if $\eta > 0$, let P_η be the set of simple closed curves J of the form

$$J = \{re^{i\theta} : r = g(\theta)\}$$

where g is a continuously differentiable function on $[0, 2\pi]$ with $g(0) = 1$, $g = 1$ on $[\eta, 2\pi]$, and $|g'| < \eta$.

Lemma 4.4. *There exists $\eta_0 > 0$ such that if a curve $J \in P_{\eta_0}$ bounds a domain D and ϕ maps Δ conformally on D with $\phi(0) = 0$, $\phi'(0) > 0$, then $|\arg\phi'| < \pi/4$ on $\overline{\Delta}$.*

Proof. If $\{\eta_n\}$ is a sequence tending to zero and $J_n \in P_{\eta_n}$ bounds the domain D_n, and ϕ_n maps Δ conformally on D_n with $\phi_n(0) = 0$ and $\phi_n'(0) > 0$, then $\phi_n(z) \to z$ uniformly for $z \in \overline{\Delta}$. It follows easily from this and the description of $\arg \phi'$ on Γ above that if η_0 is chosen small enough then $|\arg \phi'| < \pi/4$ on Γ, and hence on $\overline{\Delta}$ since $\arg \phi'$ is harmonic on Δ.

Lemma 4.5. *Let J be a hypo-Liapunov curve in P_{η_0}, bounding the domain D. Let ϕ map Δ conformally on D with $\phi(0) = 0$ and $\phi'(0) > 0$.*

Then ϕ' extends continuously to $\overline{\Delta}$ and does not vanish there, and $\arg \phi'$ is strongly continuous on $\overline{\Delta}$. Moreover there are upper bounds for $|\phi'|$, $|\phi'|^{-1}$, and the integral $\int_0^1 (\omega(t)/t)\,dt$ where ω is the modulus of continuity of $\arg \phi'$, depending only on the quantity

$$\chi(J) = \int_0^1 \frac{\tilde{\omega}(t)}{t}\,dt$$

where $\tilde{\omega}$ is the modulus of continuity of the direction of the tangent to J.

Proof. We apply Theorem IX.9 of [13], observing that in our situation the estimates in the proof of that theorem depend only on $\chi(J)$. This yields the continuity of ϕ' and the bounds for $|\phi'|$ and $|\phi'|^{-1}$. It follows that ϕ satisfies a Lipschitz condition, which together with our description of $\arg \phi'$ yields the desired estimate for the modulus of continuity of $\arg \phi'$ on Γ. A simple estimate with the Poisson kernel yields a similar estimate on all of $\overline{\Delta}$.

Combining Lemmas 4.3, 4.4 and 4.5 we obtain

Lemma 4.6. *Let J be a hypo-Liapunov curve in P_{η_0}, bounding D. Then there is a constant $M > 0$, depending only on $\chi(J)$, such that if ϕ maps Δ conformally on D with $\phi(0) = 0$ and $\phi'(0) > 0$, and E is any compact subset of Δ, then*

$$\alpha(E) \leq M\alpha(\phi(E)) \quad \textit{and} \quad \gamma(E) \leq M\gamma(\phi(E)).$$

Combining this with a theorem of Melnikov [12, Chapter 3, §1, Theorem 1] we obtain

Corollary 4.7. *Let J be a hypo-Liapunov curve in P_{η_0}, bounding D, let $E \subseteq D$ be compact, and let $f \in C(\overline{D})$ be analytic on $D \setminus E$. Then*

$$\int_J f(z)\,dz \leq M_1 \alpha(E)\,\|f\|$$

where M_1 depends only on $\chi(J)$.

Proof. Let ϕ map Δ conformally on D with $\phi(0) = 0$, $\phi'(0) > 0$. Then

$$\left|\int_J f(z)\,dz\right| = \left|\int_\Gamma f(\phi(w))\phi'(w)\,dw\right|$$

$$\leq A_1 A_2 \alpha(\phi^{-1}(E))\,\|f\| \leq A_1 A_2 M\alpha(E)\,\|f\|$$

where $A_1 = \sup|\phi'|$, and A_2 is the constant in Melnikov's theorem.

Let Ω denote the set of all simple closed curves which can be mapped onto hypo-Liapunov curves in P_{η_0} by transformations of the form $z \to az + b$, $a, b \in \mathbf{C}$. For $J \in \Omega$ define $\chi(J)$ to be $\chi(J_0)$ where J_0 is the unique curve in P_{η_0} onto which J can be so mapped. Then Corollary 4.7 clearly holds for $J \in \Omega$. We shall show later that it holds for all simple closed hypo-Liapunov curves, but the above result suffices for our applications to negligibility, so we consider these first. The methods come from [14] and [9].

Lemma 4.8. *Let $J \in \Omega$ bound D, let E be a relatively closed subset of D, and let $f \in C(\overline{D})$ be analytic on $D \setminus E$. Then $|\int_J f(z)\,dz| \leq M_1 \alpha(E)\,\|f\|$.*

Proof. For δ small enough, for each $z \in J$ we have $\gamma(\Delta(z, \delta) \setminus \overline{D}) > \delta/4$. Hence by Corollary 2.2 we can approximate f uniformly by functions in $C(\overline{D})$ analytic in D outside a compact subset of E. For these functions the required estimate holds by Corollary 4.7, hence it holds also for f.

Lemma 4.9. *Let $J \in \Omega$ bound D. Let E be a compact subset of $\overline{D}$. Let f be a bounded borel function on $\overline{D}$ continuous on $\overline{D} \setminus E$ and analytic on $D \setminus E$.*

Then $|\int_J f(z)\,dz| \leq M_1 \gamma(E)\,\|f\|$. In particular (taking $f = 1$ on E, 0 elsewhere) if $E \subset J$, $l(E) \leq M_1 \gamma(E)$.

Proof. Fix $\epsilon > 0$ and let $E_\epsilon = \{z: d(z, \epsilon) < \epsilon\}$. Choose $g \in C(D)$ with $\|g\| \leq \|f\| + \epsilon$, $g = f$ on $\overline{D} \setminus E_\epsilon$, so that $\int_J g(z)\,dz = \int_J f(z)\,dz$. Then by Lemma 4.8

$$\left|\int_J g(z)\,dz\right| \leq M_1 \alpha(E_\epsilon)(\|f\| + \epsilon).$$

Since $\alpha(E_\epsilon) \to \gamma(E)$ as $\epsilon \to 0$, the desired result follows.

We are now in a position to prove the results on negligibility. First we observe that if J is a hypo-Liapunov arc, the direction of whose tangent oscillates by less than η_0, then we can find curves J_1 and J_2 in Ω, bounding disjoint domains D_1 and D_2, so that $J_1 \cap J_2$ is an arc J_0 containing J, and the domain D^* bounded by $J^* = \overline{(J_1 \cup J_2) \setminus J_0}$ contains any preassigned bounded set.

Theorem 4.10. *Any hypo-Liapunov arc is α-negligible.*

Proof. It suffices to prove it for arcs whose tangent oscillates by less than η_0, so let J be such an arc, fix $R > 0$, and choose J_1 and J_2 as above so that D^* contains the disc $\Delta(0, R)$. Let E by any compact subset of $\Delta(0, R)$, and let $f \in C(E, 1)$. Then

$$2\pi|f'(\infty)| = \left|\int_{J^*} f(z)\,dz\right| \le \left|\int_{J_1} f(z)\,dz\right| + \left|\int_{J_2} f(z)\,dz\right|$$
$$\le M_1\gamma(E \cap D_1) + M_1\gamma(E \cap D_2) \le 2M_1\gamma(E\setminus J)$$

by Lemma 4.8. It follows that J is α-negligible.

Theorem 4.11. *Any set of zero length on a hypo-Liapunov arc is γ-negligible.*

Proof. Let F be a set of zero length on a hypo-Liapunov arc J whose tangent we assume to oscillate less than η_0. Fix $R > 0$; as before we choose J_1 and J_2 so that $\Delta(0, R) \subseteq D^*$. Let E be a compact subset of $\Delta(0, R)$.

Let $f \in B(E, 1)$, we assume f extends to be a borel function on $\mathbf{C}$, $\|f\| \le 1$. Let $J^{(n)}$ be a sequence of arcs of Ω contained in D_1, with $J^{(n)} \to J$ and $\chi(J^{(n)})$ bounded. Since f is analytic outside E, we have

$$\left|\int_{J_1} f(z)\,dz\right| \le \limsup \left|\int_{J^{(n)}} f(z)\,dz\right| + 2l(E) \le M_1\gamma(E \cap D_1) + 2l(E \cap J_0).$$

Similarly, $|\int_{J_2} f(z)\,dz| \le M_1(E \cap D_2) + 2l(E \cap J_0)$. We have

$$l(E \cap J_0) = l((E \cap J_0)\setminus F) \le M_1\gamma((E \cap J_0)\setminus F) \le M_1\gamma(E\setminus F)$$

by Lemma 4.9. Hence

$$2\pi|f'(\infty)| \le \left|\int_{J_1} f(z)\,dz\right| + \left|\int_{J_2} f(z)\,dz\right| \le (2M_1 + 4)\gamma(E\setminus F)$$

which proves that F is γ-negligible.

Finally we show that the estimate in Corollary 4.7 is valid for any simple closed hypo-Liapunov arc, following the ideas of [14, Chapter 3, §1]. If J is a hypo-Liapunov curve of diameter 1 let $\chi_0(J)$ be the quantity $\int_0^1 (\omega(t)/t)\,dt$ where ω is the modulus of continuity of the direction of the tangent to J. For any hypo-Liapunov curve J, let $\chi_0(J) = \chi_0(J')$ where J' is similar to J with diameter 1.

Theorem 4.12. *Let J be a simple closed hypo-Liapunov curve bounding the domain D, let E be a relatively closed subset of D, and let $f \in C(\overline{D})$ be analytic on $D\setminus E$. Then $|\int_J f(z)\,dz| \le M_1\alpha(E)\|f\|$ where M_1 depends only on $\chi_0(J)$.*

Proof. We may assume J has diameter 1. Topological considerations show that we can find $0 < \delta < \eta_0$, depending only on $\chi(J)$, so that if $z \in J$ then $\Delta(z, \delta) \cap J$ is an arc whose tangent oscillates by less than η_0. Then we can find a curve $J_z \in \Omega$, so that $\Delta(z, \delta) \cap J \subseteq J_z$, $\operatorname{diam}(J_z) \le 8$, and the domain D_z bounded by J_z contains $\Delta(z, \delta) \cap D$.

We construct the approximation scheme of §2 with discs of radius $\delta/2$.

Extend f to be continuous on $\mathbf{C}$, with compact support. Put $f_k = T_{\phi_k} f$. For each k such that $\Delta_k \subseteq D$ we have $|\int_J f_k(z)\,dz| = 2\,|f_k'(\infty)| \leq A\gamma(E \cap \Delta_k)\,\|f\|$. For each k such that $\Delta_k \cap \overline{D} = \emptyset$ we have $\int_J f_k(z)\,dz = 0$. For each k such that Δ_k meets J we have $\Delta_k \subseteq \Delta(z_k, \delta)$ for some $z_k \in J$. Then

$$\left|\int_J f_k(z)\,dz\right| = \left|\int_{J_{z_k}} f_k(z)\,dz\right| \leq M_0\gamma(E \cap \Delta_k)\,\|f\|$$

where M_0 depends only on $\chi(J_z) \leq 8\chi_0(J)$, by Lemma 4.8. Putting these together we obtain

$$\left|\int_J f(z)\,dz\right| \leq \sum_k \left|\int_J f_k(z)\,dz\right| \leq M_0\|f\|\left(\sum_k \gamma(E \cap \Delta_k)\right) \leq C(\delta)M_0\,\|f\|$$

by Theorem 2.7 of [7, Chapter 8].

Since δ depends only on $\chi_0(J)$ the theorem follows. One can easily deduce an analogue of Lemma 4.9. We have the following corollary on semi-additivity.

Corollary 4.13. *Let $D_1, \cdots, D_n$ be disjoint domains bounded by hypo-Liapunov arcs $J_1, \cdots, J_n$. Let E_i be a closed subset of $\overline{D}_i$, $i = 1, \cdots, n$, and let $E = \bigcup E_i$. Then*

$$\alpha(E) \leq M \sum_{i=1}^{n} \alpha(E_i)$$

where M depends only on $\sup \chi(J_i)$.

We remark finally that our estimates could be proved for curves that are piecewise hypo-Liapunov, provided the corners are not sharp. The methods used by Vituškin in [15] to treat sharp corners in the Liapunov case do not seem to carry over.

5. γ **as a set function.** We consider first the problem of comparing γ and γ^*. The only examples where $\gamma(S)$ and $\gamma^*(S)$ are known to be different are non-borel sets; however, in the positive direction we can only prove the following partial result for the special case where S is locally compact. So there is still a large gap to be filled in.

If $S \subseteq \mathbf{C}$ we define

$$\beta_\gamma(S) = \sup\left\{\left|\lim_{z\to\infty} z^2 f(z)\right| : f \text{ analytic outside a compact subset of } S,\ \|f\| \leq 1, \text{ and } f(\infty) = f'(\infty) = 0\right\}.$$

Theorem 5.1. *There is an absolute constant $A > 0$ such that $\gamma^*(S) \leq A\gamma(S)$ for all locally compact sets S.*

Proof. Fix a locally compact set S, and let $\epsilon > 0$. Let V be an open set such that $K \cap S$ is compact for every compact $K \subseteq V$. In V we construct the modified approximation scheme of §3 ($\rho = 1$). Let $\epsilon > 0$. For each $r = 1, 2, \cdots$ choose a compact subset K_r of $\overline{Q}_r$ such that $K_r \cap S = \emptyset$ and such that

(a) if $\gamma(\overline{Q}_r \cap S) \neq 0$ then

$$\gamma(\overline{Q}_r \setminus K_r) < 2\gamma(\overline{Q}_r \cap S) \quad \text{and} \quad \beta_\gamma(\overline{Q}_r \setminus K_r) < 2\beta_\gamma(\overline{Q}_r \cap S);$$

(b) if $\gamma(\overline{Q}_r \cap S) = 0$ then $\gamma(\overline{Q}_r \setminus K_r) < 2^{-r}\epsilon$.

Let $U = V \setminus \bigcup_r K_r$; then U is open and $U \supseteq S$. Let f be a bounded analytic function on $\mathbf{C} \setminus Q$, where Q is a compact subset of U, and $\|f\| \leq 1$. Let $f_r = T_{\phi_r} f$; then f_r is analytic outside a compact subset of $\overline{Q}_r \setminus K_r$. Then for each r such that $\gamma(\overline{Q}_r \cap S) > 0$ we can find g_r, analytic off $\overline{Q}_r \cap S$, such that $g_r - f_r$ has a triple zero at infinity and $\|g_r\| \leq A_1$. For each r such that $\gamma(\overline{Q}_r \cap S) = 0$ let $g_r = 0$; for such r $|f_r'(\infty)| < A_1 2^{-r}\epsilon$.

Let $g = f + \Sigma_r (g_r - f_r)$, the sum being taken over the (finite) set of r for which f_r is not identically zero. Using 3.1(b) we find $\|g\| \leq A_2$ and $|g'(\infty) - f'(\infty)| < \epsilon$. Moreover, g is analytic outside a compact subset of S, which completes the proof.

Now we turn to the problem of semiadditivity, raised by Vituškin [14].

Conjecture (A). *There exists an absolute constant $M > 0$ such that $\gamma(E \cup F) \leq M(\gamma(E) + \gamma(F))$, whenever E and F are disjoint compact subsets of* $\mathbf{C}$.

We show that a number of apparently stronger or weaker assertions are actually equivalent to (A).

Theorem 5.2. *Suppose* (A) *is false. Then for any $\epsilon > 0$ we can find subsets E and F of the unit square Q_0, with E and $E \cup F$ compact, $\gamma(E) = 0$, $\gamma(F) < \epsilon$, and $\alpha(E \cup F) > \frac{1}{2}$.*

The proof will be broken up into lemmas.

Lemma 5.3. *Suppose P and Q are disjoint compact sets with $d(P, Q) = \delta$ and* $\operatorname{diam}(P \cup Q) \leq 1$. *Then $\gamma(P \cup Q) \leq \gamma(P) + A\delta^{-2}\gamma(Q)$ where A is an absolute constant.*

Proof. Let $f \in B(P \cup Q, 1)$. Choose a continuously differentiable function ϕ with $\phi = 1$ on a neighborhood of Q, $\phi = 0$ on a neighborhood of P, $\|\operatorname{grad}\phi\| \leq A_1/\delta$, and $\operatorname{diam}(\operatorname{supp}\phi) < 2$. Put

$$f_1 = T_\phi f \in B(Q, A_2/\delta); \qquad f_2 = f - T_\phi f \in B(P, A_2/\delta).$$

Since $d(P, Q) = \delta$, on P we have $|f_1| \leq A_2\delta^{-2}\gamma(Q)$ and so $|f_2| \leq 1 + A_2\delta^{-2}\gamma(Q)$ on P, and hence everywhere. Hence $|f_2'(\infty)| \leq \gamma(P)(1 + A_2\delta^{-2}\gamma(Q))$ and

$$|f'(\infty)| \leq (1 + A_2\delta^{-2}\gamma(Q))\gamma(P) + A_2\delta^{-1}\gamma(Q) \leq \gamma(P) + A\delta^{-2}\gamma(Q)$$

which proves the lemma.

Lemma 5.4. *Let Q_0 be the closed unit square, n a positive integer, and divide Q_0 into n^2 equal squares $\{Q_{ij}\}_{i,j=1}^n$ of side $1/n$. Let K be a subset of Q_0° with $\gamma(K) \leq 1/n$. Let K_{ij} be the image of K under the natural contraction and translation map of Q_0 onto Q_{ij}. Let $K^* = \bigcup_{i,j=1}^n K_{ij}$. Then $\gamma(K^*) \geq \theta n\gamma(K)$ where $\theta > 0$ is an absolute constant.*

Proof. Let $f_{ij} \in B(K_{ij}, 1)$ with $f_{ij}'(\infty) = \gamma(K_{ij}) = n^{-1}\gamma(K)$. Let $f = \Sigma_{i,j=1}^n f_{ij}$, so $f'(\infty) = n\gamma(K)$ and f is analytic off K^*. Fix $z \in Q_0$; then for at most 9 pairs (i, j) we have $d(z, K_{ij}) < 1/n$, and for these $|f_{ij}(z)| \leq 1$. For each $r \geq 2$, at most $(2r+1)^2$ pairs (i, j) satisfy $d(z, K_{ij}) < r/n$; if $d(z, K_{ij}) \geq r/n$, $r \geq 1$, then $|f_{ij}(z)| \leq r^{-1}\gamma(K)$. Thus we obtain

$$|f(z)| \leq \sum_{i,j} |f_{ij}(z)| \leq 9 + \sum_{r=2}^{n} \frac{(2r+1)^2 - (2r-1)^2}{r-1}\gamma(K) \leq 25$$

since $\gamma(K) \leq 1/n$. So $\gamma(K^*) \geq (n/25)\gamma(K)$.

Lemma 5.5. *Let Q be a finite union of nonoverlapping closed squares whose sides have rational length. Let $\epsilon, \alpha > 0$. Let $f \in C(Q, 1)$. Then we can find $\eta > 0$ and a subdivision of Q into squares $Q_1 \cdots Q_n$ of side η such that if we choose any compact set $K_i \subseteq Q_i^\circ$ with $\gamma(K_i) > \alpha\eta$ for $i = 1, \cdots, n$, then, putting $K = \bigcup_{i=1}^n K_i$, we can find $g \in C(K, 1)$ with $\|g - f\| < \epsilon$.*

Proof. Let $\delta > 0$ and construct $\{\phi_k\}$ and $\{\Delta_k\}$ as in §2. Choose $\eta < \delta$ such that Q can be divided into squares of side η. Reducing δ if necessary we may assume $\delta/2 < \eta < \delta$. Put $f_k = T_{\phi_k} f$. Then $f_k = 0$ if Δ_k does not meet Q; for those k such that Δ_k meets Q we choose i_k so that Q_{i_k} meets Δ_k. Then choose $h_k \in C(K_{i_k}, 2)$ such that $h_k'(\infty) = \alpha\eta$, and put

$$g_k = \frac{f_k'(\infty)h_k}{\alpha\eta} + \frac{1}{\alpha^2\eta^2}\left(f_k''(\infty) - \frac{f_k'(\infty)}{\alpha\eta}h_k''(\infty)\right)h_k^2.$$

Then g_k has the same first two Laurent coefficients as f_k and $\|g_k\| \leq A_1\alpha^{-2}\|f_k\|$. Moreover, g_k is analytic outside $\Delta(z_k, 3\delta)$.

Put $g = \Sigma_k g_k$, the sum being taken over those k for which Q meets Δ_k. Then g is continuous on $\mathbf{C}$, analytic outside K, and $\|f - g\| \leq A_2\alpha^{-2}\sup_k \|f_k\|$. Since f is continuous, by choosing δ small enough we can ensure $A_2\alpha^{-2}\sup_k \|f_k\| < \epsilon$. which proves the lemma.

Lemma 5.6. *If (A) is false then for any $\epsilon > 0$ we can find compact subsets*

E and F of Q_0 with $E \cap F = \emptyset$, $\gamma(E) < \epsilon$, $\gamma(F) < \epsilon$, and $\gamma(E \cup F) \geq \theta/2$. ($\theta$ as in Lemma 5.4.) We can assume E and F are unions of nonoverlapping squares with rational sides.

Proof. Let P and R be disjoint compact subsets of Q_0°. Choose n so that $n^{-1} \geq \gamma(P \cup R) \geq (n+1)^{-1}$. Divide Q_0 into n^2 equal squares and form P^* and R^* from P and R as we formed K^* from K in Lemma 5.4. Then $\gamma(P^* \cup R^*) \geq [n/(n+1)]\theta \geq \theta/2$ by Lemma 2.

By Vituškin's extension of Melnikov's theorem [14, Chapter 3, §1], $\gamma(P^*) \leq An\gamma(P)$ and $\gamma(R^*) \leq An\gamma(R)$ where A is an absolute constant. If (A) is false we could have chosen P and R so that $\gamma(P \cup R) > A\epsilon^{-1}(\gamma(P) + \gamma(R))$ so that $\gamma(P^*) \leq An\gamma(P) < \epsilon$ and $\gamma(R^*) < \epsilon$.

Let U and V be disjoint neighborhoods of P^* and Q^* in Q_0° with $\gamma(U) < \epsilon$ and $\gamma(V) < \epsilon$. Then choose E and F with $P^* \subseteq E \subseteq U$ and $Q^* \subseteq F \subseteq V$.

Proof of Theorem 5.2. Choose $f_0 \in C(Q_0, 1)$ with $f_0'(\infty) > \frac{1}{2}$. We construct by induction on n sequences $\{f_n\}_{n=0}^\infty$ of functions and $\{E_n\}$, $\{F_n\}_{n=1}^\infty$ of sets such that E_n and F_n are compact and disjoint; $F_n \subseteq F_{n+1}$; $E_1 \cup F_1 \subseteq Q_0^\circ$; $E_{n+1} \cup F_{n+1} \subseteq E_n^\circ$, $n \geq 1$; $\gamma(E_n) < 2^{-n}$, $\gamma(F_n) < \epsilon$; $f_n \in C(E_n \cup F_n, 1)$; $\|f_{n+1} - f_n\| < 2^{-n}$; $|f_n'(\infty)| > \frac{1}{2}$.

Induction step (initial step is similar). Assume E_k, F_k, f_k have been constructed for $k \leq n$, satisfying the above conditions. Apply Lemma 5.5 with $Q = E_n \cup F_n$, $f = f_n$, $\alpha = \theta/2$, and $\epsilon = \min(2^{-n}, |f_n'(\infty)| - \frac{1}{2})$. Let N be the number of squares of side η into which E_n is thereby divided. By Lemma 4 we can find disjoint compact subsets $\tilde{E}$ and $\tilde{F}$ of Q_0°, each a union of nonoverlapping squares with rational sides, such that $\gamma(\tilde{E}) < 1/(2^{n+1}A_1N\eta)$,

$$\gamma(\tilde{F}) < (\epsilon - \gamma(F_n))d(E_n, F_n)^2/(A_1AN\eta),$$

and $\gamma(\tilde{E} \cup \tilde{F}) \geq \theta/2$ where A is the constant of Lemma 1 and A_1 is the constant in the Melnikov estimate for a square. We map Q_0 onto each square of the subdivision of E_n in the natural way and denote the unions of the images of $\tilde{E}$ and $\tilde{F}$ respectively by E^* and F^*. Define $E_{n+1} = E^*$ and $F_{n+1} = F_n \cup F^*$. Then $\gamma(E_{n+1}) < 2^{n-1}$ and by Lemma 5.3 $\gamma(F_{n+1}) < \epsilon$. By the conclusion of Lemma 5.5 we can find $f_{n+1} \in C(E_{n+1} \cup F_{n+1}, 1)$ with $\|f_{n+1} - f_n\| < \min(2^{-n}, |f_n'(\infty)| - \frac{1}{2})$ which implies $|f_{n+1}'(\infty)| > \frac{1}{2}$. This completes the induction.

Now set $E = \bigcap_n E_n$ and $F = \bigcup_n F_n$, so that $E \cup F = \bigcap_n (E_n \cup F_n)$. The sequence f_n converges uniformly to a limit $f \in C(E \cup F, 1)$ satisfying $|f'(\infty)| \geq \frac{1}{2}$, whence $\alpha(E \cup F) \geq \frac{1}{2}$. Since $\gamma(E_n) \to 0$ we have $\gamma(E) = 0$. Finally, since any compact subset of F is contained in F_n for some n, we have $\gamma(F) \leq \sup_n \gamma(F_n) \leq \epsilon$. The theorem is proved.

Theorem 5.7. *Suppose (A) is false. Then we can find sets E and F with*

$\gamma^*(E) = 0$, $\gamma^*(F) = 0$, *but* $\alpha(E \cup F) > 0$.

Proof. We construct by induction on n sets E_n, F_n and functions f_n satisfying: E_n is compact; F_n is a countable union of disjoint compact sets, each a finite union of nonoverlapping squares, clustering only on E_n; $E_n \subseteq E_{n+1} \subseteq E_n \cup F_n^\circ$; $F_{n+1} \subseteq F_n$; $\gamma(E_n) = 0$; $\gamma^*(E_n) < 2^{-n}$; $f_n \in C(E_n \cup F_n, 1)$; $|f_n'(\infty)| > 1/4$; $\|f_{n+1} - f_n\| < 2^{-n-1}$.

We obtain E_1 and F_1 by Theorem 5.2.

Induction step. Suppose f_n, E_n, F_n constructed. Let V be a neighborhood of E_n so that $\gamma(V) < 2^{-n-1}$ and $\partial V \cap F_n = \emptyset$. By the method of proof of Theorem 5.2 we can find disjoint subsets P and Q of $F_n^\circ \setminus \overline{V}$ such that P is compact, Q is a countable union of compact sets, each a finite union of nonoverlapping squares, clustering only on P, such that $\gamma(P) = 0$, $\gamma^*(Q)$ is small, and there exists $f_{n+1} \in C(P \cup Q \cup (F_n \cap V) \cup E_n, 1)$ with $|f_n'(\infty)| > 1/4$ and $\|f_{n+1} - f_n\| < 2^{-n-1}$. Put $E_{n+1} = E_n \cup P$; $F_{n+1} = (F_n \cap V) \cup Q$. If $\gamma^*(Q)$ is small enough then $\gamma^*(F_{n+1}) < 2^{-n-1}$ by Lemma 5.3, so the induction is complete.

Now set $E = \bigcup_n E_n$ and $F = \bigcap_n F_n$. Clearly $\gamma^*(F) = 0$. Let $f = \lim f_n \in C(E \cup F, 1)$. (Note that $E \cup F = \bigcap_n (E_n \cup F_n)$ is compact.) Then $|f'(\infty)| \geq 1/4$ so $\alpha(E \cup F) < 1/4$. It remains to show $\gamma^*(E) = 0$.

Let $\epsilon > 0$. We construct by induction open sets $U_1, U_2, \cdots$ such that $E_n \subseteq U_n$, $\gamma(U_n) < \epsilon$, $U_n \subseteq U_{n+1}$, and $\partial U_n \cap F_n = \emptyset$. For the induction step, suppose U_n has been constructed. Since $\partial U_n \cap F_n = \emptyset$, $d(U_n, E_{n+1} \setminus U_n) > 0$. Applying Lemma 5.3 we can then find a neighborhood V of $E_{n+1} \setminus U_n$ with $V \cap U_0 = \emptyset$ and $\gamma(V \cup U_n) < \epsilon$. Shrinking V if necessary we may assume $\partial V \cap F_{n+1} = \emptyset$. Then $U_{n+1} = V \cup U_n$ satisfies the requirements and the induction is complete.

Put $U = \bigcup_n U_n$, then $E \subseteq U$ and since each compact subset of U is contained in some U_n, $\gamma(U) \leq \epsilon$. Thus $\gamma^*(E) \leq \epsilon$ and since ϵ is arbitrary, $\gamma^*(E) = 0$.

Theorem 5.8. *Suppose* (A) *is true. Then there is an absolute constant* M *such that* $\gamma(E \cup F) \leq M(\gamma(E) + \gamma(F))$ *whenever* E *is a compact set and* F *any set.*

Proof. We may assume that $E \cup F$ is compact. Let $\epsilon > 0$. We can find a neighborhood U of E, bounded by finitely many smooth curves, with $\gamma(U) < \gamma(E) + \epsilon$. Let $f \in B(E \cup F, 1)$. We can find $\delta > 0$ such that for each $z \in \partial U$, $\Delta(z, \delta) \cap U$ contains a compact connected set with diameter $> \delta/2$, whence $\gamma(\Delta(z, \delta) \cap U) > \delta/8$. Hence for all $z \in \mathbf{C}$, $\gamma(\Delta(z, \delta) \cap (E \cup F)) \leq 8\gamma(\Delta(z, 2\delta) \cap ((U \cup F) \setminus \partial U))$. We apply Corollary 2.2 with A_1 being the functions analytic on $\mathbf{C} \setminus (E \cup F)$ and A_2 the functions analytic off a compact subset of $U \cup (F \setminus \partial U)$, and find g analytic off a compact set $K \subseteq U \cup (F \setminus \partial U)$ with $g'(\infty) = f'(\infty)$ and $\|g\| \leq A$, an absolute constant.

Thus

$$|f'(\infty)| \leq A\gamma(K) \leq AM_1(\gamma(K \cap U) + \gamma(K \cap (F \setminus U))) \leq AM_1(\gamma(U) + \gamma(F))$$

where M_1 is the constant of (A). Thus $\gamma(E \cup F) \leq AM_1(\gamma(E) + \gamma(F) + \epsilon)$; letting $\epsilon \to 0$ yields the theorem.

Finally we relate the semiadditivity problem for γ to that for α.

Theorem 5.9. (A) *is equivalent to the following*: $(\mathrm{A})_\alpha$ *There exists an absolute constant* $M > 0$ *such that* $\alpha(E \cup F) \leq M(\alpha(E) + \alpha(F))$ *whenever* E *and* F *are disjoint compact sets.*

We first prove a lemma. Recall the definition of β_γ at the beginning of this section; β_α is defined analogously.

Lemma 5.10. *Let* Δ *be a disc of radius* r. *Let* λ *and* μ *be positive numbers with* $\lambda \leq r$ *and* $\lambda^2 \leq \mu \leq \lambda r$. *Then we can find compact subsets* S *and* T *of* Δ *such that*

$$A^{-1} \leq \lambda^{-1}\gamma(S) \leq A, \qquad A^{-1} \leq \mu^{-1}\beta_\gamma(S) \leq A, \qquad \alpha(S) = 0;$$

$$A^{-1} \leq \lambda^{-1}\alpha(T) \leq A, \qquad A^{-1} \leq \mu^{-1}\beta_\alpha(T) \leq A,$$

where $A > 0$ *is an absolute constant.*

Proof. For S we can take two parallel line segments of length λ separated by a distance μ/λ. For T we can take a small compact neighborhood of S.

Proof of theorem. First assume (A). Let E and F be disjoint compact sets and let $\delta < \frac{1}{2}d(E, F)$. With this δ we construct the approximation scheme of §2, in such a way that $\Delta(z_k, \delta/5)$ does not meet $\Delta(z_i, \delta) = \Delta_i$ for $i \neq k$. For each k such that Δ_k meets E choose compact $E_k^* \subseteq \Delta(z_k, \delta/5)$ such that

$$A_1^{-1}\alpha(E \cap \Delta_k) \leq \gamma(E_k^*) \leq A_1\alpha(E \cap \Delta_k)$$

and $A_1^{-1}\beta_\alpha(E \cap \Delta_k) \leq \beta_\gamma(E_k^*) \leq A_1\beta_\alpha(E \cap \Delta_k)$ using the lemma. Let $E^* = \bigcup_k E_k^*$ and define F^* similarly. Then $E^* \cap \Delta_k = E_k^*$ and $F^* \cap \Delta_k = F_k^*$.

Given $f \in C(E, 1)$ we form $f_k = T_{\phi_k} f$ and using the above estimates choose $g_k \in B(E_k^*, A_3)$ with the same first two Laurent coefficients as f. Then $g = \Sigma g_k \in B(E^*, A_4)$ by the usual estimates, and since $g'(\infty) = f'(\infty)$ we obtain $\alpha(E) \leq A_4\gamma(E^*)$. The argument works in reverse, starting with $g \in B(E^*, 1)$ we can find $f \in C(E, A_4)$ with $f'(\infty) = g'(\infty)$. Hence

$$A_4^{-1}\alpha(E) \leq \gamma(E^*) \leq A_4\alpha(E)$$

and similar estimates hold for F and $E \cup F$. Together with (A) this yields $(\mathrm{A})_\alpha$.

A completely analogous argument with α and γ interchanged shows that $(\mathrm{A})_\alpha \Rightarrow (\mathrm{A})$.

We can summarize the results of this section as follows.

Theorem 5.11. *The following conjectures are equivalent.*

(1) *If E and F are any sets with $\gamma^*(E) = 0$ and $\gamma^*(F) = 0$ then $\alpha(E \cup F) = 0$.*

(2) *There is a constant M such that if E is compact and $\gamma(E) = 0$ then $\alpha(E \cup F) \leq M\gamma(F)$ for all F.*

(3) *If E is compact and $\gamma(E) = 0$ then $\gamma(E \cup F) = \gamma(F)$ for all sets F.*

(4) *If E is compact and $\gamma(E) = 0$ then E is γ-negligible.*

(5) *There is an absolute constant M with $\gamma(E \cup F) \leq M(\gamma(E) + \gamma(F))$ whenever E and F are compact and disjoint.*

(6) *There is an absolute constant M with $\gamma(E \cup F) \leq M(\gamma(E) + \gamma^*(F))$ for all sets E and F.*

(7) *There is an absolute constant M with $\alpha(E \cup F) \leq M(\alpha(E) + \alpha(F))$ whenever E and F are compact and disjoint.*

Proof. The only step which has not been already covered is (2) $\Rightarrow$ (3). To prove (3) we may assume that $E \cup F$ is compact. Let A_1 be the set of functions in A_0 analytic off $E \cup F$, and A_2 those analytic off E. Then (2) implies $\gamma_{A_1}(S) \leq M\gamma_{A_2}(S)$ for all sets S, so that by Corollary 2.2 for any $f \in A_1$ we can find a sequence $\{f_n\}$ in A_2 with $\|f_n\| \leq M_1 \|f\|$, $f_n \to f$ pointwise on $\mathbf{C}\setminus(E \cup F)$. Then by [9, Theorem 6.7], it follows that one can actually find such a sequence with $\|f_n\| \leq \|f\|$, which yields (3).

6. **A special case—Dirichlet algebras.** An important special case of the problem of bounded approximation by $A(U)$ or $R(K)$ occurs when each component of U or K° is simply connected. In this case it is known that $A(U)$ is boundedly pointwise dense in $H^\infty(U)$ if and only if $A(U)$ is a Dirichlet algebra on ∂U (i.e. every continuous real-valued function on ∂U is a uniform limit on ∂U of real parts of functions in $A(U)$) and a necessary and sufficient condition can be given in terms of the boundary values of conformal maps of the disc onto the components of U (see [9], [2], [18]). Likewise $R(K)$ is a Dirichlet algebra on ∂K if and only if $R(\partial K) = C(\partial K)$, $\mathbf{C}\setminus K^\circ$ is connected, and $R(K)$ is boundedly pointwise dense in $H^\infty(K^\circ)$.

In this case the capacity conditions assume a special form since we know that if U has connected complement and $z \in \partial U$ then $\Delta(z, \delta)\setminus U$ contains a compact set of diameter $\geq \delta$, if δ is small enough, and hence $\gamma(\Delta(z, \delta)\setminus U)$ is comparable with δ. The aim of this section is to establish that sets with zero ½-dimensional Hausdorff measure satisfy a negligibility condition which is applicable to these capacity conditions, and to deduce criteria for $A(U)$ or $R(K)$ to be Dirichlet algebras. The proof of the main result (Theorem 6.1) is rather circuitous and it seems reasonable to expect that a direct proof can be found. It also seems likely that the dimension ½ could be improved, possibly by replacing ½ by 1. This question is discussed at the end of the section and is also related to §7.

Theorem 6.1. *There exist positive absolute constants ϵ_0 and δ_0 such that if K is a compact connected set with diameter 1 and $\{\Delta_i\}$ is a sequence of open sets with radii $\{r_i\}$ such that $\Sigma_i r_i^{1/2} < \epsilon_0$, then $\gamma(K \setminus \bigcup_{i=1}^{\infty} \Delta_i) \geq \delta_0$.*

Proof. We split the proof up into 3 steps.

Step 1. Assuming the theorem false we construct an arc J of diameter 1, a compact subset F of J with zero $\frac{1}{2}$-dimensional Hausdorff measure, and a bounded open set $V \subseteq \mathbf{C}$ such that $V \cap J = J \setminus F$, $\mathbf{C} \setminus V$ is connected, each component of V meets J, each point of $\mathbf{C} \setminus J$ has a neighborhood meeting at most one component of V, and $\gamma(V) < \frac{1}{4}$.

Step 2. Assuming the conclusion of Step 1 and writing $K = S^2 \setminus V$ we show that each point y of K° has a unique representing measure on ∂K for $R(K)$ (i.e. a positive measure μ on ∂K such that $f(y) = \int f\, d\mu$, $f \in R(K)$). The proof goes by analyzing the logarithmic potential of μ.

Step 3. Via the theory of abstract Hardy spaces, one deduces from Step 2 that $R(K)$ is boundedly pointwise dense in $H^\infty(K^\circ)$, which contradicts $\gamma(V) < \frac{1}{4}$.

We now give the details of each step.

Step 1. Assume the theorem is false. Let $\epsilon, \delta > 0$ be given. Then we can find a compact connected set K with $\operatorname{diam}(K) = 1$, and discs $\Delta_1, \cdots, \Delta_n$ with radii $r_1, \cdots, r_n$ such that $\Sigma_{r=1}^n r_i^{1/2} < \epsilon$ but $\gamma(K \setminus \bigcup_{i=1}^n \Delta_i) < \delta$. Let V be an open neighborhood of $K \setminus \bigcup_{i=1}^n \Delta_i$ with $\gamma(V) < \delta$. Let $W = V \cup (\bigcup_{i=1}^n \Delta_i)$. Since W is an open set containing K we can find a polygonal arc J of diameter 1 contained in W, and we may assume that $J \cap \Delta_i$ is an arc J_i of length $\leq 2r_i$ for each i. Since $J \setminus \bigcup_i J_i \subseteq V$ we have $\gamma(J \setminus \bigcup_i J_i) < \delta$. Let z_1 and z_2 be two points of J with $|z_1 - z_2| = 1$ such that any two points of the subarc with endpoints z_1 and z_2 have distance ≤ 1. Replacing J by this subarc we may assume also that J lies in the set

$$Y(z_1, z_2) = \{\zeta\colon \max(|\zeta - z_1|, |\zeta - z_2|) \leq |z_1 - z_2|\}.$$

Now let $\delta > 0$ be fixed. Using the above we construct by induction on n a sequence $\{J^n\}$ of polygonal arcs with $\operatorname{diam}(J^n) = 1$, and for each n a set $\{J_1^n \cdots J_{r_n}^n\}$ of open subarcs of J^n with disjoint closures, open neighborhoods W_i^n of J_i^n for $i = 1, \cdots, r_n$, and an open neighborhood V_n of the interior of I^n, where $I^n = J^n \setminus \bigcup_{i=1}^{r_n} J_i^n$, satisfying the following conditions:

(1) $I^n \subseteq I^{n+1}$;

(2) $\overline{W_i^n} \cap \overline{W_j^n} = \emptyset$, $i \neq j$;

(3) $W_j^{n+1} \subseteq \bigcup_{i=1}^{r_n} W_i^n$, $j = 1, \cdots, r_{n+1}$;

(4) $\Sigma_{i=1}^{r_n} (\operatorname{diam}(W_i^n))^{1/2} < 2^{-n}$;

(5) $V_n \subseteq V_{n+1}$;

(6) $\gamma(V_n) \leq (1 - 2^{-n})\delta$;

(7) $\overline{V}_n \cap J^n = I^n$;

(8) each component of V_n meets J^n;

(9) $\mathbf{C} \setminus V_n$ is connected.

Induction step. Suppose J_n, etc. have been constructed. Fix i with $1 \leq i \leq r_n$, and choose open arcs $\sigma_1, \cdots, \sigma_r \subseteq J_i^n$ with disjoint closures such that (a) each endpoint of J_i^n lies in $\bigcup_{j=1}^p \overline{\sigma}_j$, (b) each corner of J_i^n lies in $\bigcup_{j=1}^p \sigma_j$, and (c) $\Sigma_{j=1}^p |\sigma_j|^{1/2} < 2^{-n-2} r_n^{-1}$. Let $K_1, \cdots, K_r$ be the components of $J_i^n \setminus \bigcup_{j=1}^p \sigma_j$. Then each K_s is a closed line segment. For $s = 1, \cdots, r$ we can find disjoint closed intervals $[a_t^s, b_t^s]_{t=1}^{q_s}$ in order on K_s such that a_1^s and $b_{q_s}^s$ are the endpoints of K_s, $Y(a_t^s, b_t^s)$ lies in $W_i^{(n)}$ and meets no other K_s, $\Sigma_{t=2}^q |b_{t-1}^s - a_t^s|^{1/2} < 2^{-n-2}(rr_n)^{-1}$, and all sets $Y(a_t^s, b_t^s)$ are disjoint.

We now apply the first paragraph to each $[a_t^s, b_t^s]$ to find a polygonal arc $\tilde{J}_t^s$ with endpoints a_t^s and b_t^s, lying in $Y(a_t^s, b_t^s)$, and subarcs $\{\tilde{J}_{t,m}^s\}_{m=1}^{M_t^s}$ with $\Sigma_m |\tilde{J}_{t,m}^s|^{1/2} < 2^{-n-2}(rr_n)^{-1}$ and

$$\gamma\left(\tilde{J}_t^s \setminus \bigcup_m \tilde{J}_{t,m}^s\right) < \delta(\rho_t^s)^2/(rr_n q_s A 2^n)$$

where ρ_t^s is the distance from $[a_t^s, b_t^s]$ to the nearest similar interval, and A is the constant of Lemma 5.3.

We do this construction for each $i = 1, \cdots, r_n$. Then we define J^{n+1} by replacing $[a_t^s, b_t^s]$ by $\tilde{J}_t^s$ in J^n for all t and s, for each i. The arcs J_i^{n+1} are all the arcs $\sigma_1 \cdots \sigma_r$, all the components of sets $K_s \setminus \bigcup_{t=1}^{q_s} [a_t^s, b_t^s]$, and all interiors of arcs $J_{t,m}^s$ obtained in this construction. Then

$$I_{n+1} = I_n \cup \left(\bigcup_{s,t} \left(\overline{J_t^s \setminus \bigcup_m J_{t,m}^s}\right)\right).$$

We have $\Sigma_{i=1}^{r_{n+1}} (\operatorname{diam}(J_i^{n+1}))^{1/2} < 2^{-n-1}$ and using Lemma 5.3, $\gamma(V_n \cup (I_{n+1} \setminus I_n)) < (1 - 2^{-n})\delta$. Then it is clear that V_{n+1} and W_i^{n+1} can be chosen to satisfy (1)–(9). The induction is complete.

Now set $J = \bigcap_n (I^{(n)} \cup (\bigcup_{i=1}^{r_n} W_i^{(n)}))$, $F = \bigcap_n (\bigcup_{i=1}^n \overline{W_i^{(n)}})$, and $V = \bigcup_n V_n$. Then (1)–(4) imply that J is an arc. (4) implies that F has zero ½-dimensional measure and (6) that $\gamma(V) \leq \delta$. By (8) each component of V meets J, by (7) $J \cap V = J \setminus F$, and by (9) $\mathbf{C} \setminus V$ is connected, so Step 1 is complete.

Step 2. Let $K = S^2 \setminus V$; we have to show that each point of K° has unique representing measure for $R(K)$ on ∂K. We make a conformal transformation of the sphere which sends K into the disc $\{|z| < ½\}$; we use the same notation K, V, J

for the transformed sets. Let $g \in K^{\circ}$, and let μ be a representing measure on ∂K for g. Let

$$P_{\mu}(z) = \int \log \frac{1}{|z-\zeta|} d\mu(\zeta), \qquad z \in \mathbf{C};$$

P_{μ} is an extended real-valued superharmonic function on $\mathbf{C}$. Let $g(z) = P_{\mu}(z) - \log(1/|z-y|)$, $z \in \mathbf{C}\setminus\{y\}$. The fact that μ is a representing measure for y means that g is a constant on each component of V; we shall show that actually $g = 0$ on V.

Let $J_1, J_2, \cdots$ be the components of $J\setminus F$; each J_i is contained in a component of V, so g is a constant, say λ_i, on J_i. To prove $g = 0$ on V it suffices to show each $\lambda_i = 0$, since each component of V meets J. We fix two components J_1, J_2. Before completing the proof we prove two lemmas.

Lemma A. *Suppose* $0 < \beta < \alpha < 1$, $\epsilon = (\alpha\beta)^{1/2}/(\alpha^{1/2} - \beta^{1/2})$, *and* $0 < x < 1$. *Let*

$$I = \int_{\beta}^{\alpha} \frac{\epsilon}{2r^{3/2}} \log \frac{1}{|x-r|} dr.$$

Then $I < \log x^{-1}(1 + \epsilon\alpha^{-1/2}) + 2\epsilon\alpha^{-1/2}$.

Proof. By direct calculation (assuming $x \neq \alpha$ or β)

$$I = \log\frac{1}{x}(1 + \epsilon\alpha^{-1/2}) + \epsilon\left(\beta^{-1/2}\log\frac{x}{|x-\beta|} - x^{1/2}\log\left|\frac{x^{1/2}+\beta^{1/2}}{x^{1/2}-\beta^{1/2}}\right|\right)$$

$$+ \epsilon\left(\alpha^{-1/2}\log|x-\alpha| + x^{-1/2}\log\left|\frac{x^{1/2}+\alpha^{1/2}}{x^{1/2}-\alpha^{1/2}}\right|\right)$$

$$= \log\frac{1}{x}(1 + \epsilon\alpha^{-1/2}) + T_1 + T_2, \text{ say.}$$

Expanding T_1 as a power series in $(\beta/x)^{1/2}$ or $(x/\beta)^{1/2}$ according as $x > \beta$ or $x < \beta$ we find $T_1 < 0$. Similarly we find $T_2 < 2\epsilon\alpha^{-1/2}$ which proves the lemma.

Now put $\rho = d(y, \partial K)$ and fix $\alpha < \min(\rho/4, \frac{1}{2}\operatorname{diam} J_2)$.

Lemma B. *There is a positive constant* A *such that if* $0 < \beta < \alpha$ *and* $\epsilon = (\alpha\beta)^{1/2}/(\alpha^{1/2} - \beta^{1/2})$, *and* $z \in \mathbf{C}$ *with* $d(z, \partial K) < \rho/4$, *then we can find* r, *with* $\beta < r < \alpha$, *such that for all* ζ *with* $|\zeta - z| = r$, *we have*

$$g(\zeta) < (1 + \epsilon\alpha^{-1/2})g(z) + A\epsilon.$$

Proof. Suppose that for all r with $\beta < r < \alpha$ we can find ζ with $|\zeta - z| = r$ and $g(\zeta) \geq (1 + \epsilon\alpha^{-1/2})g(z) + A\epsilon$. For such ζ we have

$$P_\mu(\zeta) \geq (1+\epsilon\alpha^{-1/2})P_\mu(z) - \epsilon\alpha^{1/2}\log\frac{1}{|z-y|} + A\epsilon - \left|\log\frac{1}{|z-y|} - \log\frac{1}{|\zeta-y|}\right|$$
$$\geq (1+\epsilon\alpha^{-1/2})P_\mu(z) - \epsilon\alpha^{1/2}\log\rho^{-1} + A\epsilon - 2r/\rho.$$

Hence

$$\int \log\frac{1}{|r-|w-z||}\,d\mu(w) \geq \int \log\frac{1}{|\zeta-w|}\,d\mu(w)$$
$$\geq (1+\epsilon\alpha^{-1/2})P_\mu(z) - \epsilon\alpha^{-1/2}\log\rho^{-1} + A\epsilon - 2r/\rho,$$

for $\beta < r < \alpha$, which implies

$$\int_\alpha^\beta \int \frac{\epsilon}{2r^{3/2}}\log\frac{1}{|r-|w||}\,d\mu(w)\,dr$$
$$\geq (1+\epsilon\alpha^{-1/2})P_\mu(z) + \epsilon(A - \alpha^{-1/2}\log\rho^{-1}) - \int_\beta^\alpha \epsilon\rho^{-1}\alpha^{-1/2}\,dr.$$

By Lemma A the left side is $\geq (1+\epsilon\alpha^{-1/2})P_\mu(z) + 2\epsilon\alpha^{-1/2}$ whence $2\epsilon\alpha^{-1/2} \geq \epsilon(A - \alpha^{-1/2}\log\rho^{-1}) - 2\epsilon\rho^{-1}(\alpha^{1/2} - \beta^{1/2})$ which is a contradiction if

$$A > \alpha^{1/2}(2+\log\rho^{-1}) + 2\alpha^{1/2}/\rho.$$

Lemma B is proved.

Now we complete Step 2. Let $\eta > 0$, and cover F by discs $\Delta_1, \cdots, \Delta_n$ where $\Delta_i = \Delta(z_i, \delta_i)$, such that $\Sigma_{i=1}^n (2\delta_i)^{1/2} < \alpha^{1/2}\eta$. Choose $x_i \in \bar{\Delta}_i$ so that $g(x_i) = \inf_{\bar{\Delta}_i} g$ (this is possible since g is lower semicontinuous). Let $\beta_i = 2\delta_i$ and $\tilde{\Delta}_i = \Delta(x_i, \beta_i) \supseteq \Delta_i$. By Lemma B we can find r_i with $\beta_i < r_i < \alpha$ so that

$$(*) \qquad g(\zeta) < (1+\alpha^{-1/2}\eta_i)g(x_i) + A\eta_i$$

for $\zeta \in \Gamma_i = \{\zeta : |\zeta - x_i| = r_i\}$, where $\eta_i = \alpha^{1/2}\beta_i^{1/2}/(\alpha^{1/2} - \beta_i^{1/2})$ (so that $\Sigma\eta_i < 2\eta$).

Fix a direction on J so that J_1 precedes J_2. We construct inductively a finite sequence $w_1, \cdots, w_k$ of points in order on J as follows: let ζ_1 be the last endpoint of J_1; then $\zeta_1 \in \Delta_{i_1}$, for some i_1. Let w_1 be the last point of $\Gamma_{i_1} \cap J$.

For the inductive step, suppose w_m found, $m \geq 1$. If $w_m \in J_2$ the construction terminates, and $k = m$. If $w_m \in F$ then $w_m \in \Delta_{i_{m+1}}$ for some i_{m+1} and then we take w_{m+1} to be the last point in $J \cap \Gamma_{i_{m+1}}$. Finally, if $w_m \in J_i$ for some $i \neq z$ let ζ' be the last endpoint of J_i, so $\zeta' \in \Delta_{i_{m+1}}$ for some i_{m+1}, and then take w_{m+1} to be the last point in $J \cap \Gamma_{i_{m+1}}$.

Since $r_i < \alpha < \frac{1}{2}\operatorname{diam}(J_2)$, for some k we have $w_k \in J_2$, so the construction terminates. Moreover, $i_l \neq i_m$ for $m \neq l$, $1 \leq m, l \leq k$. Recalling that $g = \lambda_i$ on J_i we have $g(x_{i_{m+1}}) \leq g(w_m)$, so that by $(*)$

$$g(w_{m+1}) \leq (1 + \alpha^{-1/2}\eta_{i_{m+1}})g(w_m) + A\eta_{i_{m+1}}$$

and $g(w_1) \leq (1 + \alpha^{-1/2}\eta_{i_1})\lambda_1 + A\eta_{i_1}$. Recalling that $\Sigma\eta_i < 2\eta$, it follows that if η were chosen small enough,

$$\lambda_2 = g(w_k) < \lambda_1(1 + 3\alpha^{-1/2}\eta) + 3A\eta.$$

Letting $\eta \to 0$ we obtain $\lambda_2 \leq \lambda_1$; similarly, $\lambda_1 \leq \lambda_2$, so $\lambda_1 = \lambda_2$, whence all the λ_j are equal. Thus g is constant on V and since g tends to zero at infinity we must have $g = 0$ on V.

It now follows as in [3, p. 377, last paragraph of Lemma 5.6], that if λ is any other representing measure then $P_\mu = P_\lambda$, whence $\mu = \lambda$. This completes Step 2.

Step 3. Let $y \in K^\circ$ and let λ be harmonic measure for y on ∂K which we now know is the unique representing measure for y. If P is the Gleason part of $R(K)$ containing y then $P \supseteq K^\circ$; moreover, by Theorem VI.7.2 of [7] there is $F \in H^\infty(\lambda)$ (the weak-star closure of $R(K)$ in $L^\infty(\lambda)$) such that $\|F\| = 1$ and F maps P (1-1) onto the unit disc, and F^{-1} is analytic. Hence P must be open, so that $P = K^\circ$ and F maps K° conformally onto Δ. Composing with a conformal mapping of the disc we may assume $F(\infty) = 0$; since $\operatorname{diam}(\mathbf{C}\setminus K^\circ) = 1$ we have $|F'(\infty)| \geq 1/4$. By Theorem VI.5.2 of [7] there is a sequence $\{f_n\}$ in $R(K)$ with $\|f_n\| \leq 1$ and $f_n \to F$ pointwise on K° so that $f_n'(\infty) \to F'(\infty)$. We may assume f_n is analytic outside a compact subset of V, whence $\gamma(V) \geq 1/4$. This is a contradiction if δ is chosen less than $1/4$, and the proof of Theorem 6.1 is complete.

The following corollary is immediate.

Corollary 6.2. *Let K be a compact connected set, and E a subset of K with zero $1/2$-dimensional outer Hausdorff measure. Then $\gamma(K\setminus E) \geq \delta_0\gamma(K)$.*

Combining this with the results of §4 we obtain the following result.

Theorem 6.3. *Let U be a bounded open plane set with connected complement. Let E be a subset of ∂U with zero $1/2$-dimensional outer measure and $\{E_n\}$ a sequence of subsets of ∂U each of which lies on a hypo-Liapunov arc and has zero length. Suppose that for each $z \in \partial U\setminus(E \cup (\bigcup_{n=1}^\infty E_n))$, we can find m, r, $\delta_0 > 0$ such that*

$$\gamma(\Delta(z, \delta)\setminus U) \leq m\alpha(\Delta(z, r\delta)\setminus U), \qquad 0 < \delta < \delta_0.$$

Then $A(U)$ is a Dirichlet algebra on ∂U.

Proof. Let $f \in H^\infty(U)$ with $\|f\| = 1$. First we apply Corollary 2.2 with $A_1 = \{f \in A_0 : f|U \in H^\infty(U)\}$ and $A_2 = \{f \in A_1 : f$ is analytic on a neighborhood of $E\}$. The hypothesis is satisfied in view of Corollary 6.2. Then we can find $\{f_n\}$ in A_2 with $\|f_n\| \leq M$ and $f_n \to f$ pointwise on U. By Theorem 4.1 each f_n is the pointwise limit on U of a sequence in $A(U)$ bounded by M; consequently, $A(U)$

is pointwise boundedly dense in $H^\infty(U)$. It then follows from [2, §2], that $A(U)$ is a Dirichlet algebra on ∂U.

In the same way one can prove the analogous result for $R(K)$.

Theorem 6.4. *Let K be a compact subset of $\mathbf{C}$ such that $\mathbf{C}\setminus K^\circ$ is connected, and suppose that for each $z \in \partial K\setminus(E \cup (\bigcup_n E_n))$, where E and E_n are as in Theorem 6.3, we can find $m, r, \delta_0 > 0$ such that*

$$\alpha(\Delta(z, \delta)\setminus K^0) \leq m\alpha(\Delta(z, r\delta)\setminus K), \qquad 0 < \delta < \delta_0.$$

Then $R(K)$ is a Dirichlet algebra on ∂K.

The last hypothesis is satisfied if each $z \in \partial K\setminus(E \cup (\bigcup_n E_n))$ lies on the boundary of some component of $\mathbf{C}\setminus K$. In this case, if moreover $E_n = \emptyset$, it is possible to prove the theorem without using Vituškin's techniques, by refining the arguments of Steps 2 and 3 in the proof of Theorem 6.1. If E is compact the construction in §5 of [3] suffices, with Lemma 5.2 replaced by our Lemma A. In the general case the proof is complicated by the fact that one needs an infinite covering of discs.

It is natural to ask whether Corollary 6.2 holds if E is assumed merely to have zero one-dimensional outer measure—this would of course follow if any of the equivalent conjectures of §5 hold. The proof of Step 1 in Theorem 6.1 with obvious modifications shows that this is equivalent to each of the following two conjectures:

(1) One can find ϵ and $\delta > 0$ such that if K is a compact connected set with diameter 1, and $\Delta_1, \cdots, \Delta_n$ are open discs the sum of whose radii is less than ϵ, then

$$\gamma\left(K\setminus\left(\bigcup_i \Delta_i\right)\right) > \delta.$$

(2) One can find $\delta > 0$ so that if J is an arc with diameter 1, and E is a compact subset of J with zero one-dimensional measure, then $\gamma(J\setminus E) > \delta$.

A slightly weaker conjecture than the above is Denjoy's conjecture, which we consider next.

7. **Denjoy's conjecture.** Denjoy's conjecture is concerned with the connection between one-dimensional measure and analytic capacity. It is elementary that sets of zero one-dimensional measure have zero analytic capacity; on the other hand Vituškin [16] has given an example of a compact set E with positive one-dimensional measure but with $\gamma(E) = 0$ (a simpler example has been given by Garnett [5]). The problem, raised by Denjoy [4], is: suppose E lies on a rectifiable arc and has positive length. Must then $\gamma(E) > 0$? The best positive result [11] (see also [6]) assumes that J satisfies a smoothness condition some-

what stronger than continuous differentiability. We show that to prove the conjecture it suffices to prove it for continuously differentiable arcs, and we give an equivalent formulation in terms of rational approximation.

Before stating the result we need a definition. If $\epsilon > 0$ we say a set S is *ϵ-angled* if whenever $z_1 \neq z_2$ and $w_1 \neq w_2$ are points of S the angle between the straight line through w_1 and w_2 and the straight line through z_1 and z_2 is less than ϵ.

Theorem 7.1. *The following four conjectures are equivalent.*

(1) *If J is a rectifiable arc and E is a compact subset of J with positive length, then $\gamma(E) > 0$.*

(2) *As in* (1) *except that J is continuously differentiable.*

(3) *One can find ϵ_0 and $\delta_0 > 0$ such that if J is an ϵ-angled arc of diameter* 1 *and $J_1, \cdots, J_n$ are subarcs with $\sum l(J_i) < \epsilon_0$, then $\gamma(J \setminus \bigcup_{i=1}^n J_i) > \delta_0$.*

(4) *Let J_1 and J_2 be continuously differentiable Jordan curves bounding the domains D_1 and D_2 respectively, such that $D_1 \cap D_2 = \emptyset$ and $J_1 \cap J_2$ has zero length. Let $K = \overline{D}_1 \cup \overline{D}_2$. Then $R(K) = A(K)$.*

Proof. We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (3) $\Leftrightarrow$ (4).

(4) $\Rightarrow$ (3). Suppose (3) is false. Imitating the construction in Step 1 of Theorem 6.1, except that we use continuously differentiable instead of polygonal arcs, and require that each arc J_i^n be 2^{-n}-angled, we can construct a continuously differentiable arc J with diameter 1 and a compact subset $E \subseteq J$ of zero length with $\gamma(V) < \delta$ for some open set $V \supseteq J \setminus E$, where δ is any preassigned positive number. We can find continuously differentiable simple closed curves J_1 and J_2 bounding disjoint domains D_1 and D_2 such that $J_1 \cap J_2 = J \cap J_1 = J \cap J_2 = E$ and such that each bounded complementary domain of $K = \overline{D}_1 \cup \overline{D}_2$ lies in V. Assuming (4) we have $R(K) = A(K)$, and since ∂K is connected it follows by [7, Chapter 8, Theorem 8.2] that $\gamma(\delta(z, \rho) \setminus K) > c\rho$, $z \in \partial K$, where c is an absolute constant. This is a contradiction if δ is small enough, so (4) $\Rightarrow$ (3).

(2) $\Rightarrow$ (3). Again suppose (3) is false. By a construction similar to that used in Theorem 6.1, Step 1, we construct inductively a sequence $\{J_n\}$ of arcs of form $J_n = \{(x, y): 0 \leq x \leq 1,\ y = f_n(x)\}$ where f_n is a real continuously differentiable function on $[0, 1]$, and for each n a partition $J_n = I_n \cup K_n$ where J_n is a finite union of disjoint closed arcs, K_n is a finite union of disjoint open arcs, and $I_n \cap K_n = \emptyset$, with the following properties: $\|f_n - f_{n+1}\| < 2^{-n}$; $\|f_n' - f_{n+1}'\| < 2^{-n}$; $K_n \subseteq K_{n+1}$; $l(I_n) > \frac{1}{2}$; there is an open neighborhood V_n of I_n with $\gamma(\overline{V}_n) < 2^{-n}$ and $V_n \supseteq I_r$ for $r \geq n$.

Then $f_n \to f$ and $J = \{(x, y): y = f(x)\}$ is a continuously differentiable arc. Put $E = \lim I_n \subseteq \bigcap \overline{V}_n$; then $l(E) \geq \frac{1}{2}$ but $\gamma(E) = 0$ which contradicts (2).

(1) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (1). Let J be a rectifiable arc and E a compact set of positive length on J. Let s denote arc length along J; parametrise J by $J = \{(\phi(s), \psi(s)), 0 \leq s \leq l\}$. Then ϕ and ψ are differentiable almost everywhere, and so by Egoroff's theorem one can find a subset $E_1 \subseteq E$ of positive length such that $(\phi(s+\delta) - \phi(s))/\delta \to \phi'(s)$ as $\delta \to 0$, uniformly for $(\phi(s), \psi(s)) \in E_1$, and similarly for ψ. Then one can find a compact ϵ_0-angled subset E_0 of E_1 with $l(E_0) > 0$. To each open subarc I of J whose endpoints are both in E we associate the line segment I' joining these two points; the union of E_0 with all such segments is an ϵ_0-angled arc J'. Let z be a density point for E_0 with respect to arc length on J'. Then if J'' is a sufficiently small subarc of J' containing z we have $l(J'' \setminus E_0)/l(J'') < \epsilon_0$, so that by (3) any neighborhood U of $J'' \cap E_0$ has $\gamma(U) \geq \delta_0 l(J'')$, whence $\gamma(E_0) \geq \delta_0 l(S'') > 0$ as required.

(3) $\Rightarrow$ (4). We can find a continuously differentiable arc J such that $J \cap K = J_1 \cap J_2$. Let $z \in J_1 \cap J_2$. We can choose $\rho_0 > 0$ so that $\Delta(z, \rho_0) \cap J$ is ϵ_0-angled. Then for $\rho < \rho_0$, (3) implies $\gamma(\Delta(z, \rho) \cap J) > \delta_0 \rho$. Hence for each $z \in \partial K$, $\liminf_{\rho \to 0} \rho^{-1} \gamma(\Delta(z, \rho) \cap J) > 0$ whence $R(K) = A(K)$ by Theorem 2.3.

8. **α as a set function and uniform approximation.** It is more difficult to handle α than γ because it does not obey any continuity condition from above. For this reason we have been unable to obtain analogues of the results of §5. We do show, however, that one version of semiadditivity of α is equivalent to a general conjecture on uniform approximation. We have already seen the relevance of semiadditivity of α when considering problems of uniform convergence when the given function is continuous everywhere; in this section we approximate functions which are continuous only on certain subsets.

Conjecture 8.1. *Let U be a bounded open subset of $\mathbf{C}$, E any bounded set. Let f be a bounded borel function on $\mathbf{C}$, analytic on U, continuous at each point of E. Let $\epsilon > 0$. Then we can find a bounded borel function g on $\mathbf{C}$, analytic on U, and continuous on a neighborhood of E, with $\|f - g\| < \epsilon$.*

Theorem 8.2. *The following are equivalent.*

(1) *There is an absolute constant M such that $\alpha(E \cup F) \leq M(\alpha(E) + \alpha(F))$ whenever E is compact (and F arbitrary).*

(2) 8.1 *holds whenever E is compact.*

Proof. (1) $\Rightarrow$ (2). For any compact set T define

$$\gamma_E(T) = \sup\{|f'(\infty)| : f \text{ is analytic on } \mathbf{C} \setminus T, \text{ continuous at each point of } E, \|f\| \leq 1\},$$

and

$$\tilde{\gamma}_E(T) = \sup\{|f'(\infty)|: f \text{ is analytic on } \mathbf{C}\setminus T, \text{ continuous on a neighborhood of } E, \|f\| \le 1\}.$$

By Lemma 2.1 and note (1) following it suffices to show that for all compact T, $\gamma_E(T) \le A\tilde{\gamma}_E(T)$, where A is an absolute constant. We may assume $E \subseteq T$.

Let Δ be an open disc containing T and set $V = \Delta\setminus E$. In V we construct the modified approximation scheme of §3 with $\rho = 1$. We denote by Q_k' the open square of center z_k and side $\delta_k/3$, so that $Q_k' \cap Q_r = \emptyset$ for $k \ne r$ (Proposition 3.1(3)). By Lemma 5.10 we can find for each k a compact set $T_k \subseteq Q_k'$ such that

$$A_1^{-1}\gamma(Q_k \cap T) \le \alpha(T_k) \le A_1\gamma(Q_k \cap T)$$

and

$$A_1^{-1}\beta_\gamma(Q_k \cap T) \le \beta_\alpha(T_k) \le A_1\beta_\gamma(Q_k \cap T)$$

where A_1 is an absolute constant. (Take $T_k = \emptyset$ if $\gamma(Q_k \cap T) = 0$.) Let $T^* = E \cup (\bigcup_k T_k)$, a compact set.

Now let f be analytic on $\mathbf{C}\setminus T$, continuous at each point of T, with $\|f\| \le 1$. Let $f_k = T_{\phi_k} f$ and choose $g_k \in C(T_k, A_2)$ with the same first two Laurent coefficients as f_k. Let $g = f + \Sigma_k (g_k - f_k)$; it follows easily from Proposition 3.1(6) that g is continuous at each point of E, hence $g \in C(T^*, A_3)$. Moreover, $g'(\infty) = f'(\infty)$. Now we invoke (1) with $F = \bigcup_k T_k$ and deduce that either we can find $\tilde{g} \in C(E, 2MA_3)$ or we can find $\tilde{g} \in C(K, 2MA_3)$ where $K \subseteq \bigcup_k T_k$ is compact, with $\tilde{g}'(\infty) = f'(\infty)$. In the former case $\tilde{g}$ is analytic off T so we obtain $|f'(\infty)| \le 2MA_3\tilde{\gamma}_E(T)$. In the latter case we consider each k for which $\tilde{g}_k = T_{\phi_k}\tilde{g}$ is nonzero (there are only finitely many) and observe that $\tilde{g}_k \in C(T_k, MA_4)$ (since $Q_k \cap (\bigcup_j T_j) = T_k$). Then we can find $\tilde{f}_k \in B(Q_k \cap T, MA_5)$ with the same first two Laurent coefficients as $\tilde{g}_k$.

Put $\tilde{f} = \tilde{g} + \Sigma_k (\tilde{f}_k - \tilde{g}_k)$; then $\|\tilde{f}\| \le MA_6$ and $\tilde{f}$ is analytic outside $T \cap$ (a finite union of Q_k's); in particular $\tilde{f}$ is analytic on E so $|f'(\infty)| \le MA_6\tilde{\gamma}_E(T)$, in either case.

It follows that $\gamma_E(T) \le MA_6\tilde{\gamma}_E(T)$, as required.

(2) $\Rightarrow$ (1). We may assume $E \cup F$ is compact. Let $f \in C(E \cup F, 1)$. We choose an open disc $\Delta \supset E \cup F$, set $V = \Delta\setminus E$, and do the modified approximation scheme in V. We choose compact sets $S_k \subseteq Q_k'$ by Lemma 5.10, so that

$$A_1^{-1}\alpha(Q_k \cap F) \le \gamma(S_k) \le A_1\alpha(Q_k \cap F),$$

with a similar inequality for $\beta_\gamma(S_k)$, and so that $\alpha(S_k) = 0$. Let $S = E \cup (\bigcup_k S_k)$. By the same arguments in (1) $\Rightarrow$ (2), we find $g \in B(S, A_2)$, continuous at each point of E, with $g'(\infty) = f'(\infty)$. Using (2) approximate g by $\tilde{g} \in$

$B(S, A_2)$ with $g'(\infty) = f'(\infty)$ and $\tilde{g}$ continuous on an open neighborhood W of E. Since $\alpha(S_k) = 0$ $\tilde{g}$ must be analytic on $W \setminus E$. Then we can find $\tilde{f} \in C(E \cup F_0, A_3)$ with $\tilde{f}'(\infty) = \tilde{g}'(\infty) = f'(\infty)$, where F_0 is a compact subset of $F \setminus E$. Hence it suffices to prove semiadditivity of α for disjoint compact sets.

In the case where $\alpha(E) = 0$, we have $\tilde{f} \in C(F_0, A_3)$, so that $\alpha(E \cup F) \leq A_3\alpha(F)$, i.e. under the assumption (2) we obtain condition (2) of Theorem 5.11. Then condition (7) of that theorem gives what we want.

Now we consider the general case of 8.1 and show that it suffices to have a pointwise capacity condition on T.

Theorem 8.3. *Let U and E be bounded sets with U open such that for each $z \in E$ we can find $m, r, \delta_0 > 0$ with*

$$\gamma_E(\Delta(z, \delta) \setminus U) \leq m\tilde{\gamma}_E(\Delta(z, r\delta) \setminus U), \qquad 0 < \delta < \delta_0.$$

Then U and E satisfy (8.1).

Proof. Let $\{\epsilon_n\}$ be a sequence of positive numbers tending to zero, to be specified below. Let V_n be an open neighborhood of E so that the oscillation of f at any point of V_n is $< \epsilon_n$; we assume $V_{n+1} \subseteq V_n$. Let

$$K_n = \{z \in V_n : \gamma_E(\Delta(z, \delta) \setminus U) \leq n\tilde{\gamma}_E(\Delta(z, n\delta) \setminus U),\ 0 < \delta < 1/n\}.$$

Then K_n is relatively closed in V_n, and $E \subseteq \bigcup_n K_n$.

We construct by induction on n a sequence of open sets $\{W_n\}$ such that $K_n \subseteq W_n \subseteq V_n$, and a sequence of functions $\{f_n\}$ which are bounded on $\mathbf{C}$, analytic on U, such that f_n is continuous on E and on a neighborhood of the intersection of E with a neighborhood of $\overline{W}_r \cap V_r$, for each $r \leq n$, and satisfies $\|f_n - f_{n-1}\| \leq A(n)\epsilon_n$, where $A(n)$ depends only on n.

Induction step. Suppose W_r and f_r have been constructed for $r = 1, \cdots, n$. For some neighborhood W of $\bigcup_{r=1}^n W_r$, f_n is continuous on a neighborhood of $W \cap E$. We construct the modified approximation scheme $\{Q_k, \phi_k\}$ of §3 in V_{n+1}, in such a way that $\operatorname{diam}(Q_k) < 1/2n$, the oscillation of f_n on Q_k is less than ϵ_n for each k, and any $\overline{Q}_k$ which meets $\bigcup_{i=1}^n \overline{W}_i$ is contained in W (this can be done by an appropriate choice of ρ). Put $f_{nk} = T\phi_k f_n$. For each k such that Q_k meets K_{n+1} but is not contained in W we choose g_{nk}, analytic off $\Delta(z_k, 2n\delta_k)$, continuous on a neighborhood of E, analytic on U, with the same first two Laurent coefficients as f_{nk}, and $\|g_{nk}\| \leq A_1(n)\|f_{nk}\| \leq \epsilon_n A_2(n)$. Put $f_{n+1} = f_n + \Sigma(g_{nk} - f_{nk})$, the sum being taken over all such k. By virtue of Proposition 3.1 (6) the series converges uniformly on compact subsets of V_{n+1} and pointwise on $\mathbf{C}$, and

$$|f_{n+1}(z) - f_n(z)| \leq \sum_k |g_{nk}(z) - f_{nk}(z)| \leq A(n)\epsilon_n, \qquad z \in \mathbf{C}.$$

Then each point of $\bigcup_{i=1}^{n} (\overline{W}_r \cap V_r)$ has a neighborhood which meets no Q_k involved in this sum, so f_{n+1} is continuous in a neighborhood of the intersection of E with some neighborhood of $\bigcup_{i=1}^{n} (\overline{W}_r \cap V_r)$. Moreover, since f_{n+1} is continuous on a neighborhood of the intersection of E with some neighborhood of K_{n+1}, we can choose a neighborhood W_{n+1} of K_{n+1} so that f_{n+1} is continuous on a neighborhood of the intersection of E with some neighborhood of $\overline{W}_{n+1} \cap V_n$. Finally each g_{nk} is continuous on E, hence so is f_{n+1}, and everything is analytic on U, hence so is f_{n+1}. The induction is complete.

If we choose $\epsilon_n < \epsilon/(2^n A(n))$, then $f_n \to g$ say, uniformly on $\mathbf{C}$, and g is analytic on U, continuous on $\bigcup_n W_n \supseteq T$, and $\|f - g\| \leq \Sigma_n \epsilon_n A_n < \epsilon$. The theorem is proved.

By a completely analogous argument one can prove the following theorem which extends results of Stray [12] and Gamelin and Garnett [10].

Theorem 8.4. *Let U and E be bounded sets, U being open, and suppose that for each $z \in E$ one can find $m, r, \delta_0 > 0$ with*

$$\gamma_E(\Delta(z, \delta) \setminus U) \leq m\gamma(\Delta(z, r\delta) \setminus (U \cup E)), \qquad 0 < \delta < \delta_0.$$

Let f be bounded on $\mathbf{C}$, analytic on U, and continuous at each point of E. Let $\epsilon > 0$. Then we can find g analytic in U and on a neighborhood of E, with $\|f - g\| < \epsilon$.

9. **Approximation of Lipschitz functions.** We now consider approximation of continuous functions when we are given some information of the modulus of continuity. Let $h(t)$, $t > 0$, be a positive continuous increasing function with $\int_0^1 h(t)/t^2 < \infty$. The following theorem has also been proved by Garnett [6] by different methods.

Theorem 9.1. *Let T be a bounded subset of $\mathbf{C}$ with zero h-measure (i.e. for any $\epsilon > 0$ T can be covered by discs $\Delta_1, \Delta_2, \cdots$ with radii $r_1, r_2, \cdots$ such that $\Sigma\, h(r_i) < \epsilon$). Let f be a continuous function on $\mathbf{C}$, analytic on a bounded open set, such that the modulus of continuity ω of f satisfies $\omega(t) \leq t^{-1}h(t)$, $t > 0$. Let $\epsilon > 0$.*

Then we can find a continuous function g on $\mathbf{C}$, analytic on U and on a neighborhood of T, with $\|f - g\| < \epsilon$.

Proof. Let p be a positive integer to be fixed below. Fix $\eta > 0$. For $n = 0, 1, \cdots$ we divide the plane into a lattice $\mathfrak{U}_n$ of closed squares of side p^{-n} in the natural way ($\mathfrak{U}_n$ consists of all squares with sides parallel to the axes, with side p^{-n}, whose vertices are of the form (rp^{-m}, sp^{-m}) where r and s are integers). We choose squares $\{Q_i\}_{i=1}^{\infty}$, $Q_i \in \mathfrak{U}_{n(i)}$, so that $T \subseteq (\bigcup Q_i)^\circ$ and

$\Sigma_i\, h(p^{-n(i)}) < \eta$. We may assume that for any square $Q \in \mathfrak{U}_n$, $\Sigma_{Q_i \subseteq Q}\, h(p^{-n(i)}) \leq h(p^{-n})$, for otherwise we can replace those Q_i contained in Q by Q itself.

Let $\tilde{Q}_i$ be the open square with the same center as Q_i and twice the side. Let ϕ_i be continuously differentiable, with support in $\tilde{Q}_i$, such that $0 \leq \phi_i \leq 1$, $\phi_i = 1$ on a neighborhood of Q_i, and $\|\operatorname{grad} \phi_i\| \leq A_1 p^{n(i)}$. Define $\psi_i = (1 - \phi_1) \cdots (1 - \phi_{i-1})\phi_i$. Then $\Sigma \psi_i = 1$ on $\bigcup Q_i$, the convergence being uniform on each Q_i. Moreover, if p is large enough, $\|\operatorname{grad} \psi_i\| \leq A_2 p^{n(i)}$.

Let $f_i = T_{\phi_i} f$. We claim that the series $\Sigma_i f_i$ converges uniformly on $\mathbf{C}$. To see this let μ be the measure obtained by spreading a mass $h(p^{-n(i)})$ evenly over Q_i for each i. Then for $z \in \mathbf{C}$,

$$|f_i(z)| \leq A_3 p^{n(i)} h(p^{-n(i)}) \min\left(1, \frac{p^{-n}}{|z - z_i|}\right) \leq A_4 \int_{Q_i} \frac{d\mu(\zeta)}{|z - \zeta|}$$

where z_i is the center of Q_i. Hence

$$\left|\sum_{i=n}^{\infty} f_i(z)\right| \leq A_4 \int \frac{d\mu_n(\zeta)}{|z - \zeta|} = A_4 \int_0^{\infty} \frac{\psi_n(r)}{r^2}\, dr$$

where $\mu_n = \mu \,|\, \bigcup_{i=n}^{\infty} Q_i$ and $\psi_n(r) = \mu_n(\{\zeta : |\zeta - z| < r\})$. We have $\psi_n(r) \leq \min(\|\mu_n\|, h(r))$, so

$$\left|\sum_{i=n}^{\infty} f_i(z)\right| \leq A_4 \left(\int_0^{\|\mu_n\|^{1/2}} \frac{h(r)}{r^2}\, dr + \int_{\|\mu_n\|^{1/2}}^{\infty} \frac{\|\mu_n\|}{r^2}\, dr\right)$$

$$\longrightarrow 0 \text{ as } \|\mu_n\| \longrightarrow 0.$$

Since $\|\mu_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain uniform convergence. Moreover, since $\|\mu_n\| \leq \|\mu\| \leq \eta$, we can make $|\Sigma_{i=1}^{\infty} f_i(z)| < \epsilon$ by making η small enough.

Put $g = f - \Sigma_i f_i$. Then g is continuous on $\mathbf{C}$, analytic in U, and analytic in a neighborhood of T since $\Sigma\, \phi_i = 1$ there. Since $\|f - g\| < \epsilon$ the theorem is proved.

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