HYPONORMAL OPERATORS
HAVING REAL PARTS WITH SIMPLE SPECTRA\(^{(1)}\)

BY

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ABSTRACT. Let \( T^*T - TT^* = D \geq 0 \) and suppose that the real part of \( T \) has a simple spectrum. Then \( D \) is of trace class and \( \pi \text{ trace}(D) \) is a lower bound for the measure of the spectrum of \( T \). This latter set is specified in terms of the real and imaginary parts of \( T \). In addition, the spectra are determined of self-adjoint singular integral operators on \( L^2(E) \) of the form \( A(x)f(x) + \sum b_j(x)H(b_j)(x) \), where \( E \subset (-\infty, \infty) \), \( A(x) \) is real and bounded, \( \sum |b_j(x)|^2 \) is positive and bounded, and \( H \) denotes the Hilbert transform.

1. A bounded operator \( T \) on a Hilbert space \( \mathfrak{H} \) (which in this paper will be assumed to be separable) is said to be hyponormal if

\[
T^*T - TT^* = D \geq 0,
\]

or, equivalently, if \( T \) has the Cartesian form \( T = H + iJ \),

\[
HJ - JH = -iC, \quad C = \frac{1}{2}D \geq 0.
\]

In this case, the spectra of \( H \) and \( J \) are the (real) projections of the spectrum of \( T \) onto the coordinate axes, thus

\[
\text{sp}(H) = \text{Re}(\text{sp}(T)) \quad \text{and} \quad \text{sp}(J) = \text{Im}(\text{sp}(T));
\]

see Putnam [15, p. 46]. It was shown in Putnam [17] that if \( T \) is hyponormal then

\[
\pi \|D\| \leq \text{meas}_2(\text{sp}(T));
\]

in the particular case in which \( D \) is completely continuous, the inequality \( (1.4) \) was proved by Clancey [2]. If \( T \) is hyponormal and if its real part, \( H \), satisfies

\[
H = \frac{1}{2}(T + T^*)
\]

has a simple spectrum,

or, more generally, if \( H \) has finite spectral multiplicity, then \( C \) belongs to trace class; Kato [8]. In general, the inequality \( (1.4) \) is nontrivially optimal (e.g., equality holds if \( T \) is the unilateral shift). In certain instances, however, a sharpening of the inequality can be obtained as in the following

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Theorem 1. Let $T$ satisfy (1.1) and (1.5). Then
\begin{equation}
\pi \text{ trace } (D) \leq \text{ meas}_2(\text{sp}(T)).
\end{equation}

An operator $T$ satisfying (1.1) is said to be completely hyponormal if there is no (nontrivial) subspace reducing $T$ and on which $T$ is normal. In case $T$ is completely hyponormal then its real and imaginary parts must be absolutely continuous; see [15, p. 42]. It is well known that if a bounded selfadjoint operator $H$ has a simple spectrum and is absolutely continuous, then there is a set $E$, where
\begin{equation}
E \text{ is a bounded subset of } (-\infty, \infty) \text{ of positive measure},
\end{equation}
such that $H$ is unitarily equivalent to the coordinate multiplication operator, $x$, on $L^2(E)$. The next theorem concerns completely hyponormal operators $T = H + ij$ for which (1.5) holds. Without loss of generality it can be supposed therefore that
\begin{equation}
(Hf)(x) = xf(x) \quad \text{and} \quad (Cf)(x) = \pi^{-1} \sum \lambda_j \phi_j(x), \quad f \in L^2(E),
\end{equation}
where
\begin{equation}
\{\phi_1, \phi_2, \ldots\} \text{ is orthonormal on } L^2(E); \quad \lambda_1 \geq \lambda_2 \geq \cdots > 0 \text{ and } \sum \lambda_j < \infty.
\end{equation}
There will be proved the following
\begin{theorem}
Let $T$ of (1.1) be completely hyponormal and suppose that (1.7), (1.8) and (1.9) hold. Then
\begin{equation}
0 < b(x) \leq \text{const} < \infty \quad \text{a.e. on } E, \quad \text{where } b(x) = \sum \lambda_j |\phi_j(x)|^2.
\end{equation}
Further, $J$ of (1.2) is given by
\begin{equation}
(Jf)(x) = -[a(x) + (im)^{-1} \sum \lambda_j \phi_j(x)] \int_E f(t) \overline{\phi_j(t)} (t - x)^{-1} dt,
\end{equation}
where the summation operator is the strong limit of its partial sums. Also, $z = s + it$ (s, t real) is in sp$(T)$ if and only if
\begin{equation}
\text{meas}_1 \{x \in E \cap \Delta : -a(x) - b(x) - \epsilon < t < -a(x) + b(x) + \epsilon \} > 0
\end{equation}
holds whenever $\epsilon > 0$ and $\Delta$ is any open interval containing $s$.

Remark. Conversely, if $E$ is any set satisfying (1.7) and if the selfadjoint operators $H$ and $J$ are defined as above by (1.8)–(1.11), then it is easily verified that (1.2) holds with $C$ defined by (1.8), so that $T$ is hyponormal. (Cf. §4, below, and note that $J = -a + J_0$, where $J_0$ is defined by (4.2).) That, in fact, $T$ is completely hyponormal can be seen as follows. Let $S_1$ ($\neq 0$) $\subset L^2(E)$ reduce $T$, hence also $H$ and $J$, and suppose that $T$ is normal on $S_1$. Then $Cf = 0$ for
all \( f \) in \( \mathcal{H}_1 \). But if \( f_1 \in \mathcal{H}_1 \) and \( f_1 \neq 0 \), then the set \( \{ x : f_1(x) \neq 0 \} \) has positive measure. Since \( \sum \lambda |\phi_j(x)|^2 > 0 \) on \( E \) there exists some \( \phi_k \) such that \( \{ x : \phi_k \neq 0 \} \cap \{ x : f_1(x) \neq 0 \} \) has positive measure. But \( \rho(x)/f_1(x) \) also belongs to \( \mathcal{H}_1 \) for any polynomial \( \rho(x) \), and it is clear from the Weierstrass approximation theorem that there must exist some \( g \) in \( \mathcal{H}_1 \) such that \( (g, \phi_k) \neq 0 \). Since \( Cg = 0 \), then \( \phi_k = \sum_{i \neq k} a_i \phi_i (\Sigma|a_i|^2 < \infty) \), in contradiction to the supposed orthonormality of \( \{ \phi_i \} \), \( i = 1, 2, \ldots \).

As noted above, both operators \( H \) and \( J \) satisfying (1.8)–(1.11) are absolutely continuous. Also, concerning Theorem 2, see the Remark at the beginning of §7 below. In case \( \text{rank}(D) = \text{rank}(C) = 1 \), the assertion (1.11) concerning the form of \( J \) is due to Xa Dao-xeng [21]; for a simpler proof, using a result in [19, p. 326].

Some corollaries of Theorems 1, 2 together with a lemma and some remarks will be stated in §2. The proofs of Theorems 1, 2 will be given in §§3, 4 respectively. A connection between certain operators considered by Kato [8] and those in the present paper will be discussed in §5. In §6, some applications of Theorems 1, 2 will be made and the results stated as Theorems 3–5. §§7, 8 will deal with generalized selfadjoint singular integral operators.

2. If \( T = V \), the unilateral shift, then \( \text{sp}(V) \) is the closed unit disk and (1.4) becomes an equality. Since \( V \) is isometric (hence hyponormal), so are its powers \( V^n \), \( n = 1, 2, \ldots \). Further, it is easily verified that \( (V^n)^* V^n - V^n(V^n)^* \) has norm 1 and trace \( n \). Since \( \text{sp}(V^n) = \text{sp}(V) \) then (1.5), with \( T = V^n \), holds only if \( n = 1 \) (although even equality holds in (1.4) for all \( n \)). It follows from Theorem 1 that \( \text{Re}(V^n) = \frac{1}{2}(V^n + V^n^*) \) does not have a simple spectrum for \( n \geq 2 \). That, incidentally, \( \frac{1}{2}(V + V^*) \) does have a simple spectrum is easily verified directly. In fact, on \( l^2 = \{ (x_1, x_2, \cdots) : \sum |x_i|^2 < \infty \} \), \( V(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots) \) and \( (1, 0, 0, \cdots) \) is readily seen to be a cyclic vector of \( \frac{1}{2}(V + V^*) \).

More generally, one has the following obvious

Corollary of Theorem 1. If \( T \) satisfies (1.1), if equality holds in (1.4), and if (1.5) holds, then \( \text{rank}(D) \leq 1 \).

An explicit formulation of Theorem 2 in the special case in which \( C \) of (1.8) has rank 1, so that \( C\phi = \lambda_1(f, \phi_1)\phi_1 \), occurs in Clancey and Putnam [4]. Another way of giving the spectrum of \( T \), somewhat different from that specified in Theorem 2 using (1.12), and involving a “determining set,” appears in Clancey [3], where again \( C \) has rank 1 and \( H \) has a simple spectrum, and in Pincus [12], where \( C \) is of trace class and \( H \) has arbitrary (not necessarily simple) spectral multiplicity. These papers, as well as the present one, use a result obtained in
Putnam [18] for determining the "cross sections" of the spectrum of a hyponormal operator $T$, and which will be stated below as a lemma.

Remark (added November 6, 1971). Professor Pincus has pointed out to the author that Theorem 2 can be deduced from the general results of his papers [10] and [12]. The proof of Theorem 2 as given below will be used later (Theorem 6, below) along with [16] to yield corresponding results for singular integral operators in which the set $E$ satisfies only $E \notin (-\infty, \infty)$.

First, let $T$ be hyponormal on $\mathcal{H}$ and let $H$ of (1.2) have the spectral resolution
\begin{equation}
H = \int \lambda \, dE_{\lambda},
\end{equation}
and, for any open interval $\Delta$, let $E(\Delta)$ denote the associated projection operator. Then $T_{\Delta} = E(\Delta)TE(\Delta)$ is hyponormal on $E(\Delta)\mathcal{H}$ and, as was shown in [17] (and in [2] in case $C$ is completely continuous),
\begin{equation}
\text{sp}(T_{\Delta}) \subset \text{sp}(T).
\end{equation}
Moreover, as was shown in [18],
\begin{equation}
\text{sp}(T_{\Delta}) \cap \{z : \text{Re}(z) = s\} = \text{sp}(T) \cap \{z : \text{Re}(z) = s\},
\end{equation}
where $s$ is any number in $\Delta$. In view of (1.3), one obtains [18] the following

Lemma. If $T$ is hyponormal then
\begin{equation}
\text{Im}[\text{sp}(T) \cap \{z : \text{Re}(z) = s\}] = \bigcap_{\Delta} \text{sp}(E(\Delta)JE(\Delta)), \quad s \in \Delta,
\end{equation}
where the intersection is taken over all open intervals $\Delta$ containing $s$.

In [4], in which $C$ of (1.8) was of rank 1, the spectrum of $H + iJ$ ($H = x$) was determined from (2.4) using a knowledge of the spectrum of $J$. This latter information could be determined from (2.4) using a knowledge of the spectrum of $J$. This latter information could be determined from Pincus [9, p. 375], or Rosenblum [19, p. 323] (see also Pincus and Rovnyak [13, p. 620]). In the present paper certain general properties of hyponormal operators obtained in [17], [18] will be used to obtain the set on the right of (2.4) when $H = \text{Re}(T)$ has a simple spectrum. From this information, relation (2.4) will then be used to determine the spectrum of $T = H + iJ$. As a consequence, the spectrum of $J$ can then, if desired, be determined from the projection properties (1.3).

Thus, if $H$ and $J$ are defined by (1.8) and (1.11), where (1.9) and (1.10) are also assumed, then, by the remark following the statement of Theorem 2, $T = H + iJ$ is completely hyponormal and satisfies (1.2) with $C$ given by (1.8). In view of (1.12) and the second relation of (1.3), one obtains the following

Corollary 1 of Theorem 2. If $E$ satisfies (1.7) and if $J$ is defined by (1.11) and (1.9) then a real number $t$ is in $\text{sp}(J)$ if and only if
meas \{x \in E: -a(x) - b(x) - \epsilon < t < -a(x) + b(x) + \epsilon \} > 0

holds for every \(\epsilon > 0\).

For any measurable subset, \(E\), of the real line define the (measurable) real set \(E^*\) by

\[ E^* = \{x \in (-\infty, \infty) : \text{meas}_1(\Delta \cap E) > 0 \} \]

for every open interval \(\Delta\) containing \(x\).

Since all points of \(E\) having positive metric density are contained in \(E^*\), the set \(E \cap E^*\) differs from \(E\) by a null set. Next, for any real-valued function \(c(x)\) defined on \(E\), define \(c^*(x)\) on \(E^*\) by

\[ c^*(x) = \text{ess lim sup}_{t \to x} c(t) = \lim_{\Delta \to 0} \text{ess sup}_{\Delta \cap E} c(t), \]

where \(\Delta\) is any open interval containing \(x\).

**Corollary 2 of Theorem 2.** For all \(x\) in \(E^* (= \text{sp}(H))\) there exists a real number \(a_x\) such that

\[ \{x + iy : a_x - b^*(x) < y < a_x + b^*(x)\} \subset \text{sp}(T). \]

It may be noted that \(E\) in Theorem 2 is determined to within a null set and that the spectrum of the multiplication operator, \(x\), on \(L^2(E)\) is the set \(E^*\) of (2.5). The assertion of the corollary is that for any \(x\) in \(E^* (= \text{sp}(H) = \text{Re}(\text{sp}(T)))\), the set \(\text{sp}(T)\) contains a vertical segment (possibly a point) of length \(2b^*(x)\).

One need only note that there exists a set \(F = F_x\) such that \(F \subset E\) with the property that \(\text{meas}_1(F \cap \Delta) > 0\) for every open interval containing \(x\), and such that \(b(t) \to b^*(x)\) as \(t \to x\), \(t \in F\). Then let \(a_x\) denote, say, the essential limit superior of \(-a(t)\) at \(x\) when \(t\) is restricted to the set \(F\). The assertion (2.7) then follows from the criterion (1.12). (See also Theorem 2 and its proof in [4].)

3. **Proof of Theorem 1.** If \(\Delta\) is an open interval then relation (1.4) applied to \(T_\Delta = E(\Delta)T\delta(\Delta)\) (cf. (2.1)) yields \(\pi\|E(\Delta)D\delta(\Delta)\| \leq \text{meas}_2(\text{sp}(T_\Delta))\). But, in view of (2.3) (or even (2.2)), one has

\[ \pi\|E(\Delta)D\delta(\Delta)\| \leq \int_\Delta F(x)\,dx, \]

where

\[ F(x) = \text{meas}_1\{y : x + iy \in \text{sp}(T)\}, \]

that is, \(F(x)\) is the measure of a vertical cross section of \(\text{sp}(T)\). Hence,

\[ 2\pi \lim_{|\Delta| \to 0} \|C_\Delta\|/|\Delta| \leq F(x), \quad x \in \Delta \text{ (a.e.)}, \]

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where \( C_\Delta = E(\Delta)CE(\Delta) \).

Now, it is clear that it is sufficient to establish (1.6) if \( T \) is completely hyponormal. As already noted in §1, it can therefore be assumed that \( H \) and \( C \) are of the form (1.8). Since, by (1.8), \( (C_\Delta f, f) = \pi^{-1} \sum \lambda_j \left| (E(\Delta)f, \phi_j) \right|^2 \), it is clear that

\[
\| C_\Delta \| = \pi^{-1} \sup_{\| f \| = 1} \sum \lambda_j \left| \int_{\Delta \cap E} f(t) \overline{\phi_j(t)} dt \right|^2.
\]

Hence, if one defines \( f(t) \) on \( E \) (or, more precisely, on \( E^* \), so that, in any case, \( f \in L^2(E) \)) by \( f(t) = |\Delta \cap E|^{-1/2} \) or 0 according as \( t \in \Delta \cap E \) or \( t \notin \Delta \cap E \), then \( \| f \| = 1 \) and

\[
\pi^{-1} \sum \lambda_j \left| \int_{\Delta \cap E} \overline{\phi_j(t)} dt \right|^2 \leq \| C_\Delta \| / |\Delta|.
\]

On letting \( |\Delta| \to 0 \) and noting, by Lebesgue's metric density theorem, that \( |\Delta \cap E| / |\Delta| \to 1 \) as \( |\Delta| \to 0 \) holds a.e. on \( E \), one obtains from (3.3) and (3.5),

\[
2 \sum \lambda_j |\phi_j(x)|^2 \leq F(x) \quad \text{a.e. on } E.
\]

But

\[
2 \int_E \left( \sum \lambda_j |\phi_j(x)|^2 \right) dx = 2 \sum \lambda_j \int_E |\phi_j|^2(x) dx = 2 \pi \text{trace}(C),
\]

and

\[
\int_E F(x) dx \leq \int_{-\infty}^{\infty} F(x) dx = \text{meas}_2(\text{sp}(T)),
\]

and so (1.6) follows from (3.6).

It is clear that even

\[
\pi \text{trace}(D) \leq \int_E F(x) dx \leq \int_{\text{sp}(H)} F(x) dx = \text{meas}_2(\text{sp}(T))
\]

has been established. (Note that \( E^* = \text{sp}(H) = \text{Re}(\text{sp}(T)) \).) Thus, if the set \( \text{sp}(H) - E \) has positive measure and if \( F(x) > 0 \) on a subset of \( \text{sp}(H) - E \) also having positive measure, then the first inequality of (3.7) is sharper than (1.6).

4. Proof of Theorem 2. In view of (2.3) (or (2.2)) one has \( \text{sp}(J_{\Lambda_1}) \subset \text{sp}(J_{\Lambda_2}) \) if \( \Lambda_1 \subset \Lambda_2 \) and hence, by (2.4),

\[
F(x) = \lim_{|\Delta| \to 0} \left[ \text{meas}_1(\text{sp}(E(\Delta))J_\Delta) \right],
\]

where \( x \in \Delta \) and \( J \) is any solution of (1.2) with \( H \) and \( C \) defined by (1.8). Now, let \( J_0 \) be defined by

\[
(J_0f)(x) = -\sum \lambda_j \phi_j(x) H / \overline{\phi}_j(l), \quad f \in L^2(E),
\]
where \( H[g] \) denotes the (unitary and selfadjoint) Hilbert transform on \( L^2(-\infty, \infty) \),
\[
(4.3) \quad H[g](x) = (i\pi)^{-1} \int_{-\infty}^{\infty} g(t)(t-x)^{-1} \, dt.
\]

It will next be shown that the summation of (4.2) is strongly convergent.

In view of (3.6), each \( \phi_j \) is (essentially) bounded on \( E \), so that \( H[\phi_j] \) is defined for \( f \in L^2(E) \). (As is customary we regard \( f = 0 \) on \( (-\infty, \infty) - E \).) Let \( m < n \). Then

\[
\left| \sum_{m}^{n} \lambda_j \phi_j H[\phi_j] \right| \leq \sum_{m}^{n} \lambda_j |\phi_j|^2 \leq \text{const} \sum_{m}^{n} \lambda_j \|H[\phi_j]\|^2 \quad \text{a.e. on } E.
\]

Hence

\[
\left\| \sum_{m}^{n} \lambda_j \phi_j H[\phi_j] \right\|^2 \leq \text{const} \sum_{m}^{n} \lambda_j \|H[\phi_j]\|^2 \leq \text{const} \sum_{m}^{n} \lambda_j \|\phi_j\|^2
\]

\[
= \text{const} \int_{E} |f|^2 \left( \sum_{m}^{n} \lambda_j |\phi_j|^2 \right) dx.
\]

Since \( \sum_{m}^{n} \lambda_j |\phi_j|^2 \to 0 \) a.e. on \( E \) as \( m, n \to \infty \), it follows from (3.6) and Lebesgue’s dominated convergence theorem that the last integral tends to 0 and hence that the summation defining \( J_0 \) in (4.2), and which occurs also in (1.11), converges strongly. It is clear also from the above calculations that

\[
(4.4) \quad \|J_0\| \leq \text{ess sup}_{E} b(x).
\]

A straightforward calculation shows that \( HJ_0 - J_0 H = -iC \) where \( H \) and \( C \) are defined by (1.8). If \( J \) is any solution of (1.2), then \( J_0 - J \) commutes with \( H \) and, since \( H = x \) has a simple spectrum, \( J_0 - J = a(x) \), where \( a \in L^\infty(E) \). (See Achieser and Glasmann [1, p. 220], also Rosenblum [19, p. 326].) This establishes (1.11). That \( b(x) \leq \text{const} < \infty \) a.e. follows from (3.6); since \( T \) is completely hyponormal, \( b(x) > 0 \) a.e. on \( E \) and so (1.10) holds.

Next, let \( F_0(x) \) denote the measure of the vertical cross section of the spectrum of \( T_0 = H + iJ_0 \), where \( H \) and \( J_0 \) are defined by (1.8) and (4.2), so that

\[
(4.5) \quad F_0(x) = \text{meas}_{1} \{ y : x + iy \in \sigma(T_0) \}, \quad T_0 = x + iJ_0.
\]

It follows from (4.4) applied to \( E(\Delta)J_0E(\Delta) \) that

\[
(4.6) \quad \|E(\Delta)J_0E(\Delta)\| \leq \text{ess sup}_{\Delta} b(t).
\]

Consequently, relation (2.4) implies that

\[
(4.7) \quad \{ y : x + iy \in \sigma(T_0) \} \subseteq \{ -b^{*}(x), b^{*}(x) \},
\]

where \( b^{*}(x) = \text{ess lim sup}_{t\to x} b(t) \) is defined on \( E^{*} \) by (2.6) with \( c(x) = b(x) \) on
E. Since, by (3.6) with $F$ replaced by $F_0$, $2b(x) \leq F_0(x)$ a.e. on $E$, then $2b^*(x) \leq F_0^*(x)$ on $E^*$. By (4.7), $F_0^*(x) \leq 2b^*(x)$ on $E^*$. But $F_0^*(x)$ is upper semicontinuous and hence $F_0^*(x) \leq F_0^*(x)$ for all $x$ on $(-\infty, \infty)$. It then follows that

$$F_0(x) = 2b^*(x) \text{ for all } x \text{ in } E^*.$$  

Relations (4.7) and (4.8) now yield

$$\{y : x + iy \in \text{sp}(T_0)\} = [-b^*(x), b^*(x)], \quad x \in \text{Re}\,(\text{sp}(T_0)).$$

Next, we prove the last part of Theorem 2 concerning the spectrum of $T$. First, let $z = s + it$ be a number for which the assertion of Theorem 2 concerning (1.12) fails to hold. It will be shown that $z$ is not in $\text{sp}(T)$. There exist some open interval $\Lambda$ containing $s$ and some number $\epsilon > 0$ such that, for almost all $x$ in $E \cap \Lambda$, either $b(x) + \epsilon \leq -a(x) - t$ or $b(x) + \epsilon \leq a(x) + t$, so that

$$0 < b(x) + \epsilon \leq |a(x) + t| \text{ for a.a. } x \text{ on } \Lambda \cap E.$$

Now, if $s + it \in \text{sp}(T)$ then, by (2.4), there exist $f_n = E(\Lambda)/f_n$ in $L^2(E)$, where $\|f_n\| = 1$, such that $\|E(\Lambda)/f_n - f_n\| \to 0$ as $n \to \infty$. But $a(x) + t = J_0 - (J - t)$ and hence

$$\int_\Lambda (a(x) + t)^2|f_n(x)|^2\,dx = \|E(\Lambda)(J_0 - (J - t))f_n\|^2 = \|E(\Lambda)f_n\|^2 + o(1)$$

as $n \to \infty$. It follows from (4.6) and (4.10) that $\text{ess sup}_\Lambda b + \epsilon \leq \text{ess sup}_\Lambda b$, a contradiction. This proves the "only if" part of the assertion of Theorem 2 concerning (1.12).

Next, let $z = s + it$ be a number for which (1.12) holds for any $\epsilon > 0$ and every open interval $\Lambda$ containing $s$. It will be shown that $z \in \text{sp}(T)$. To this end, first note that there exists a set $F \subseteq E$ such that $\text{meas}_1(F \cap \Lambda) > 0$ and $-a(x) - b(x) - \epsilon < t < -a(x) + b(x) + \epsilon$ holds for a.a. $x$ in $F \cap \Lambda$, where $\Lambda$ is any open interval containing $s$. Clearly, for every $\delta > 0$, there exist a set $G = G_\delta \subseteq F$ and a constant $\lambda$ (e.g., with $\lambda = \text{essential limit superior of } a(x)$ at $s$ where $x$ is restricted to $F$) such that

$$|a(x) - \lambda| < \delta \text{ on } G \cap \Lambda \text{ and } \text{meas}_1(G \cap \Lambda) > 0.$$

Then $-\lambda - \delta - b(x) - \epsilon < t < -\lambda + \delta + b(x) + \epsilon$ on $G \cap \Lambda$, that is,

$$-b(x) - (\delta + \epsilon) < t + \lambda < b(x) + (\delta + \epsilon) \text{ on } G \cap \Lambda.$$

Now, it follows from (4.9) and the projection properties of the spectra of hyponormal operators that

$$\text{sp}(J_0) = \left[-\text{ess sup}_E b(x), \text{ess sup}_E b(x)\right].$$

If this result is applied to $E(G \cap \Lambda) J_0 E(G \cap \Lambda)$ it is seen that
(4.14) \[
\text{sp}(E(G \cap \Delta)J_0E(G \cap \Delta)) = \left[\text{ess sup}_{Gn\Delta} b, \text{ess sup}_{G\Delta} b\right].
\]

Hence, by (4.12), there exist a real number \( \mu \) and unit vectors \( f_n = E(G \cap \Delta)f_n \) for which, as \( n \to \infty \),

(4.15) \[
(E(G \cap \Delta)f_nE(G \cap \Delta) - \mu)f_n \to 0 \quad \text{where } |\mu - (t + \lambda)| \leq \delta + \epsilon.
\]

It follows from (4.11) that, for large \( n \),

(4.16) \[
\|J(a - \lambda)f_n\| = \left( \int (a(x) - \lambda)^2|f_n|^2\,dx \right)^{1/2} < \delta.
\]

Since \( J = J_0 - a(x) \), it follows from (4.15) and (4.16) that

(4.17) \[
\|[E(G \cap \Delta)\delta E(G \cap \Delta) - t]/f_n\| \leq 2\delta + \epsilon + o(1) \quad \text{as } n \to \infty.
\]

It follows from (4.17) that

\[
\|[E(G \cap \Delta)\delta E(G \cap \Delta) - (s + it)]/f_n\| \leq |\Delta| + 2\delta + \epsilon + o(1), \quad n \to \infty.
\]

Now, it is clear from [17] that the relation (2.2) holds if \( \Lambda \) is replaced by an \( E_\lambda \)-measurable (hence, in the present case, Lebesgue measurable) set, so that, in particular,

(4.18) \[
\text{sp}(E(G \cap \Delta)\delta E(G \cap \Delta)) \subset \text{sp}(T).
\]

But, for an arbitrary hyponormal operator \( A \) on a Hilbert space, \( \|Ax\| \geq \text{dist}(0, \text{sp}(A))\|x\| \) for all \( x \) in the space. Therefore, there exists a number \( z_\Delta \) in \( \text{sp}(E(G \cap \Delta)\delta E(G \cap \Delta)) \), hence in \( \text{sp}(T) \), such that \( |z_\Delta - (s + it)| \leq |\Delta| + 2\delta + \epsilon \).

Since \( |\Delta| \), \( \delta \) and \( \epsilon \) can be chosen arbitrarily small, there exist \( z_n \) \((n = 1, 2, \ldots)\) in \( \text{sp}(T) \) for which \( z_n \to s + it \) as \( n \to \infty \). Hence, \( s + it \) belongs to \( \text{sp}(T) \) as was to be shown.

5. Remarks. It may be noted that Kato has determined necessary and sufficient conditions that, for a given \( H \) and \( C \), the equation (1.2) have a solution \( J \); see [7, p. 552] and [8, p. 537 ff]. In particular, when \( C \geq 0 \) and \( H \) has a simple spectrum, his special solutions, corresponding to the "canonical" \( J_0 \) in (4.2) above, are

\[
M^+ = - \int_0^{t_0} e^{itx}Ce^{-itx}dt
\]

(the integrals converging strongly).

To see the connection with, say \( M^+ \), note that if \( C \) is given by (1.8) then a straightforward calculation shows that: \( M^+f = -2\Sigma \lambda_j\phi_j(\phi_j)^* \), where, for \( g \in L^2(-\infty, \infty) \),

\[
g^+(x) = (2\pi)^{-1/2} \int_0^{\infty} g^*(y)e^{ixy}\,dy
\]
and $\hat{g}(y)$ is the Fourier transform of $g$. If

$$g^+(x) = (2\pi)^{-1/2} \int_{-\infty}^{0} \hat{g}(y)e^{ixy} \, dy,$$

then $g = g^+ + g^-$ and $H[g] = g^+ - g^-$. (See Titchmarsh [20], Hilgevoord [6]; for an application to commutators and singular integral operators, see Putnam [16].)

Thus, $g = \frac{1}{2}(H + i)g$. It is seen therefore that

$$M^+f = -\sum \lambda_j |\phi_j|^2/ (H + i)\phi_j = J_0f - \sum \lambda_j |\phi_j|^2f = (J_0 - b(x))f.$$

It is easy to determine the spectra of the operators $T_0 = H + iJ_0$ ($H = x$) and $T^+ = H + iM^+$ from Theorem 2. First, note that both $T_0$ and $T^+$ are completely hyponormal (cf. the remark after Theorem 2). If $E^*$ is defined by (2.5) then

$$\text{sp}(T_0) = \{x + iy : x \in E^*, -b^*(x) < y < b^*(x)\},$$

and

$$\text{sp}(T^+) = \{x + iy : x \in E^*, -2b^*(x) < y < 0\}.$$

6. Some applications of Theorems 1, 2 will be obtained below. Consider the special case in which $T$ satisfies (1.1) and (1.5) and in which the first inequality of (3.7) is an equality, so that

$$\pi \text{trace}(D) = \int_E F(x) \, dx.$$

Thus, $\pi \text{trace}(D)$ equals the measure of that part of $\text{sp}(T)$ lying over the set $E$. In addition, suppose that $T$ is completely hyponormal, so that $T$ is unitarily equivalent to (and will, up to but not including the statement of Theorem 5 below, simply be taken to be equal to) $H + iJ$ on $L^2(E)$ defined by (1.8)–(1.11).

It is clear that equality holds a.e. in (3.6), thus

$$2b(x) = F(x) \quad \text{a.e. on } E,$$

where $b(x)$ is defined in (1.10). Now, by Theorem 2 and Corollary 2 to Theorem 2, one has in general

$$2b^*(x) = F_0(x) \leq F(x) \quad \text{for all } x \in E^*,$$

where $F(x)$ and $F_0(x)$ are defined by (3.2) and (4.5). Hence, by (6.2) and (6.3), $b^*(x) \leq b(x)$ a.e. on $E$. But $b(x) \leq b^*(x)$ a.e. on $E$, so that

$$2b^*(x) = F_0(x) = F(x) = 2b(x) \quad \text{a.e. on } E.$$

Since $F(x)$ is upper semicontinuous, then $F(x)$ is continuous except possibly on a set of the first category; cf. Goffman [5, p. 110]. It is possible of course that $E$ itself is of the first category so that the continuity of $b(x)$ at...
a point of $E$ cannot be inferred. In any case, suppose, in addition to (6.1), that
\[ F(x) \text{ is continuous a.e. on } E. \]
Then it is easy to see that the function $a(x)$ of (1.11) is (essentially) continuous a.e. on $E$, that is,
\[ \text{ess lim sup}_{t \to x} a(t) = \text{ess lim inf}_{t \to x} a(t) \quad \text{a.e. on } E, \]
the second expression being defined by (2.6) with "sup" replaced by "inf."
For, if (6.6) does not hold, then by (6.4) and (6.5), there exists a set $P \subseteq E$ of positive measure such that, for $c$ in $P$, $b(x)$ is continuous, $F(c) = 2b(c) > 0$ and $\alpha = \text{ess lim sup}_{t \to c} a(t) > \beta = \text{ess lim inf}_{t \to c} a(t)$. It follows from the last part of Theorem 2, however, that each of the vertical segments $\{x + iy : -\alpha - b(c) < y < -\alpha + b(c)\}$ and $\{x + iy : -\beta - b(c) < y < -\beta + b(c)\}$ belongs to $\text{sp}(T)$, and hence, in particular, $F(c) > 2b(c)$ on $P$, a contradiction to (6.4).

These results can be summarized as follows:

**Theorem 3.** Suppose that $T = H + ij$ on $L^2(E)$ is defined by (1.7)-(1.11). In addition, suppose that (6.1) and (6.5) hold. Then, in the definition (1.11) of $J$, both $a(x)$ and $b(x)$ can be assumed to be continuous a.e. on $E$. Further, at all points $x$ in $E$ where $a(x)$ and $b(x)$ are continuous,
\[ \text{sp}(T) \cap \{z : \text{Re}(z) = x\} = \{x + iy : -a(x) - b(x) < y < -a(x) + b(x)\}. \]

The assertion (6.7), which follows immediately from (1.12), is simply that the spectrum of $T$ lying over any point $x$ in $E$ at which both $a(x)$ and $b(x)$ are continuous is a closed interval centered at $x - ia(x)$ and of length $2b(x)$. The functions $a(x)$ and $b(x)$ are thus uniquely determined a.e. on $E$ by $\text{sp}(T)$.

**Theorem 4.** Suppose that $T = H + ij$ on $L^2(E)$ is defined by (1.7)-(1.11) and that $F(x)$ of (3.2) satisfies
\[ F(x) > 0 \quad \text{and is continuous for a.a. } x \text{ in } \text{sp}(H) (= \text{Re}(\text{sp}(T))). \]
In addition, suppose that equality holds in (1.6), so that
\[ \pi \text{trace}(D) = \text{meas}_2(\text{sp}(T)). \]
Then, in the definition (1.11) of $J$, $E = \text{sp}(H)$, and both $a(x)$ and $b(x)$ can be assumed to be continuous a.e. on $\text{sp}(H)$. Also, for all $x$ in $\text{sp}(H)$ at which both $a(x)$ and $b(x)$ are continuous, relation (6.7) holds.

It is clear that the hypotheses of Theorem 4 are stronger than those of Theorem 3. That $\text{Re}(\text{sp}(T)) = \text{sp}(H)$ is simply the projection property (1.3). Since $F(x) > 0$ on $\text{sp}(H)$ it follows from (3.7) and (6.9) that $E = \text{sp}(H)$ (to within a null set) and the proof of Theorem 4 is complete.
Theorem 5. Let $T = H + iJ$ satisfy (1.1) and (1.5) and suppose that $T$ is completely hyponormal. In addition, suppose that equality holds in (1.4), that is,

$$\pi\|D\| = \text{meas}_2(\text{sp}(T)),$$

and, further, that (6.8) holds, where $F(x)$ is defined by (3.2). Then $T$ is unitarily equivalent to the operator $T_1$ on $L^2(\text{sp}(H))$ ($\text{sp}(H) = \text{Re}(\text{sp}(T))$) defined by

$$(6.11) \quad (T_1f)(x) = xf(x) - i\left[ a(x)f(x) + (i\pi)^{-1}b^{1/2}(x) \int_{\text{sp}(H)} f(t)b^{1/2}(t)(t - x)^{-1} \, dt \right],$$

where $2b(x) = F(x)$ and $a(x)$ are continuous a.e. on $\text{sp}(H)$. Also for all $x$ in $\text{sp}(H)$ at which both $a(x)$ and $b(x)$ are continuous, relation (6.7) holds.

It follows from the Corollary of Theorem 1 and the complete hyponormality of $T$ that rank($C$) (= rank($D$)) = 1. Thus (6.10) reduces to (6.9). It then follows from Theorem 4 that $T$ is unitarily equivalent to $T_2$ on $L^2(\text{sp}(H))$ where

$$(6.12) \quad (T_2f)(x) = xf(x) - i\left[ a(x)f(x) + (i\pi)^{-1} \int_{\text{sp}(H)} f(t)\phi(t)(t - x)^{-1} \, dt \right],$$

where $a, \phi \in L^\infty(\text{sp}(H))$ and $b(x) = |\phi(x)|^2 > 0$ on $\text{sp}(H)$, and $a(x), b(x)$ can be taken to be continuous a.e. on $\text{sp}(H)$. (Note that (6.2) holds.) But $b^{1/2}(x) = m(x)\phi(x)$, where $m(x)$ is measurable on $E$ and $|m(x)| = 1$. Since the unitary operator $U: f(x) \rightarrow m(x)f(x)$ of $L^2(E)$ onto itself obviously commutes with $x$ and $a(x)$, it follows that $T_2$, hence also $T$, is unitarily equivalent to $T_1$ of (6.11). That (6.7), with $T$ replaced by $T_1$, holds is clear from Theorem 3 and the proof of Theorem 5 is now complete.

It is seen that $\text{sp}(T)$ is a complete unitary invariant for operators $T$ satisfying the hypotheses of Theorem 5. (Concerning complete unitary invariants for hyponormal operators under other hypotheses, see Pincus [11, Theorem 22]; also [12].) As a simple application, one has the following

Corollary of Theorem 5. Let $T$ be isometric and completely hyponormal, and suppose that $\frac{1}{2}(T + T^*)$ has a simple spectrum. Then $T$ is unitarily equivalent to the unilateral shift.

Since the closed unit disk is the spectrum of both $T$ and the unilateral shift, it is easily verified (see also the beginning of §2 above) that all hypotheses of Theorem 5 are satisfied by both operators.

7. The assertions concerning the singular integral operator $J$ of (1.11) can be generalized to the case where $E$ need not satisfy (1.7) but, more generally, is subject only to
Further, it will no longer be supposed that \( \{ \phi_j \} \) is an orthonormal system.

**Remark.** The above-mentioned orthonormality hypothesis was used in Theorem 1. It could have been omitted in Theorem 2 however (cf. below) if, say, relation (1.10) was simply hypothesized.

For \( k = 1, 2, \ldots \), let \( b_k \in L^\infty(E) \), where \( E \) satisfies (7.1). In addition, suppose that
\[
(7.2) \quad A(x) \text{ is a real-valued, measurable function on } E,
\]
and that
\[
(7.3) \quad 0 < B(x) \leq \text{const (} < \infty \text{) a.e. on } E, \quad \text{where } B(x) = \sum |b_j(x)|^2.
\]
Define the singular integral operator \( L \) on \( L^2(E) \) by
\[
(7.4) \quad (Lf)(x) = - \left[ A(x)f(x) + (i\pi)^{-1} \sum b_j(x) \int_E f(t) \overline{b}_j(t)(t-x)^{-1} dt \right],
\]
that is, \( L = L_0 - A \), where
\[
(7.5) \quad (L_0f)(x) = - \sum b_j(x) H[\overline{\phi}_j](x), \quad f \in L^2(E).
\]
An argument similar to that used in the beginning of §4, but with \( \lambda_j^{1/2} \phi_j \) replaced by \( b_j \), shows that the summation of (7.4) converges strongly. (Note however that the \( b_j \) are in \( L^\infty(E) \) but not necessarily in \( L^2(E) \).) It follows that \( L_0 \) of (7.5) is bounded and selfadjoint on \( L^2(E) \) and that (cf. (4.4))
\[
(7.6) \quad \|L_0\| \leq \text{ess sup}_E B(x).
\]

The multiplication operator \( A(x) \) is clearly selfadjoint (but not necessarily bounded) and so \( L \) of (7.4) is a selfadjoint, in general, unbounded operator on \( L^2(E) \). It follows from [16] that \( L \) is absolutely continuous. (If \( A(x) \) is also bounded from below and if the summation of (7.4) reduces to a single term, this result, and, in fact, a complete spectral analysis, was obtained by Rosenblum [19]. He also treats the case, again for the single integral operator, where \( E = (-\infty, \infty) \) and in which eigenvalues may occur.) It will be shown below that the methods of [16] can be used, at least if
\[
(7.7) \quad A \in L^\infty(E),
\]
to obtain for \( L \) an analogue (and generalization) of the assertion of Corollary 1 of Theorem 2 for \( J \). It is clear that if (7.7) holds then \( L \) of (7.4) is bounded on \( L^2(E) \). However, it is clear that if the set \( E \) is not (essentially) bounded, then the selfadjoint multiplication operator \( x \), hence also the operator \( x + iL \), is
unbounded on $L^2(E)$. It turns out however that $x$ can be replaced by another multiplication operator $c(x) \in L^\infty(E)$ and such that $c + iL$ is hyponormal. This fact will be used to obtain the following

**Theorem 6.** Assume conditions (7.1), (7.3) and (7.7) and define the (bounded) selfadjoint operator $L$ on $L^2(E)$ by (7.4). Then a real number $t$ is in $\text{sp}(L)$ if and only if

$$\text{meas}_1 \{x \in E : -A(x) - B(x) - \epsilon < t < -A(x) + B(x) + \epsilon\} > 0$$

holds for every $\epsilon > 0$.

**Remark.** The minus sign in (7.4) is used, as in the definition of $J$ in (1.11), for convenience in regarding $L$ as the imaginary part of a certain hyponormal operator. Note that the spectrum of $-L$ can be obtained from Theorem 6 simply by replacing $-A(x)$ by $A(x)$ in the measure condition.

8. Proof of Theorem 6. It was shown in [16, Lemma 2] that, for any set $E$ satisfying (7.1), there exists a real-valued function $\psi(x)$ on $(-\infty, \infty)$, depending on $E$ but independent of $L$ in (7.4), for which

$$0 < \psi(x) \leq \text{const} < \infty \quad \text{on} \quad (-\infty, \infty) - E,$$

$$\psi(x) = 0 \quad \text{on} \quad E, \quad \psi \in L^2(-\infty, \infty) \quad \text{and}$$

$$|H[\psi](x)| \leq \text{const} < \infty \quad \text{on} \quad (-\infty, \infty),$$

where $H[f]$ denotes the Hilbert transform of (4.3). Further (cf. §3 of [16]), if $c(x) = iH[\psi](x)$, so that $c(x)$ is real, then, regarding $c$ as a selfadjoint operator on $L^2(E),

$$(8.2) \quad cL - Lc = -iG, \quad G \geq 0$$

(that is, $S = c + iL$ is hyponormal) and

$$0 \not\in \text{point spectrum of } G. \quad (8.3)$$

Since $G \geq 0$ (for any $L$, in particular for $b_1 = 1$ and $b_k = 0$ for $k = 2, 3, \ldots$) then $k(x, t) = \pi^{-1}[c(x) - c(x)](t - x)^{-1}$ is the kernel of a (bounded) nonnegative integral operator $K$ on $L^2(E)$. In fact, $(Kf)(x) = iH[c\gamma f](x) - ic(x)H[f\gamma](x)$, where $H$ is the Hilbert transform of (4.3) and $f \in L^2(E)$. That $k(x, t)$ is (essentially) bounded on $E \times E$ follows from the boundedness of the operator $K$. As noted in [16, p. 459], one has the representation

$$(8.4) \quad \frac{[c(t) - c(x)](t - x)^{-1}}{} = \sum c_j(x)\bar{c}_j(t) \quad \text{for a.a. } x, t \quad (x \neq t) \quad \text{in } E,$$

where $\sum |c_j(x)|^2 \leq \text{const} < \infty$ a.e. on $E$. 
In the end of the proof of Lemma 2 of [16], and in the notation of that paper, the following correction may be noted. The functions \( b(z) \) and \( k(z) \) satisfy \( k(z) \equiv b(z) + \text{const} \) (rather than \( k(z) \equiv b(z) \)) and one may conclude that \( r(x) \equiv q(x) + \text{const} \) and hence \( H[p](x) = i[q(x) + \text{const}] \). It is readily verified that, in fact, const = \( \frac{1}{2} \).

It follows (cf. [16, pp. 456, 458]) that the function \( c(x) = iH[\psi](x) \) can then be chosen as

\[
\tag{8.5} c(x) = (e^u \cos u + 1)/(e^{2u} + 2e^u \cos u + 1) - \frac{1}{2},
\]

where

\[
\tag{8.6} u(x) = \begin{cases} 
(\pi/4) \exp(-x^2) & \text{if } x \notin E \\
0 & \text{if } x \in E
\end{cases} \quad \text{and} \quad \nu(x) = -iH[u](x).
\]

Thus,

\[
\tag{8.7} c(x) = (e^u + 1)^{-1} - \frac{1}{2}, \quad x \in E.
\]

If one restricts the quantities of (8.4) only to those \( x, t \) in \( E \) for which the asserted relations hold (i.e., one avoids an exceptional null set), then \( c'(x) = \lim_{t \to x} [c(t) - c(x)](t - x)^{-1} \quad (t \in E) \) exists a.e. on \( E \). It will next be shown that

\[
\tag{8.8} c'(x) = \sum |c_j(x)|^2 \quad \text{a.e. on } E.
\]

To this end, note that, for almost all \( x \), \( c'(x) = \Sigma c_j(x) \overline{c}_j(i) + b_x(i) \), where (for \( x \) fixed) \( b_x(i) \to 0 \) as \( t \to x \). Let \( \delta \) be an open interval containing \( x \) and let \( Q = \delta \cap E \). Then the Lebesgue density is 1 (that is, \( |Q| |\delta|^{-1} \to 1 \) as \( |\delta| \to 0 \)) for almost all \( x \in E \), and, in particular, \( |Q| > 0 \). Choose \( x \) to be such a point and for which \( c'(x) \) exists. Then

\[
c'(x) = |Q|^{-1} \int_Q c'(x) dt = |Q|^{-1} \int_Q \Sigma c_j(x) \overline{c}_j(i) dt + |Q|^{-1} \int_Q b_x(i) dt.
\]

The last term tends to 0 as \( |\delta| \to 0 \) and so

\[
c'(x) = \lim_{|\delta| \to 0} |Q|^{-1} \int_Q \Sigma c_j(x) \overline{c}_j(i) dt.
\]

The Schwarz inequality and the boundedness of \( \Sigma |c_j(x)|^2 \) on \( E \) make it clear that the integral and limit signs may be moved inside the summation, so that

\[
c'(x) = \sum c_j(x) \left( \lim_{|\delta| \to 0} |Q|^{-1} \int_Q \overline{c}_j(i) dt \right),
\]

and (8.8) follows.

Let \( c(x) \), as an operator on \( L^2(E) \), have the spectral resolution
Then, for any open interval $\Delta$ and any $f \in L^2(E)$, $E(\Delta)f = f(x)$ if $x \in M(\Delta)$, where $M(\Delta) = \{ t \in E: c(t) \in \Delta \}$, and $E(\Delta)f = 0$ otherwise. In view of (8.3), it follows from [15, p. 42], that the operator $c$ of (8.6) (and (8.9)) is absolutely continuous on $L^2(E)$ and hence $c'(x) > 0$ a.e. on $E$. Since $E$ has Lebesgue density 1 a.e. on $E$ then clearly

(8.10) both $c'(x) > 0$ and $E$ has density 1 at $x$ hold a.e. on $E$.

Let $x$ satisfy (8.10) and let $\delta$ be any open interval containing $x$. By (8.10), \[ \text{ess inf}_Q c(t) < \text{ess sup}_Q c(t), \] where $Q = \delta \cap E$; let $\Delta = (\text{ess inf}_Q c, \text{ess sup}_Q c)$. Clearly, $0 < |Q| \to 0$ as $|\delta| \to 0$.

If $f(t) = |Q|^{-1/2}$ or 0 according as $t \in Q$ or $t \in E - Q$, then it is clear that

\[
\left( \frac{G_{\Delta}f}{|\Delta|} \right) = \pi^{-1}|Q||\Delta|^{-1} \sum_j \left| Q \right|^{-1} \int_Q b_j(t) c_k(t) \, dt \right|^2,
\]

where $G_\Delta = E(\Delta) G E(\Delta)$. But $|Q||\Delta|^{-1} = |Q||\delta|^{-1}|\delta||\Delta|^{-1} \to 1/c'(x) > 0$ as $|\delta| \to 0$, and hence

(8.12) \[ \limsup_{|\Delta| \to 0} \frac{\|G_{\Delta}\|}{|\Delta|} \geq \pi^{-1} \sum |b_j(x)|^2 \text{ for a.a. } x \in E,
\]

where $c(x) \in \Delta$. An argument similar to that used in §3 yields

(8.13) \[ 2B(x) \leq F(c(x)) \text{ a.e. on } E,
\]

where

(8.14) \[ F(X) = \text{meas } \{ y: X + iy \in \text{sp}(S) \}, \quad S = c + iL.
\]

For any function $g(x)$ defined on $E$ and any open interval $\Delta$, let $g_\Delta = \text{ess sup}_{M(\Delta)} g(x)$, where $M(\Delta) = \{ x \in E: c(x) \in \Delta \}$. Then define $g^*(X)$ on the essential range, $R$, of $c$ on $E$ by $g^*(X) = \lim_{|\Delta| \to 0} g_\Delta$, $X \in \Delta$.

Next, let

(8.15) \[ F_0(X) = \text{meas } \{ y: X + iy \in \text{sp}(S_0) \}, \quad S_0 = c + iL_0,
\]

where $L_0$ is defined by (7.5). It follows from (7.6) applied to $E(\Delta)L_0$ that $\|E(\Delta)L_0E(\Delta)\| \leq B_\Delta$. It follows from the Lemma of §2 applied now to $S_0 = c + iL_0$ that

(8.16) \[ \{ y: X + iy \in \text{sp}(S_0) \} \subset [-B^*(X), B^*(X)].
\]

(Note (cf. (1.3)) that $\text{Re(} \text{sp}(S_0)) = \text{sp}(c) = R$.)
By (8.13), with $F$ replaced by $F_0$, we have $2B^*(X) \leq F_0^*(X)$ on $R$. Since $F_0(X)$ is upper semicontinuous, then $F_0^*(X) \leq F_0(X)$ for all $X$. Also, by (8.16), $F_0(X) \leq 2B^*(X)$ on $R$. Thus, $F_0(X) = B^*(X)$ on $R$, and (8.16) now implies that

$$\{y : x + iy \in \text{sp}(S_0)\} = [-B^*(X), B^*(X)] \quad \text{for } X \in \text{Re}(\text{sp}(S_0)).$$

It follows from (1.3) that $\text{sp}(L_0) = [-m, m]$, where $m = \text{ess sup}_E B(x)$, so that Theorem 6 is proved in the special case when $L = L_0$. The proof for the general operator $L = L_0 - A$ of (7.4) is similar to the argument given in §4 following (4.1) and will be omitted.

REFERENCES


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