TOPOLOGICAL PROPERTIES OF
PARANORMAL OPERATORS ON HILBERT SPACE

BY
GLENN R. LUECKE

ABSTRACT. Let $B(H)$ be the set of all bounded endomorphisms (operators) on the complex Hilbert space $H$. $T \in B(H)$ is paranormal if $\|(T - zI)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \notin \sigma(T)$ where $d(z, \sigma(T))$ is the distance from $z$ to $\sigma(T)$, the spectrum of $T$. If $\mathcal{P}$ is the set of all paranormal operators on $H$, then $\mathcal{P}$ contains the normal operators, $\mathcal{N}$, and the hyponormal operators; and $\mathcal{P}$ is contained in $\mathcal{L}$, the set of all $T \in B(H)$ such that the convex hull of $\sigma(T)$ equals the closure of the numerical range of $T$. Thus, $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L} \subseteq B(H)$. Give $B(H)$ the norm topology. The main results in this paper are (1) $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{L}$ are nowhere dense subsets of $B(H)$ when $\dim H > 2$, (2) $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{L}$ are arcwise connected and closed, and (3) $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$ when $\dim H = \infty$.

Paranormal operators have received considerable attention in the current literature ([16], [17], [19], [20], [21], [23], [24], [25]). However, only the various spectral properties of paranormal operators have been discussed. In this paper, the topological properties of the set of all paranormal operators $\mathcal{P}$ on a Hilbert space $H$ are investigated.

We begin by giving the notation to be used and by defining some of the more specialized terminology. The point spectrum and approximate point spectrum of an operator $T$ are denoted by $\sigma_p(T)$ and $\sigma_a(T)$, respectively. $z \in \sigma_p(T)$ is a normal eigenvalue if $\{x \in H : (T - zI)x = 0\} = \{x \in H : (T - zI)^*x = 0\}$ where $I$ denotes the identity operator on $H$. $z \in \sigma_a(T)$ is a normal approximate eigenvalue of $T$ when (1) $\|x\| = 1$ and $\|(T - zI)x\| \to 0$ as $n \to \infty$ imply $\|(T - zI)^*y\| \to 0$ as $n \to \infty$, and (2) $\|y\| = 1$ and $\|(T - zI)y\| \to 0$ as $n \to \infty$ imply $\|(T - zI)^*y\| \to 0$ as $n \to \infty$. If $S$ is a set of complex numbers, then $\partial S$ denotes the boundary of $S$ and $\text{co}(S)$ denotes the convex hull of $S$. Let $W(T) = \text{closure}\{(Tx, x) : x \in H, \|x\| = 1\}$ denote the closure of the numerical range of $T$. Let $R(T, z) = (T - zI)^{-1}$ for each $z \in \rho(T)$, the resolvent set of $T$. The spectral radius of $T$ is denoted by $R_{sp}(T)$. $T \in B(H)$ is hyponormal if $T^*T - TT^* \geq 0$.

I. Elementary properties of paranormal operators. Stampfli [19] has shown that every hyponormal operator is paranormal. Therefore, since every normal
operator is hyponormal, \( H \subset P \). G. Orland [13] proved that \( T \in \mathcal{L} \) if and only if \( \| R(T, z) \| \leq 1/d(z, \sigma(T)) \) for all \( z \not\in \sigma(T) \). From this it follows immediately that if \( T \) is a paranormal operator, then \( \sigma(T) = \mathcal{W}(T) \), i.e. \( P \subset \mathcal{L} \).

**Theorem 1.1.** If \( T \) is a paranormal operator and if \( \alpha \) and \( \beta \) are complex numbers, then \( \alpha T + \beta I \) and \( T^* \) are paranormal.

The proof is a simple computation that depends on \( \sigma(\alpha T + \beta I) = \alpha \sigma(T) + \beta \) and \( \sigma(T^*) = \sigma(T)^* \).

The following theorem is very useful in constructing examples.

**Theorem 1.2.** If \( A \) is any operator on \( H \), then there exists a Hilbert space \( K \) and a normal operator \( N \) on \( K \) such that \( T = A \otimes N \in \mathcal{B}(H \otimes K) \) is paranormal.

**Proof.** Since \( \mathcal{W}(A) \) is a compact set of complex numbers, there exists a Hilbert space \( K \) and a normal operator \( N \) on \( K \) such that \( \sigma(N) = \mathcal{W}(A) \) [3, p. 581]. Let \( T = A \otimes N \). Then \( \sigma(T) = \sigma(N) = \mathcal{W}(A) \). For \( z \not\in \sigma(T) = \mathcal{W}(A) \), it is well known that \( \| R(A, z) \| \leq 1/d(z, \mathcal{W}(A)) = 1/d(z, \sigma(T)) \). Since \( N \) is normal and \( z \in \rho(N) \), \( \| R(N, z) \| = 1/d(z, \sigma(T)) \). Therefore

\[
\| R(T, z) \| = \max \{ \| R(A, z) \|, \| R(N, z) \| \} = 1/d(z, \sigma(T)).
\]

Thus \( T \) is paranormal.

As we shall see, the class of all hyponormal operators on \( H \) is distinct, in general, from the class \( P \) of paranormal operators. We know [19] that if \( T \) is hyponormal, then \( \| T \| = R_{sp}(T) \) and if \( T \) is also invertible, then \( T^{-1} \) is hyponormal. These properties do not generalize to paranormal operators.

**Theorem 1.3.** There exists an invertible paranormal operator \( T \) such that

1. \( T \) is not hyponormal,
2. \( T^2 \) is not paranormal,
3. \( \| T \| > R_{sp}(T) \), and
4. \( T^{-1} \) is not paranormal.

**Proof.** Let \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \). Let \( N \) be a normal operator such that \( \sigma(N) = \mathcal{W}(A) \), and let \( T = A \otimes N \). Then by Theorem 1.2, \( T \) is paranormal. We know from [19] that a hyponormal operator is hyponormal on invariant subspaces. Therefore, since \( A \) is not hyponormal, \( T \) is not hyponormal. By [1], \( \mathcal{W}(A) \) is the closed disc of radius 1/2 about \( z = 1 \), and \( \mathcal{W}(A^2) \) is the closed disc of radius 1 about \( z = 1 \). Therefore

\[
0 \in \mathcal{W}(A^2) \subseteq \mathcal{W}(T^2) \quad \text{and} \quad 0 \not\in \sigma(T^2) = \sigma(T)^2.
\]

Therefore, \( \sigma(T^2) \neq \mathcal{W}(T^2) \) and so \( T^2 \) is not paranormal. Let \( x = \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix} \), then \( \| x \| = 1 \) and \( \| Ax \| = \sqrt{10}/2 \). Then

\[
\| T \| > \| Ax \| = \sqrt{10}/2 > 3/2 = R_{sp}(T).
\]
Therefore $\|T\| > R_{sp}(T)$. If $T^{-1}$ were paranormal, then
$$\|T\| = \|R(T^{-1}, 0)\| = 1/d(0, \sigma(T^{-1})) = R_{sp}(T).$$

Contradiction. Hence $T^{-1}$ is not paranormal.

II. Topological properties of $\mathcal{P}$. Recall that $\mathfrak{N}$ is the set of all normal operators on $H$, $\mathcal{P}$ is the set of all paranormal operators on $H$, $\mathfrak{L}$ is the set of all $T \in B(H)$ such that $\sigma(\sigma(T) = \mathfrak{W}(T)$, and $\mathfrak{N} \subseteq \mathcal{P} \subseteq \mathfrak{L}$. It will always be assumed that $B(H)$ has the uniform operator (norm) topology.

The following notation will be used in this section: If $S$ is a compact subset of the complex numbers $\mathbb{C}$ and if $\epsilon > 0$, then let $S + (\epsilon) = \{z : d(z, S) < \epsilon\}$. If $S$ and $S_n$, $n = 1, 2, 3, \ldots$, are compact sets in $\mathbb{C}$, then the sequence $\{S_n\}$ approaches $S$, written $S_n \to S$, if for every $\epsilon > 0$ there exists a positive integer $N$ such that, for $n > N$, $S_n \subseteq S + (\epsilon)$ and $S \subseteq S_n + (\epsilon)$.

In general $\sigma(T)$ is not a continuous function of $T$ in $B(H)$ (see [7, problem 85]), but $\sigma(T)$ is continuous if we restrict $T$ to $\mathfrak{P}$.

**Theorem 2.1.** If $\{T_n\}$ is a sequence of paranormal operators approaching the operator $T$ in norm, then $\sigma(T_n) \to \sigma(T)$ as $n \to \infty$.

To prove this theorem we need the following lemma from [7, problem 86].

**Lemma.** If $T \in B(H)$ and $\epsilon > 0$, then there exists $\delta > 0$ such that if $S \in B(H)$ and $\|T - S\| < \delta$, then $\sigma(S) \subseteq \sigma(T) + (\epsilon)$.

**Proof of Theorem 2.1.** We know by the lemma that for each $\epsilon > 0$ there exists a positive integer $N$ such that, for $n \geq N$, $\sigma(T_n) \subseteq \sigma(T) + (\epsilon)$. Therefore, to show $\sigma(T_n) \to \sigma(T)$, it suffices to show that for each $\epsilon > 0$ there exists a positive integer $N$ such that $\sigma(T) \subseteq \sigma(T_n) + (\epsilon)$ for all $n \geq N$. If this does not hold, then without loss of generality we may assume that there exists $\epsilon > 0$ and a sequence $\{z_n\}$ $\subseteq \sigma(T)$ such that $d(z_n, \sigma(T_n)) \geq \epsilon$ for all $n$. Since $\sigma(T)$ is compact, we may assume $z_n \to z \in \sigma(T)$. If $|z_n - z| < \epsilon/2$, then
$$d(z, \sigma(T_n)) \geq d(z_n, \sigma(T_n)) - |z - z_n| \geq \epsilon - \epsilon/2 = \epsilon/2.$$  

Hence
$$\|R(T_n, z)\| = 1/d(z, \sigma(T_n)) \leq 2/\epsilon.$$

Now choose $m$ so that $\|(T_m - T)R(T_m, z)\| < 1$, then $I - (T_m - T)R(T_m, z)$ is invertible [7, problem 173]. Let
$$A = R(T_m, z)(I - (T_m - T)R(T_m, z))^{-1}.$$  

Then $A(T - zI) = (T - zI)A = I$ so that $z \in \rho(T)$. Contradiction.

**Remark.** Note that the full strength of the paranormality assumption was not
used in this proof, and that one can easily devise various larger classes of operators on which $\sigma(T)$ is continuous.

**Theorem 2.2.** $\mathcal{P}$ is an arcwise connected, closed subset of $B(H)$.

**Proof.** Since $T \in \mathcal{P}$ implies $aT \in \mathcal{P}$ for every complex number $a$, we see that the ray in $B(H)$ through $T$ is contained in $\mathcal{P}$. Therefore $\mathcal{P}$ is arcwise connected.

Suppose $T_n \to T, \{T_n\}$ a sequence of operators in $\mathcal{P}$, and $T \in B(H)$. Let $z \in \rho(T)$. By the lemma to Theorem 2.1,

$$\limsup_{n \to \infty} \frac{1}{d(z, \sigma(T_n))} \leq \frac{1}{d(z, \sigma(T))}.$$ 

Therefore, since $\|R(T_n, z)\| = 1/d(z, \sigma(T_n))$ whenever $z \in \rho(T_n)$, there exists a positive integer $N$ such that the sequence $\{\|R(T_n, z)\|: n \geq N\}$ is bounded. Then, since $R(T, z) - R(T_n, z) = R(T, z)(T - T_n)R(T_n, z), \|R(T_n, z)\| \to \|R(T, z)\|$ as $n \to \infty$. Consequently,

$$\|R(T, z)\| = \lim_{n \to \infty} \|R(T_n, z)\| = \lim_{n \to \infty} \frac{1}{d(z, \sigma(T_n))} \leq \frac{1}{d(z, \sigma(T))}.$$ 

Since the general $\|R(T, z)\| \geq 1/d(z, \sigma(T)), \ T$ is paranormal.

**Theorem 2.3.** $\mathcal{L}$ is an arcwise connected, closed subset of $B(H)$.

**Proof.** Since $T \in \mathcal{L}$ implies that $aT \in \mathcal{L}$ for every complex number $a$, $\mathcal{L}$ is arcwise connected.

Let $T_n \to T, \{T_n\} \subseteq \mathcal{L}$ and $T \in B(H)$. Since $\|(T_n x, x) - (T x, x)\| \leq \|T_n - T\|$ for $\|x\| = 1, W(T_n) \subseteq W(T) + (2\|T - T_n\|) \text{ and } W(T) \subseteq W(T_n) + (2\|T - T_n\|)$. Consequently, $W(T_n) \to W(T)$. Let $\epsilon > 0$, then by the lemma to Theorem 2.1 there exists a positive integer $N$ such that $\sigma(T_n) \subseteq \sigma(T) + (\epsilon)$ for all $n \geq N$. Therefore, for $n \geq N, \co \sigma(T_n) \subseteq \co \sigma(T) + (\epsilon)$ and hence

$$W(T) = \lim_{n \to \infty} W(T_n) = \lim_{n \to \infty} \co \sigma(T_n) \subseteq \co \sigma(T) + (\epsilon).$$ 

Since $\epsilon > 0$ is arbitrary, $W(T) \subseteq \co \sigma(T)$. Since in general $\co \sigma(T) \subseteq W(T), T \in \mathcal{L}$.

Let $\mathcal{N}$ be the set of all normal operators on $H$. Since $\|T_n - T\| \to 0$ implies $\|T_n^* - T^*\| \to 0$, $\mathcal{N}$ is closed in the uniform operator topology on $B(H)$. Since $T \in \mathcal{N}$ implies $aT \in \mathcal{N}$ for any complex $a$, $\mathcal{N}$ is arcwise connected.

We already know that $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L} \subseteq B(H)$. We will now investigate the relative topological properties of these four sets. Stampfli [20, Theorem C] has shown that $\mathcal{N} = \mathcal{P}$ when $\dim H < \infty$. However, when $\dim H = \infty$, then $\mathcal{N}$ is a very "thin" subset of $\mathcal{P}$.

**Theorem 2.4.** $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$ when $\dim H = \infty$.

**Proof.** Since $\mathcal{N}$ is closed, to show that $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$,
it suffices to show that $\mathcal{H}$ has empty interior in $\mathcal{P}$. Let $T \in \mathcal{H}$ and let $\epsilon > 0$.

First suppose that $T$ has an eigenvalue of infinite multiplicity. We may assume that the eigenvalue is zero. Let $M$ be the eigenspace of zero. Then $\dim M = \infty$, $M$ reduces $T$, and we can write $T = B \oplus Z$ where $Z$ is the zero operator on $M$. Let $S = \left[ \begin{smallmatrix} 0 & \epsilon \\ 0 & 0 \end{smallmatrix} \right] \oplus N$ be a nonnormal paranormal operator [see Theorem 1.2] on $M$ with $N$ a normal operator such that $\sigma(N) = W[0, \epsilon]^1$. Then $B \oplus S$ is a nonnormal paranormal operator such that $\|T - B \oplus S\| = \|B \oplus Z - B \oplus S\| = \|S\| = \epsilon$. The last equality holds since $\|N\| = Rsp(N) = \epsilon/2$ and $\|\left[ \begin{smallmatrix} 0 \\ \epsilon/2 \end{smallmatrix} \right]\| = \epsilon$. Therefore, since $\epsilon > 0$ is arbitrary, $T$ is not contained in the interior of $\mathcal{H}$ in $\mathcal{P}$.

If $\sigma(T)$ is finite and $T \in \mathcal{H}$, then $\sigma(T) = \sigma_p(T)$ and $T$ has an eigenvalue of infinite multiplicity. We therefore assume that $\sigma(T)$ is infinite and that zero is an accumulation point of $\sigma(T)$. Let $D$ be the open disc about zero of radius $\epsilon/2$. Let $E$ be the resolution of the identity for $T$ so that $T = \int_{\sigma(T)} z dE_z$. Let $M = (E(D))(H)$, $P = \sigma(T) - D$, and let $A = \int_{\sigma_p(T)} z dE_z$. Then $M$ reduces $T$, $\dim M = \infty$, and $A$ is a normal operator. Let $Z$ be the zero operator on $M$. Then $A \oplus Z$ is a normal operator with zero an eigenvalue of infinite multiplicity, and

$$\|T - A \oplus Z\| = \left\| \int_D z dE_z \right\| \leq \epsilon/2.$$ 

By the first part of this proof, there exists a nonnormal paranormal operator $S$ such that $\|A \oplus Z - S\| < \epsilon/2$. Then

$$\|T - S\| \leq \|T - A \oplus Z\| + \|A \oplus Z - S\| < \epsilon.$$

Therefore, since $\epsilon > 0$ is arbitrary, $T$ is not contained in the interior of $\mathcal{H}$ in $\mathcal{P}$. Hence the interior of $\mathcal{H}$ in $\mathcal{P}$ is empty.

Define $C_2$ to be the set of all operators $T \in \mathcal{L}$ with $W(T)$ a closed line segment or a point. For $k = 3, 4, 5, \ldots$, let $C_k$ be the set of all operators $T \in \mathcal{L}$ such that $W(T)$ is the convex hull of a nondegenerate polygon with $k$ sides. If $T \in C_k$, then each vertex of $W(T)$ must be in the spectrum of $T$ [1]. S. Hildebrandt [10, Theorem 2] has shown that, if $z \in \sigma_p(T) \cap \partial W(T)$, then $z$ is a normal eigenvalue of $T$. Thus for $T \in C_k$, the vertices of $W(T)$ are normal eigenvalues of $T$, when $\dim H < \infty$. Hence, all the operators in $C_n \cup C_{n-1}$ are normal operators when $\dim H = n < \infty$.

In [10, Theorem 9] S. Hildebrandt showed that $\mathcal{H} = \mathcal{P} = \mathcal{L}$ when $\dim H \leq 4$, and that $\mathcal{H} \neq \mathcal{L}$ for $5 \leq \dim H < \infty$. The following theorem gives more information on how $\mathcal{P}$ and $\mathcal{L}$ are related when $5 \leq \dim H < \infty$. Recall that $\mathcal{H} = \mathcal{P}$ for $\dim H < \infty$.

**Theorem 2.5.** If $5 \leq \dim H = n < \infty$, then the interior of $\mathcal{P}$ in $\mathcal{L}$ equals $C_n \cup C_{n-1}$.

**Proof.** Suppose $T \in C_n \cup C_{n-1}$. Since $C_n \cup C_{n-1} \subseteq \mathcal{H}$, $T$ is normal. First, we show that there exists $\epsilon > 0$ such that whenever $S \in \mathcal{L}$ and $\|T - S\| < \epsilon$, then...
$S \in \mathcal{C}_n \cup \mathcal{C}_{n-1}$. To show this, suppose the statement were false. Then there would exist \( \{S_n\} \subseteq \mathcal{L} \) such that \( \|T - S_n\| \to 0 \) and \( S_n \in \bigcup_{i=2}^{n-2} C_i \). Then, since \( W(S_n) \to W(T) \), \( T \in \bigcup_{i=2}^{n-2} C_i \). Contradiction. Hence, \( T \) is an interior point of \( \mathcal{P} \) in \( \mathcal{L} \).

Let \( T \) be contained in the interior of \( \mathcal{P} = \mathcal{N} \) in \( \mathcal{L} \). Suppose \( T \notin \mathcal{C}_n \cup \mathcal{C}_{n-1} \).

Let \( \varepsilon > 0 \). Since \( \text{co} \sigma(T) = W(T) \), \( T \in C_k \), for some \( k \leq n - 2 \). Since \( \text{dim } H \geq 5 \) and since \( T \) is a normal operator in \( C_k \), there exists a normal operator \( N \) such that
1. \( \|T - N\| < \varepsilon/2 \),
2. \( W(N) \) is a polygon with at least three sides, and
3. \( N \) has at least two eigenvalues \( z, w \) contained in the interior of \( W(N) \).

Write \( N = A \oplus B \) where \( B \) can be written as \( B = [\begin{smallmatrix} \varepsilon & 0 \\ 0 & \omega \end{smallmatrix}] \). Let \( a > 0 \) and let \( C = [\begin{smallmatrix} \varepsilon & a \\ 0 & \omega \end{smallmatrix}] \); then \( \|B - C\| = \|[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}]\| = a \). Choose \( a > 0 \) small enough so that \( W(C) \subseteq W(N) \) and so that \( a < \varepsilon/2 \). Then since \( W(A) = W(N) \) and \( \sigma(A \oplus C) = \sigma(N) \), \( \text{co} \sigma(A \oplus C) = W(A \oplus C) \). Hence \( A \oplus C \in \mathcal{P} \). Since \( A \oplus C \) is not normal, and since
\[
\|T - A \oplus C\| \leq \|T - N\| + \|N - A \oplus C\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,
\]
\( T \) is not an interior point of \( \mathcal{P} \) in \( \mathcal{L} \). Contradiction. Hence \( T \in \mathcal{C}_n \cup \mathcal{C}_{n-1} \).

It is an open question as to what the interior of \( \mathcal{P} \) in \( \mathcal{L} \) is when \( \text{dim } H = \infty \). However, it can be shown that \( \mathcal{P} \neq \mathcal{L} \) when \( \text{dim } H = \infty \).

**Theorem 2.6.** \( \mathcal{P} \neq \mathcal{L} \) when \( \text{dim } H = \infty \).

**Proof.** Write \( H = M \oplus M^\perp \) where \( \text{dim } M = 5 \). Let
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}
\]
where \( a, b, c \) are three complex numbers that form a triangle with \( W(A) \) contained in the interior of the triangle. Consider \( A \oplus N \) as an operator on \( M \) and observe that \( \text{co} \sigma(A \oplus N) = W(N) = W(A \oplus N) \). Since \( A \oplus N \) is not normal and since \( \text{dim } M < \infty \), \( A \oplus N \) is not paranormal. Hence there exists \( z \in \rho(A \oplus N) \) such that
\[
\|R(A \oplus N, z)\| > 1/d(z, \sigma(A \oplus N)).
\]
Let \( I \) be the identity operator on \( M^\perp \) and let \( T = A \oplus N \oplus aI \). Then, \( \sigma(T) = \sigma(A \oplus N) \) and \( W(T) = W(A \oplus N) \). Therefore \( T \in \mathcal{L} \). Since \( d(z, \sigma(T)) \leq |z - a| \),
\[
\|R(T, z)\| = \max \{\|R(A \oplus N, z)\|, |z - a|^{-1}\}
= \|R(A \oplus N, z)\| > 1/d(z, \sigma(T)).
\]
Therefore \( T \) is not paranormal.

We now show that if \( \text{dim } H \geq 2 \), then \( \mathcal{L} \) is a nowhere dense ("thin") subset of \( B(H) \). Once this is shown, it follows immediately that \( \mathcal{N} \) and \( \mathcal{P} \) are also nowhere dense subsets of \( B(H) \).

**Theorem 2.7.** \( \mathcal{L} \) is a nowhere dense subset of \( B(H) \) when \( \text{dim } H \geq 2 \).
To prove this theorem we need the following two technical lemmas.

Lemma 1. If \( z_1 \) and \( z_2 \) are distinct, normal approximate eigenvalues of \( T \in B(H) \), then there exist sequences \( \{x_n\} \) and \( \{y_n\} \) of unit vectors in \( H \) such that

1. \( \langle x_n, y_n \rangle = 0 \) for all \( n \),
2. \( \| (T - z_1) x_n \| \to 0 \) as \( n \to \infty \), and
3. \( \| (T - z_2) y_n \| \to 0 \) as \( n \to \infty \).

Lemma 2. If \( T \in \mathcal{L} \) such that there exists distinct \( a, b \in \partial \sigma(T) \cap \sigma_p(T) \), then \( T \) is not contained in the interior of \( \mathcal{L} \).

Proof of Theorem 2.7. Since \( \mathcal{L} \) is closed (Theorem 2.3), to show that \( \mathcal{L} \) is nowhere dense it suffices to show that \( \mathcal{L} \) has empty interior.

We first show that if \( T \) is in the interior of \( \mathcal{L} \), then \( \sigma(T) \) must contain at least two points. Suppose \( T \in \mathcal{L} \) and \( \sigma(T) = \{a\} \). Then \( \langle (T - a) x, x \rangle = 0 \) for all \( x \in H \) so that \( T = aI \). Since \( \dim H \geq 2 \), write \( H = M \oplus M^\perp \) where \( \dim M = 2 \). Let \( b > 0 \) and define \( A \in B(H) \) as

\[
A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \text{ on } M, \quad \text{and } A = 0 \text{ on } M^\perp.
\]

Then \( \sigma(T + A) = \{a\} \) and, since \( b \neq 0 \), \( \{a\} \notin \sigma(T + A) \). Therefore \( T + A \notin \mathcal{L} \).

Since \( \|A\| = b > 0 \) is arbitrary, \( T \notin \text{interior } \mathcal{L} \).

With the above remark completed, we can now finish the proof of Theorem 2.7. Suppose the theorem were false and there exists \( T \in \text{interior } \mathcal{L} \). Then there exists \( \epsilon > 0 \) such that whenever \( V \in B(H) \) and \( \|T - V\| < \epsilon \), then \( V \in \mathcal{L} \). From the above remark \( \sigma(T) \) must contain at least two points. There must be at least two extreme points of \( \partial \sigma(T) \), since extreme points of \( \partial \sigma(T) \) for \( T \notin \mathcal{L} \) are extreme points of \( \sigma(T) \). Hence, after a rotation, if necessary, we may assume there exists \( z_1, z_2 \in \partial \sigma(T) \cap \sigma_p(T) \) such that

1. \( \Re z_1 = \inf \Re \sigma(T) \),
2. \( \Re z_2 = \sup \Re \sigma(T) \), and
3. \( \Re z_1 < \Re z_2 \).

Since \( z_1, z_2 \in \partial \sigma(T) \cap \sigma_p(T) \), \( z_1 \) and \( z_2 \) are normal approximate eigenvalues of \( T \) \cite{10, Theorem 2, p. 233}. By Lemma 1 there exist unit vectors \( x, y \in H \) such that \( \langle x, y \rangle = 0 \), \( \| (T - z_1) x \| < \epsilon/8 \), and \( \| (T - z_2) y \| < \epsilon/8 \). Let \( M \) be the closed subspace spanned by \( \{x, y\} \). Define \( C \in B(H) \) as

\[
Cx = -\left(\epsilon/4\right)x, \\
Cy = +\left(\epsilon/4\right)y, \text{ and} \\
Cz = 0 \text{ for all } z \in M^\perp.
\]

Since \( \|C\| \leq \epsilon/2 \), \( T + C \in \mathcal{L} \). Since

\[
\langle (T + C)x, x \rangle = \langle Tx, x \rangle - \epsilon/4 \quad \text{and} \\
\| (Tx, y) - z_1 \| \leq \| (T - z_1) x \| < \epsilon/8,
\]

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we obtain $\inf \Re W(T + C) < \Re z_1$. Since $T + C \in \mathcal{P}$, there exists $a \in \sigma_{\partial}(T + C) \cap \partial W(T + C)$ such that $\Re a = \inf \Re W(T + C)$. Since $C$ is a compact operator, Weyl’s spectral inclusion theorem [7, problem 143] yields $\sigma(T + C) - \sigma_p(T + C) \subseteq \sigma(T)$. Therefore, $a \in \sigma_p(T + C)$ so that $a \in \sigma_p(T + C) \cap \partial W(T + C)$. Similarly one shows there exists

$$b \in \sigma_p(T + C) \cap \partial W(T + C)$$

such that $\Re b = \sup W(T + C) > \Re z_2$, and hence $a \neq b$. By Lemma 2, there exists $S \in B(H)$ such that $\|S\| < \epsilon/2$ and $T + C + S \not\in \mathcal{P}$. However, $\|T - (T + C + S)\| \leq \|C\| + \|S\| < \epsilon$ and so by assumption $T + C + S \not\in \mathcal{P}$. Contradiction.

Proof of Lemma 1. There exist sequences $\{w_n\}$ and $\{y_n\}$ of unit vectors in $H$ such that $\|(T - z_1)w_n\| \to 0$ and $\|(T - z_2)\| \to 0$ as $n \to \infty$. Then

$$\left| (z_1 - z_2)(w_n, y_n) \right| = \left| (z_1w_n, y_n) - (w_n, z_2y_n) \right|$$

$$\leq \|(T - z_1)w_n\| + \|(T - z_2)\| \leq \|(T - z_1)w_n\| + \|(T - z_2)\|.$$  

Therefore, $\left| (z_1 - z_2)(w_n, y_n) \right| \to 0$ as $n \to \infty$. Since $z_1 \neq z_2$, $(w_n, y_n) \to 0$ as $n \to \infty$.

There exist complex numbers $a_n$ and $b_n$ and unit vectors $x_n$ in $H$ such that $w_n = a_n y_n + b_n x_n$, $|a_n|^2 + |b_n|^2 = 1$, and $(x_n, y_n) = 0$. From the above paragraph we have that $a_n \to 0$, so $|b_n| \to 1$. Therefore, $\|(T - z_1)x_n\| \to 0$ as $n \to \infty$.

Proof of Lemma 2. Let $\epsilon > 0$. Since $a, b \in \sigma_p(T) \cap \partial W(T)$, $a$ and $b$ are normal eigenvalues of $T$ [10, Theorem 2, p. 233]. Let $u, v \in H$ be unit vectors such that $Tu = au$ and $Tv = bv$. Then $(u, v) = 0$ and the closed subspace $N$ spanned by $\{u, v\}$ reduces $T$. Define $S \in B(H)$ as

$$Su = eu, \quad Sv = 0, \quad Sz = 0 \quad \text{for all} \ z \in N^\perp.$$  

Then we may write $T + S = A \oplus B$ corresponding to $H = N \oplus N^\perp$. Then the matrix representation of $A$ relative to $\{u, v\}$ is $A = \left[ \begin{array}{c|c} a & b \\ \hline 0 & b \end{array} \right]$. Hence $\sigma(A) = \{a, b\} \subseteq \sigma(T)$. Clearly $\sigma(B) \subseteq \sigma(T)$. Therefore,

$$\sigma(T + S) \subseteq \sigma(T) = W(T).$$

Since $A$ is not a normal operator, $W(A)$ is the convex hull of a nondegenerate ellipse (i.e., not a straight line) with foci at $a$ and $b$ (see [1]). Since $W(A) \subseteq W(T + S)$, we must have $\sigma(A) \neq W(T + S)$. Therefore, $T + S \not\in \mathcal{P}$. Thus, since $\|S\| = \epsilon > 0$ is arbitrary, $T \not\in \text{interior } \mathcal{P}$.

**BIBLIOGRAPHY**