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HYPONORMAL OPERATORS HAVING REAL PARTS WITH SIMPLE SPECTRA[1]

BY

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ABSTRACT. Let $T^*T - TT^* = D \geq 0$ and suppose that the real part of T has a simple spectrum. Then D is of trace class and π trace(D) is a lower bound for the measure of the spectrum of T. This latter set is specified in terms of the real and imaginary parts of T. In addition, the spectra are determined of self-adjoint singular integral operators on $L^2(E)$ of the form $A(x)f(x) + \sum b_j(x)H[f\bar{b}_j](x)$, where $E \neq (-\infty, \infty)$, $A(x)$ is real and bounded, $\sum|b_j(x)|^2$ is positive and bounded, and H denotes the Hilbert transform.

1. A bounded operator T on a Hilbert space $\mathfrak{H}$ (which in this paper will be assumed to be separable) is said to be hyponormal if

$$T^*T - TT^* = D \geq 0, \tag{1.1}$$

or, equivalently, if T has the Cartesian form $T = H + iJ$,

$$HJ - JH = -iC, \quad C = \tfrac{1}{2}D \geq 0. \tag{1.2}$$

In this case, the spectra of H and J are the (real) projections of the spectrum of T onto the coordinate axes, thus

$$\mathrm{sp}(H) = \mathrm{Re}(\mathrm{sp}(T)) \quad \text{and} \quad \mathrm{sp}(J) = \mathrm{Im}(\mathrm{sp}(T)); \tag{1.3}$$

see Putnam [15, p. 46]. It was shown in Putnam [17] that if T is hyponormal then

$$\pi\|D\| \leq \mathrm{meas}_2(\mathrm{sp}(T)); \tag{1.4}$$

in the particular case in which D is completely continuous, the inequality (1.4) was proved by Clancey [2]. If T is hyponormal and if its real part, H, satisfies

$$H = \tfrac{1}{2}(T + T^*) \quad \text{has a simple spectrum,} \tag{1.5}$$

or, more generally, if H has finite spectral multiplicity, then C belongs to trace class; Kato [8]. In general, the inequality (1.4) is nontrivially optimal (e.g., equality holds if T is the unilateral shift). In certain instances, however, a sharpening of the inequality can be obtained as in the following

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Theorem 1. *Let T satisfy* (1.1) *and* (1.5). *Then*

$$\pi \operatorname{trace}(D) \leq \operatorname{meas}_2(\operatorname{sp}(T)). \tag{1.6}$$

An operator T satisfying (1.1) is said to be completely hyponormal if there is no (nontrivial) subspace reducing T and on which T is normal. In case T is completely hyponormal then its real and imaginary parts must be absolutely continuous; see [15, p. 42]. It is well known that if a bounded selfadjoint operator H has a simple spectrum and is absolutely continuous, then there is a set E, where

(1.7) $\qquad E$ is a bounded subset of $(-\infty, \infty)$ of positive measure,

such that H is unitarily equivalent to the coordinate multiplication operator, x, on $L^2(E)$. The next theorem concerns completely hyponormal operators $T = H + iJ$ for which (1.5) holds. Without loss of generality it can be supposed therefore that

$$(Hf)(x) = xf(x) \quad \text{and} \quad (Cf)(x) = \pi^{-1}\sum \lambda_j (f, \phi_j)\phi_j(x), \qquad f \in L^2(E), \tag{1.8}$$

where

(1.9) $\{\phi_1, \phi_2, \cdots\}$ is orthonormal on $L^2(E)$; $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ and $\sum \lambda_j < \infty$.

There will be proved the following

Theorem 2. *Let T of* (1.1) *be completely hyponormal and suppose that* (1.7), (1.8) *and* (1.9) *hold. Then*

$$0 < b(x) \leq \text{const} \ (< \infty) \quad a.e. \text{ on } E, \quad \text{where } b(x) = \sum \lambda_j |\phi_j(x)|^2. \tag{1.10}$$

Further, J of (1.2) *is given by*

$$(Jf)(x) = -[a(x) + (i\pi)^{-1}\sum \lambda_j \phi_j(x) \int_E f(t)\bar{\phi}_j(t)(t-x)^{-1}dt], \qquad a(x) \text{ real}, \ \in L^\infty(E), \tag{1.11}$$

where the summation operator is the strong limit of its partial sums. Also, $z = s + it$ (s, t real) is in $\operatorname{sp}(T)$ *if and only if*

$$\operatorname{meas}_1 \{x \in E \cap \Delta : -a(x) - b(x) - \epsilon < t < -a(x) + b(x) + \epsilon\} > 0 \tag{1.12}$$

holds whenever $\epsilon > 0$ and Δ is any open interval containing s.

Remark. Conversely, if E is any set satisfying (1.7) and if the selfadjoint operators H and J are defined as above by (1.8)–(1.11), then it is easily verified that (1.2) holds with C defined by (1.8), so that T is hyponormal. (Cf. §4, below, and note that $J = -a + J_0$, where J_0 is defined by (4.2).) That, in fact, T is completely hyponormal can be seen as follows. Let $\mathfrak{H}_1$ $(\neq 0) \subset L^2(E)$ reduce T, hence also H and J, and suppose that T is normal on $\mathfrak{H}_1$. Then $Cf = 0$ for all f in $\mathfrak{H}_1$. But if $f_1 \in \mathfrak{H}_1$ and $f_1 \neq 0$, then the set $\{x: f_1(x) \neq 0\}$ has positive measure. Since $\Sigma\lambda_j|\phi_j(x)|^2 > 0$ on E then there exists some ϕ_k such that $\{x: \phi_k \neq 0\} \cap \{x: f_1(x) \neq 0\}$ has positive measure. But $p(x)/f_1(x)$ also belongs to $\mathfrak{H}_1$ for any polynomial $p(x)$, and it is clear from the Weierstrass approximation theorem that there must exist some g in $\mathfrak{H}_1$ such that $(g, \phi_k) \neq 0$. Since $Cg = 0$, then $\phi_k = \Sigma_{i \neq k} a_i \phi_i$ $(\Sigma|a_i|^2 < \infty)$, in contradiction to the supposed orthonormality of $\{\phi_i\}$, $i = 1, 2, \cdots$.

As noted above, both operators H and J satisfying (1.8)–(1.11) are absolutely continuous. Also, concerning Theorem 2, see the Remark at the beginning of §7 below. In case $\operatorname{rank}(D)$ $(= \operatorname{rank}(C)) = 1$, the assertion (1.11) concerning the form of J is due to Xa Dao-xeng [21]; for a simpler proof, using a result in [14], see Rosenblum [19, p. 326].

Some corollaries of Theorems 1, 2 together with a lemma and some remarks will be stated in §2. The proofs of Theorems 1, 2 will be given in §§3, 4 respectively. A connection between certain operators considered by Kato [8] and those in the present paper will be discussed in §5. In §6, some applications of Theorems 1, 2 will be made and the results stated as Theorems 3–5. §§7, 8 will deal with generalized selfadjoint singular integral operators.

2. If $T = V$, the unilateral shift, then $\operatorname{sp}(V)$ is the closed unit disk and (1.4) becomes an equality. Since V is isometric (hence hyponormal), so are its powers V^n, $n = 1, 2, \cdots$. Further, it is easily verified that $(V^n)^* V^n - V^n (V^n)^*$ has norm $= 1$ and trace $= n$. Since $\operatorname{sp}(V^n) = \operatorname{sp}(V)$ then (1.5), with $T = V^n$, holds only if $n = 1$ (although even equality holds in (1.4) for all n). It follows from Theorem 1 that $\operatorname{Re}(V^n) = \frac{1}{2}(V^n + V^{n*})$ does not have a simple spectrum for $n \geq 2$. That, incidentally, $\frac{1}{2}(V + V^*)$ does have a simple spectrum is easily verified directly. In fact, on $l^2 = \{(x_1, x_2, \cdots): \Sigma|x_j|^2 < \infty\}$, $V(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ and $(1, 0, 0, \cdots)$ is readily seen to be a cyclic vector of $\frac{1}{2}(V + V^*)$.

More generally, one has the following obvious

Corollary of Theorem 1. *If T satisfies* (1.1), *if equality holds in* (1.4), *and if* (1.5) *holds, then* $\operatorname{rank}(D) \leq 1$.

An explicit formulation of Theorem 2 in the special case in which C of (1.8) has rank 1, so that $Cf = \lambda_1(f, \phi_1)\phi_1$, occurs in Clancey and Putnam [4]. Another way of giving the spectrum of T, somewhat different from that specified in Theorem 2 using (1.12), and involving a "determining set," appears in Clancey [3], where again C has rank 1 and H has a simple spectrum, and in Pincus [12], where C is of trace class and H has arbitrary (not necessarily simple) spectral multiplicity. These papers, as well as the present one, use a result obtained in

Putnam [18] for determining the "cross sections" of the spectrum of a hyponormal operator T, and which will be stated below as a lemma.

Remark (added November 6, 1971). Professor Pincus has pointed out to the author that Theorem 2 can be deduced from the general results of his papers [10] and [12]. The proof of Theorem 2 as given below will be used later (Theorem 6, below) along with [16] to yield corresponding results for singular integral operators in which the set E satisfies only $E \neq (-\infty, \infty)$.

First, let T be hyponormal on $\mathfrak{H}$ and let H of (1.2) have the spectral resolution

$$H = \int \lambda \, dE_\lambda, \tag{2.1}$$

and, for any open interval Δ, let $E(\Delta)$ denote the associated projection operator. Then $T_\Delta = E(\Delta)TE(\Delta)$ is hyponormal on $E(\Delta)\mathfrak{H}$ and, as was shown in [17] (and in [2] in case C is completely continuous),

$$\mathrm{sp}(T_\Delta) \subset \mathrm{sp}(T). \tag{2.2}$$

Moreover, as was shown in [18],

$$\mathrm{sp}(T_\Delta) \cap \{z : \mathrm{Re}(z) = s\} = \mathrm{sp}(T) \cap \{z : \mathrm{Re}(z) = s\}, \tag{2.3}$$

where s is any number in Δ. In view of (1.3), one obtains [18] the following

Lemma. *If T is hyponormal then*

$$\mathrm{Im}[\mathrm{sp}(T) \cap \{z : \mathrm{Re}(z) = s\}] = \bigcap_\Delta \mathrm{sp}(E(\Delta)JE(\Delta)), \qquad s \in \Delta, \tag{2.4}$$

where the intersection is taken over all open intervals Δ containing s.

In [4], in which C of (1.8) was of rank 1, the spectrum of $H + iJ$ $(H = x)$ was determined from (2.4) using a knowledge of the spectrum of J. This latter information could be determined from Pincus [9, p. 375], or Rosenblum [19, p. 323] (see also Pincus and Rovnyak [13, p. 620]). In the present paper certain general properties of hyponormal operators obtained in [17], [18] will be used to obtain the set on the right of (2.4) when $H = \mathrm{Re}(T)$ has a simple spectrum. From this information, relation (2.4) will then be used to determine the spectrum of $T = H + iJ$. As a consequence, the spectrum of J can then, if desired, be determined from the projection properties (1.3).

Thus, if H and J are defined by (1.8) and (1.11), where (1.9) and (1.10) are also assumed, then, by the remark following the statement of Theorem 2, $T = H + iJ$ is completely hyponormal and satisfies (1.2) with C given by (1.8). In view of (1.12) and the second relation of (1.3), one obtains the following

Corollary 1 of Theorem 2. *If E satisfies* (1.7) *and if J is defined by* (1.11) *and* (1.9) *then a real number t is in* $\mathrm{sp}(J)$ *if and only if*

$$\text{meas}_1\{x \in E: -a(x) - b(x) - \epsilon < t < -a(x) + b(x) + \epsilon\} > 0$$

holds for every $\epsilon > 0$.

For any measurable subset, E, of the real line define the (measurable) real set E^* by

$$E^* = \{x \in (-\infty, \infty): \text{meas}_1(\Delta \cap E) > 0 \tag{2.5}$$
$$\text{for every open interval } \Delta \text{ containing } x\}.$$

Since all points of E having positive metric density are contained in E^*, the set $E \cap E^*$ differs from E by a null set. Next, for any real-valued function $c(x)$ defined on E, define $c^*(x)$ on E^* by

$$c^*(x) = \text{ess} \limsup_{t=x} c(t) = \lim_{|\Delta|\to 0} \operatorname*{ess\,sup}_{\Delta \cap E} c(t), \tag{2.6}$$

where Δ is any open interval containing x.

Corollary 2 of Theorem 2. *For all x in E^* (= sp(H)) there exists a real number a_x such that*

$$\{x + iy : a_x - b^*(x) \le y \le a_x + b^*(x)\} \subset \text{sp}(T). \tag{2.7}$$

It may be noted that E in Theorem 2 is determined to within a null set and that the spectrum of the multiplication operator, x, on $L^2(E)$ is the set E^* of (2.5). The assertion of the corollary is that for any x in $E^*(= \text{sp}(H) = \text{Re}(\text{sp}(T)))$, the set sp$(T)$ contains a vertical segment (possibly a point) of length $2b^*(x)$. One need only note that there exists a set $F = F_x$ such that $F \subset E$ with the property that $\text{meas}_1(F \cap \Delta) > 0$ for every open interval containing x, and such that $b(t) \to b^*(x)$ as $t \to x$, $t \in F$. Then let a_x denote, say, the essential limit superior of $-a(t)$ at x when t is restricted to the set F. The assertion (2.7) then follows from the criterion (1.12). (See also Theorem 2 and its proof in [4].)

3. **Proof of Theorem 1.** If Δ is an open interval then relation (1.4) applied to $T_\Delta = E(\Delta)TE(\Delta)$ (cf. (2.1)) yields $\pi\|E(\Delta)DE(\Delta)\| \le \text{meas}_2(\text{sp}(T_\Delta))$. But, in view of (2.3) (or even (2.2)), one has

$$\pi\|E(\Delta)DE(\Delta)\| \le \int_\Delta F(x)\,dx, \tag{3.1}$$

where

$$F(x) = \text{meas}_1\{y : x + iy \in \text{sp}(T)\}, \tag{3.2}$$

that is, $F(x)$ is the measure of a vertical cross section of sp(T). Hence,

$$2\pi \limsup_{|\Delta|\to 0} \|C_\Delta\|/|\Delta| \le F(x), \qquad x \in \Delta \text{ (a.e.)}, \tag{3.3}$$

where $C_\Delta = E(\Delta)CE(\Delta)$.

Now, it is clear that it is sufficient to establish (1.6) if T is completely hyponormal. As already noted in §1, it can therefore be assumed that H and C are of the form (1.8). Since, by (1.8), $(C_\Delta f, f) = \pi^{-1}\Sigma\lambda_j|(E(\Delta)f, \phi_j)|^2$, it is clear that

$$\|C_\Delta\| = \pi^{-1} \sup_{\|f\|=1} \sum \lambda_j \left| \int_{\Delta\cap E} f(t)\bar{\phi}_j(t)\,dt \right|^2. \tag{3.4}$$

Hence, if one defines $f(t)$ on E (or, more precisely, on E^*, so that, in any case, $f \in L^2(E)$) by $f(t) = |\Delta \cap E|^{-1/2}$ or 0 according as $t \in \Delta \cap E$ or $t \notin \Delta \cap E$, then $\|f\| = 1$ and

$$\pi^{-1}\sum\lambda_j \left|\int_{\Delta\cap E} \bar{\phi}_j(t)\,dt\right|^2 \Big/ |\Delta||\Delta \cap E| \leq \|C_\Delta\|/|\Delta|. \tag{3.5}$$

On letting $|\Delta| \to 0$ and noting, by Lebesgue's metric density theorem, that $|\Delta \cap E|/|\Delta| \to 1$ as $|\Delta| \to 0$ holds a.e. on E, one obtains from (3.3) and (3.5),

$$2\sum\lambda_j|\phi_j(x)|^2 \leq F(x) \quad \text{a.e. on } E. \tag{3.6}$$

But

$$2\int_E \left(\sum\lambda_j|\phi_j(x)|^2\right) dx = 2\sum\lambda_j\int_E |\phi_j|^2(x)\,dx = 2\pi\,\mathrm{trace}\,(C),$$

and

$$\int_E F(x)dx \leq \int_{-\infty}^{\infty} F(x)dx = \mathrm{meas}_2\,(\mathrm{sp}(T)),$$

and so (1.6) follows from (3.6).

It is clear that even

$$\pi\,\mathrm{trace}\,(D) \leq \int_E F(x)dx \leq \int_{\mathrm{sp}\,(H)} F(x)dx = \mathrm{meas}_2\,(\mathrm{sp}\,(T)) \tag{3.7}$$

has been established. (Note that $E^* = \mathrm{sp}\,(H) = \mathrm{Re}\,(\mathrm{sp}\,(T))$.) Thus, if the set $\mathrm{sp}(H) - E$ has positive measure and if $F(x) > 0$ on a subset of $\mathrm{sp}(H) - E$ also having positive measure, then the first inequality of (3.7) is sharper than (1.6).

4. **Proof of Theorem 2.** In view of (2.3) (or (2.2)) one has $\mathrm{sp}\,(J_{\Delta_1}) \subset \mathrm{sp}\,(J_{\Delta_2})$ if $\Delta_1 \subset \Delta_2$ and hence, by (2.4),

$$F(x) = \lim_{|\Delta|\to 0} [\mathrm{meas}_1\,(\mathrm{sp}\,(E(\Delta)JE(\Delta)))], \tag{4.1}$$

where $x \in \Delta$ and J is any solution of (1.2) with H and C defined by (1.8). Now, let J_0 be defined by

$$(J_0 f)(x) = -\sum\lambda_j\phi_j(x)H[f\bar{\phi}_j](x), \qquad f \in L^2(E), \tag{4.2}$$

where $H[g]$ denotes the (unitary and selfadjoint) Hilbert transform on $L^2(-\infty, \infty)$,

$$H[g](x) = (i\pi)^{-1}\int_{-\infty}^{\infty} g(t)(t-x)^{-1}\,dt. \tag{4.3}$$

It will next be shown that the summation of (4.2) is strongly convergent.

In view of (3.6), each ϕ_j is (essentially) bounded on E, so that $H[f\bar{\phi}_j]$ is defined for $f \in L^2(E)$. (As is customary we regard $f = 0$ on $(-\infty, \infty) - E$.) Let $m < n$. Then

$$\left|\sum_m^n \lambda_j\phi_j H[f\bar{\phi}_j]\right|^2 \le \sum_m^n \lambda_j|\phi_j|^2 \sum_m^n \lambda_j|H[f\bar{\phi}_j]|^2 \le \text{const} \sum_m^n \lambda_j|H[f\bar{\phi}_j]|^2 \quad \text{a.e. on } E.$$

Hence

$$\left\|\sum_m^n \lambda_j\phi_j H[f\bar{\phi}_j]\right\|^2 \le \text{const} \sum_m^n \lambda_j\|H[f\bar{\phi}_j]\|^2 \le \text{const} \sum_m^n \lambda_j\|f\bar{\phi}_j\|^2$$

$$= \text{const} \int_E |f|^2\left(\sum_m^n \lambda_j|\phi_j|^2\right)dx.$$

Since $\Sigma_m^n \lambda_j|\phi_j|^2 \to 0$ a.e. on E as $m, n \to \infty$, it follows from (3.6) and Lebesgue's dominated convergence theorem that the last integral tends to 0 and hence that the summation defining J_0 in (4.2), and which occurs also in (1.11), converges strongly. It is clear also from the above calculations that

$$\|J_0\| \le \operatorname*{ess\,sup}_E b(x). \tag{4.4}$$

A straightforward calculation shows that $HJ_0 - J_0H = -iC$ where H and C are defined by (1.8). If J is any solution of (1.2), then $J_0 - J$ commutes with H and, since $H = x$ has a simple spectrum, $J_0 - J = a(x)$, where $a \in L^\infty(E)$. (See Achieser and Glasmann [1, p. 220], also Rosenblum [19, p. 326].) This establishes (1.11). That $b(x) \le \text{const} < \infty$ a.e. follows from (3.6); since T is completely hyponormal, $b(x) > 0$ a.e. on E and so (1.10) holds.

Next, let $F_0(x)$ denote the measure of the vertical cross section of the spectrum of $T_0 = H + iJ_0$, where H and J_0 are defined by (1.8) and (4.2), so that

$$F_0(x) = \text{meas}_1\{y : x + iy \in \text{sp}(T_0)\}, \qquad T_0 = x + iJ_0. \tag{4.5}$$

It follows from (4.4) applied to $E(\Delta)J_0E(\Delta)$ that

$$\|E(\Delta)J_0E(\Delta)\| \le \operatorname*{ess\,sup}_\Delta b(t). \tag{4.6}$$

Consequently, relation (2.4) implies that

$$\{y : x + iy \in \text{sp}(T_0)\} \subset [-b^*(x), b^*(x)], \tag{4.7}$$

where $b^*(x) = \text{ess lim sup}_{t=x} b(t)$ is defined on E^* by (2.6) with $c(x) = b(x)$ on

E. Since, by (3.6) with F replaced by F_0, $2b(x) \leq F_0(x)$ a.e. on E, then $2b^*(x) \leq F_0^*(x)$ on E^*. By (4.7), $F_0(x) \leq 2b^*(x)$ on E^*. But $F_0(x)$ is upper semicontinuous and hence $F_0^*(x) \leq F_0(x)$ for all x on $(-\infty, \infty)$. It then follows that

$$F_0(x) = 2b^*(x) \quad \text{for all } x \text{ in } E^*. \tag{4.8}$$

Relations (4.7) and (4.8) now yield

$$\{y : x + iy \in \mathrm{sp}(T_0)\} = [-b^*(x), b^*(x)], \qquad x \in \mathrm{Re}(\mathrm{sp}(T_0)). \tag{4.9}$$

Next, we prove the last part of Theorem 2 concerning the spectrum of T. First, let $z = s + it$ be a number for which the assertion of Theorem 2 concerning (1.12) fails to hold. It will be shown that z is not in $\mathrm{sp}(T)$. There exist some open interval Δ containing s and some number $\epsilon > 0$ such that, for almost all x in $E \cap \Delta$, either $b(x) + \epsilon \leq -a(x) - t$ or $b(x) + \epsilon \leq a(x) + t$, so that

$$0 < b(x) + \epsilon \leq |a(x) + t| \quad \text{for a.a. } x \text{ on } \Delta \cap E. \tag{4.10}$$

Now, if $s + it \in \mathrm{sp}(T)$ then, by (2.4), there exist $f_n = E(\Delta)f_n$ in $L^2(E)$, where $\|f_n\| = 1$, such that $\|[E(\Delta)JE(\Delta) - t]f_n\| \to 0$ as $n \to \infty$. But $a(x) + t = J_0 - (J - t)$ and hence

$$\int_\Delta (a(x) + t)^2 |f_n(x)|^2 \, dx = \|E(\Delta)(J_0 - (J - t))f_n\|^2 = \|E(\Delta)J_0 f_n\|^2 + o(1)$$

as $n \to \infty$. It follows from (4.6) and (4.10) that $\text{ess sup}_\Delta\, b + \epsilon \leq \text{ess sup}_\Delta\, b$, a contradiction. This proves the "only if" part of the assertion of Theorem 2 concerning (1.12).

Next, let $z = s + it$ be a number for which (1.12) holds for any $\epsilon > 0$ and every open interval Δ containing s. It will be shown that $z \in \mathrm{sp}(T)$. To this end, first note that there exists a set $F \subset E$ such that $\mathrm{meas}_1(F \cap \Delta) > 0$ and $-a(x) - b(x) - \epsilon < t < -a(x) + b(x) + \epsilon$ holds for a.a. x in $F \cap \Delta$, where Δ is any open interval containing s. Clearly, for every $\delta > 0$, there exist a set $G = G_\delta \subset F$ and a constant λ (e.g., with $\lambda =$ essential limit superior of $a(x)$ at s where x is restricted to F) such that

$$|a(x) - \lambda| < \delta \quad \text{on } G \cap \Delta \quad \text{and} \quad \mathrm{meas}_1(G \cap \Delta) > 0. \tag{4.11}$$

Then $-\lambda - \delta - b(x) - \epsilon < t < -\lambda + \delta + b(x) + \epsilon$ on $G \cap \Delta$, that is,

$$-b(x) - (\delta + \epsilon) < t + \lambda < b(x) + (\delta + \epsilon) \quad \text{on } G \cap \Delta. \tag{4.12}$$

Now, it follows from (4.9) and the projection properties of the spectra of hyponormal operators that

$$\mathrm{sp}(J_0) = \left[-\operatorname*{ess\,sup}_E b(x), \operatorname*{ess\,sup}_E b(x)\right]. \tag{4.13}$$

If this result is applied to $E(G \cap \Delta) J_0 E(G \cap \Delta)$ it is seen that

$$\text{sp}(E(G \cap \Delta)J_0E(G \cap \Delta)) = \left[-\operatorname*{ess\,sup}_{G\cap\Delta} b,\ \operatorname*{ess\,sup}_{G\cap\Delta} b\right]. \tag{4.14}$$

Hence, by (4.12), there exist a real number μ and unit vectors $f_n = E(G \cap \Delta)f_n$ for which, as $n \to \infty$,

$$(E(G \cap \Delta)J_0E(G \cap \Delta) - \mu)f_n \to 0 \quad \text{where } |\mu - (t + \lambda)| \leq \delta + \epsilon. \tag{4.15}$$

It follows from (4.11) that, for large n,

$$\|(a - \lambda)f_n\| = \left(\int (a(x) - \lambda)^2 |f_n|^2\, dx\right)^{1/2} < \delta. \tag{4.16}$$

Since $J = J_0 - a(x)$, it follows from (4.15) and (4.16) that

$$\|[E(G \cap \Delta)JE(G \cap \Delta) - t]f_n\| \leq 2\delta + \epsilon + o(1) \qquad \text{as } n \to \infty. \tag{4.17}$$

It follows from (4.17) that

$$\|[E(G \cap \Delta)TE(G \cap \Delta) - (s + it)]f_n\| \leq |\Delta| + 2\delta + \epsilon + o(1), \qquad n \to \infty.$$

Now, it is clear from [17] that the relation (2.2) holds if Δ is replaced by an E_λ-measurable (hence, in the present case, Lebesgue measurable) set, so that, in particular,

$$\text{sp}(E(G \cap \Delta)TE(G \cap \Delta)) \subset \text{sp}(T). \tag{4.18}$$

But, for an arbitrary hyponormal operator A on a Hilbert space, $\|Ax\| \geq \text{dist}(0, \text{sp}(A))\|x\|$ for all x in the space. Therefore, there exists a number z_Δ in $\text{sp}(E(G \cap \Delta)TE(G \cap \Delta))$, hence in $\text{sp}(T)$, such that $|z_\Delta - (s + it)| \leq |\Delta| + 2\delta + \epsilon$. Since $|\Delta|$, δ and ϵ can be chosen arbitrarily small, there exist z_n $(n = 1, 2, \cdots)$ in $\text{sp}(T)$ for which $z_n \to s + it$ as $n \to \infty$. Hence, $s + it$ belongs to $\text{sp}(T)$ as was to be shown.

5. **Remarks.** It may be noted that Kato has determined necessary and sufficient conditions that, for a given H and C, the equation (1.2) have a solution J; see [7, p. 552] and [8, p. 537 ff]. In particular, when $C \geq 0$ and H has a simple spectrum, his special solutions, corresponding to the "canonical" J_0 in (4.2) above, are

$$M^{\pm} = -\int_0^{\pm\infty} e^{itx} C e^{-itx}\, dt$$

(the integrals converging strongly).

To see the connection with, say M^+, note that if C is given by (1.8) then a straightforward calculation shows that $M^+ f = -2\Sigma \lambda_j \phi_j (f\bar{\phi}_j)^+$, where, for $g \in L^2(-\infty, \infty)$,

$$g^+(x) = (2\pi)^{-1/2} \int_0^\infty g\hat{\ }(y) e^{ixy}\, dy$$

and $g^{\wedge}(y)$ is the Fourier transform of g. If

$$g^{-}(x) = (2\pi)^{-1/2} \int_{-\infty}^{0} g^{\wedge}(y) e^{ixy}\, dy,$$

then $g = g^{+} + g^{-}$ and $H[g] = g^{+} - g^{-}$. (See Titchmarsh [20], Hilgevoord [6]; for an application to commutators and singular integral operators, see Putnam [16].) Thus, $g^{+} = \frac{1}{2}(H + I)g$. It is seen therefore that

$$M^{+}f = -\sum \lambda_j \phi_j (H + I)(f\bar{\phi}_j) = J_0 f - \sum \lambda_j |\phi_j|^2 f = (J_0 - b(x))f.$$

It is easy to determine the spectra of the operators $T_0 = H + iJ_0$ $(H = x)$ and $T^{+} = H + iM^{+}$ from Theorem 2. First, note that both T_0 and T^{+} are completely hyponormal (cf. the remark after Theorem 2). If E^{*} is defined by (2.5) then

$$\operatorname{sp}(T_0) = \{x + iy : x \in E^{*}, -b^{*}(x) \leq y \leq b^{*}(x)\}$$

and

$$\operatorname{sp}(T^{+}) = \{x + iy : x \in E^{*}, -2b^{*}(x) \leq y \leq 0\}.$$

6. Some applications of Theorems 1, 2 will be obtained below. Consider the special case in which T satisfies (1.1) and (1.5) and in which the first inequality of (3.7) is an equality, so that

$$\pi \operatorname{trace}(D) = \int_E F(x)\, dx. \tag{6.1}$$

Thus, π trace (D) equals the measure of that part of $\operatorname{sp}(T)$ lying over the set E. In addition, suppose that T is completely hyponormal, so that T is unitarily equivalent to (and will, up to but not including the statement of Theorem 5 below, simply be taken to be equal to) $H + iJ$ on $L^2(E)$ defined by (1.8)–(1.11).

It is clear that equality holds a.e. in (3.6), thus

$$2b(x) = F(x) \quad \text{a.e. on } E, \tag{6.2}$$

where $b(x)$ is defined in (1.10). Now, by Theorem 2 and Corollary 2 to Theorem 2, one has in general

$$2b^{*}(x) = F_0(x) \leq F(x) \quad \text{for all } x \text{ in } E^{*}, \tag{6.3}$$

where $F(x)$ and $F_0(x)$ are defined by (3.2) and (4.5). Hence, by (6.2) and (6.3), $b^{*}(x) \leq b(x)$ a.e. on E. But $b(x) \leq b^{*}(x)$ a.e. on E, so that

$$2b^{*}(x) = F_0(x) = F(x) = 2b(x) \quad \text{a.e. on } E. \tag{6.4}$$

Since $F(x)$ is upper semicontinuous, then $F(x)$ is continuous except possibly on a set of the first category; cf. Goffman [5, p. 110]. It is possible of course that E itself is of the first category so that the continuity of $b(x)$ at a point of E cannot be inferred. In any case, suppose, in addition to (6.1), that

$$F(x) \text{ is continuous a.e. on } E. \tag{6.5}$$

Then it is easy to see that the function $a(x)$ of (1.11) is (essentially) continuous a.e. on E, that is,

$$\operatorname{ess}\limsup_{t=x} a(t) = \operatorname{ess}\liminf_{t=x} a(t) \quad \text{a.e. on } E, \tag{6.6}$$

the second expression being defined by (2.6) with "sup" replaced by "inf." For, if (6.6) does not hold, then by (6.4) and (6.5), there exists a set $P \subset E$ of positive measure such that, for c in P, $b(x)$ is continuous, $F(c) = 2b(c) > 0$ and $\alpha = \operatorname{ess}\limsup_{t=c} a(t) > \beta = \operatorname{ess}\liminf_{t=c} a(t)$. It follows from the last part of Theorem 2, however, that each of the vertical segments $\{x + iy: -\alpha - b(c) \leq y \leq -\alpha + b(c)\}$ and $\{x + iy: -\beta - b(c) \leq y \leq -\beta + b(c)\}$ belongs to $\operatorname{sp}(T)$, and hence, in particular, $F(c) > 2b(c)$ on P, a contradiction to (6.4).

These results can be summarized as follows:

Theorem 3. *Suppose that $T = H + iJ$ on $L^2(E)$ is defined by* (1.7)–(1.11). *In addition, suppose that* (6.1) *and* (6.5) *hold. Then, in the definition* (1.11) *of J, both $a(x)$ and $b(x)$ can be assumed to be continuous a.e. on E. Further, at all points x in E where $a(x)$ and $b(x)$ are continuous,*

$$\operatorname{sp}(T) \cap \{z : \operatorname{Re}(z) = x\} = \{x + iy : -a(x) - b(x) \leq y \leq -a(x) + b(x)\}. \tag{6.7}$$

The assertion (6.7), which follows immediately from (1.12), is simply that the spectrum of T lying over any point x in E at which both $a(x)$ and $b(x)$ are continuous is a closed interval centered at $x - ia(x)$ and of length $2b(x)$. The functions $a(x)$ and $b(x)$ are thus uniquely determined a.e. on E by $\operatorname{sp}(T)$.

Theorem 4. *Suppose that $T = H + iJ$ on $L^2(E)$ is defined by* (1.7)–(1.11) *and that $F(x)$ of* (3.2) *satisfies*

$$F(x) > 0 \quad \text{and is continuous for a.a. } x \text{ in } \operatorname{sp}(H) \quad (= \operatorname{Re}(\operatorname{sp}(T))). \tag{6.8}$$

In addition, suppose that equality holds in (1.6), *so that*

$$\pi \operatorname{trace}(D) = \operatorname{meas}_2(\operatorname{sp}(T)). \tag{6.9}$$

Then, in the definition (1.11) *of J, $E = \operatorname{sp}(H)$, and both $a(x)$ and $b(x)$ can be assumed to be continuous a.e. on* $\operatorname{sp}(H)$. *Also, for all x in* $\operatorname{sp}(H)$ *at which both $a(x)$ and $b(x)$ are continuous, relation* (6.7) *holds.*

It is clear that the hypotheses of Theorem 4 are stronger than those of Theorem 3. That $\operatorname{Re}(\operatorname{sp}(T)) = \operatorname{sp}(H)$ is simply the projection property (1.3). Since $F(x) > 0$ on $\operatorname{sp}(H)$ it follows from (3.7) and (6.9) that $E = \operatorname{sp}(H)$ (to within a null set) and the proof of Theorem 4 is complete.

Theorem 5. *Let* $T = H + iJ$ *satisfy* (1.1) *and* (1.5) *and suppose that* T *is completely hyponormal. In addition, suppose that equality holds in* (1.4), *that is,*

$$\pi\|D\| = \text{meas}_2(\text{sp}(T)), \tag{6.10}$$

and, further, that (6.8) *holds, where* $F(x)$ *is defined by* (3.2). *Then* T *is unitarily equivalent to the operator* T_1 *on* $L^2(\text{sp}(H))$ $(\text{sp}(H) = \text{Re}(\text{sp}(T)))$ *defined by*

$$(T_1 f)(x) = xf(x) - i\left[a(x)f(x) + (i\pi)^{-1} b^{1/2}(x)\int_{\text{sp}(H)} f(t)b^{1/2}(t)(t-x)^{-1}\,dt\right], \tag{6.11}$$

where $2b(x) = F(x)$ *and* $a(x)$ *are continuous a.e. on* $\text{sp}(H)$. *Also for all* x *in* $\text{sp}(H)$ *at which both* $a(x)$ *and* $b(x)$ *are continuous, relation* (6.7) *holds.*

It follows from the Corollary of Theorem 1 and the complete hyponormality of T that $\text{rank}(C)$ $(= \text{rank}(D)) = 1$. Thus (6.10) reduces to (6.9). It then follows from Theorem 4 that T is unitarily equivalent to T_2 on $L^2(\text{sp}(H))$ where

$$(T_2 f)(x) = xf(x) - i\left[a(x)f(x) + (i\pi)^{-1}\int_{\text{sp}(H)} f(t)\bar{\phi}(t)(t-x)^{-1}\,dt\right], \tag{6.12}$$

where $a, \phi \in L^\infty(\text{sp}(H))$ and $b(x) = |\phi(x)|^2 > 0$ on $\text{sp}(H)$, and $a(x)$, $b(x)$ can be taken to be continuous a.e. on $\text{sp}(H)$. (Note that (6.2) holds.) But $b^{1/2}(x) = m(x)\phi(x)$, where $m(x)$ is measurable on E and $|m(x)| = 1$. Since the unitary operator $U: f(x) \to m(x)f(x)$ of $L^2(E)$ onto itself obviously commutes with x and $a(x)$, it follows that T_2, hence also T, is unitarily equivalent to T_1 of (6.11). That (6.7), with T replaced by T_1, holds is clear from Theorem 3 and the proof of Theorem 5 is now complete.

It is seen that $\text{sp}(T)$ is a complete unitary invariant for operators T satisfying the hypotheses of Theorem 5. (Concerning complete unitary invariants for hyponormal operators under other hypotheses, see Pincus [11, Theorem 22]; also [12].) As a simple application, one has the following

Corollary of Theorem 5. *Let* T *be isometric and completely hyponormal, and suppose that* $\frac{1}{2}(T + T^*)$ *has a simple spectrum. Then* T *is unitarily equivalent to the unilateral shift.*

Since the closed unit disk is the spectrum of both T and the unilateral shift, it is easily verified (see also the beginning of §2 above) that all hypotheses of Theorem 5 are satisfied by both operators.

7. The assertions concerning the singular integral operator J of (1.11) can be generalized to the case where E need not satisfy (1.7) but, more generally, is subject only to

(7.1) E measurable and $E \neq (-\infty, \infty)$; that is, $\text{meas}_1 [(-\infty, \infty) - E] > 0.$

Further, it will no longer be supposed that $\{\phi_j\}$ is an orthonormal system.

Remark. The above-mentioned orthonormality hypothesis was used in Theorem 1. It could have been omitted in Theorem 2 however (cf. below) if, say, relation (1.10) was simply hypothesized.

For $k = 1, 2, \cdots$, let $b_k \in L^\infty(E)$, where E satisfies (7.1). In addition, suppose that

(7.2) $A(x)$ is a real-valued, measurable function on E,

and that

(7.3) $0 < B(x) \leq \text{const } (<\infty)$ a.e. on E, where $B(x) = \sum |b_j(x)|^2$.

Define the singular integral operator L on $L^2(E)$ by

$$(Lf)(x) = -\left[A(x)f(x) + (i\pi)^{-1} \sum b_j(x) \int_E f(t)\bar{b}_j(t)(t-x)^{-1}\,dt\right], \tag{7.4}$$

that is, $L = L_0 - A$, where

$$(L_0 f)(x) = -\sum b_j(x)\, H[f\bar{b}_j](x), \qquad f \in L^2(E). \tag{7.5}$$

An argument similar to that used in the beginning of §4, but with $\lambda_j^{1/2}\phi_j$ replaced by b_j, shows that the summation of (7.4) converges strongly. (Note however that the b_j are in $L^\infty(E)$ but not necessarily in $L^2(E)$.) It follows that L_0 of (7.5) is bounded and selfadjoint on $L^2(E)$ and that (cf. (4.4))

$$\|L_0\| \leq \operatorname*{ess\,sup}_E B(x). \tag{7.6}$$

The multiplication operator $A(x)$ is clearly selfadjoint (but not necessarily bounded) and so L of (7.4) is a selfadjoint, in general, unbounded operator on $L^2(E)$. It follows from [16] that L is absolutely continuous. (If $A(x)$ is also bounded from below and if the summation of (7.4) reduces to a single term, this result, and, in fact, a complete spectral analysis, was obtained by Rosenblum [19]. He also treats the case, again for the single integral operator, where $E = (-\infty, \infty)$ and in which eigenvalues may occur.) It will be shown below that the methods of [16] can be used, at least if

$$A \in L^\infty(E), \tag{7.7}$$

to obtain for L an analogue (and generalization) of the assertion of Corollary 1 of Theorem 2 for J. It is clear that if (7.7) holds then L of (7.4) is bounded on $L^2(E)$. However, it is clear that if the set E is not (essentially) bounded, then the selfadjoint multiplication operator x, hence also the operator $x + iL$, is unbounded on $L^2(E)$. It turns out however that x can be replaced by another multiplication operator $c(x) \in L^\infty(E)$ and such that $c + iL$ is hyponormal. This fact will be used to obtain the following

Theorem 6. *Assume conditions* (7.1), (7.3) *and* (7.7) *and define the* (*bounded*) *selfadjoint operator* L *on* $L^2(E)$ *by* (7.4). *Then a real number* t *is in* $\mathrm{sp}(L)$ *if and only if*

$$\mathrm{meas}_1\{x \in E : -A(x) - B(x) - \epsilon < t < -A(x) + B(x) + \epsilon\} > 0$$

holds for every $\epsilon > 0$.

Remark. The minus sign in (7.4) is used, as in the definition of J in (1.11), for convenience in regarding L as the imaginary part of a certain hyponormal operator. Note that the spectrum of $-L$ can be obtained from Theorem 6 simply by replacing $-A(x)$ by $A(x)$ in the measure condition.

8. **Proof of Theorem 6.** It was shown in [16, Lemma 2] that, for any set E satisfying (7.1), there exists a real-valued function $\psi(x)$ on $(-\infty, \infty)$, depending on E but independent of L in (7.4), for which

$$\begin{gathered} 0 < \psi(x) \leq \text{const } (<\infty) \quad \text{on } (-\infty, \infty) - E, \\ \psi(x) = 0 \quad \text{on } E, \quad \psi \in L^2(-\infty, \infty) \text{ and} \\ |H[\psi](x)| \leq \text{const } (<\infty) \quad \text{on } (-\infty, \infty), \end{gathered} \tag{8.1}$$

where $H[g]$ denotes the Hilbert transform of (4.3). Further (cf. §3 of [16]), if $c(x) = iH[\psi](x)$, so that $c(x)$ is real, then, regarding c as a selfadjoint operator on $L^2(E)$,

$$cL - Lc = -iG, \qquad G \geq 0 \tag{8.2}$$

(that is, $S = c + iL$ is hyponormal) and

$$0 \notin \text{point spectrum of } G. \tag{8.3}$$

Since $G \geq 0$ (for any L, in particular for $b_1 = 1$ and $b_k = 0$ for $k = 2, 3, \cdots$) then $k(x, t) = \pi^{-1}[c(t) - c(x)](t - x)^{-1}$ is the kernel of a (bounded) nonnegative integral operator K on $L^2(E)$. In fact, $(Kf)(x) = iH[cf](x) - ic(x)H[f](x)$, where H is the Hilbert transform of (4.3) and $f \in L^2(E)$. That $k(x, t)$ is (essentially) bounded on $E \times E$ follows from the boundedness of the operator K. As noted in [16, p. 459], one has the representation

$$\begin{gathered} [c(t) - c(x)](t - x)^{-1} = \sum c_j(x)\bar{c}_j(t) \quad \text{for a.a. } x, t \ (x \neq t) \text{ in } E, \\ \text{where } \sum |c_j(x)|^2 \leq \text{const } (<\infty) \text{ a.e. on } E. \end{gathered} \tag{8.4}$$

In the end of the proof of Lemma 2 of [16], and in the notation of that paper, the following correction may be noted. The functions $h(z)$ and $k(z)$ satisfy $k(z) \equiv h(z) + \text{const}$ (rather than $k(z) \equiv h(z)$) and one may conclude that $r(x) \equiv q(x) + \text{const}$ and hence $H[p](x) = i[q(x) + \text{const}]$. It is readily verified that, in fact, $\text{const} = \frac{1}{2}$.

It follows (cf. [16, pp. 456, 458]) that the function $c(x) = iH[\psi](x)$ can then be chosen as

$$c(x) = (e^{v} \cos u + 1)/(e^{2v} + 2e^{v} \cos u + 1) - \tfrac{1}{2}, \tag{8.5}$$

where

$$u(x) = \begin{cases} (\pi/4) \exp(-x^2) & \text{if } x \notin E \\ 0 & \text{if } x \in E \end{cases} \quad \text{and } v(x) = -iH[u](x). \tag{8.6}$$

Thus,

$$c(x) = (e^{v} + 1)^{-1} - \tfrac{1}{2}, \qquad x \in E. \tag{8.7}$$

If one restricts the quantities of (8.4) only to those x, t in E for which the asserted relations hold (i.e. one avoids an exceptional null set), then $c'(x) = \lim_{t \to x} [c(t) - c(x)](t - x)^{-1}$ $(t \in E)$ exists a.e. on E. It will next be shown that

$$c'(x) = \sum |c_j(x)|^2 \quad \text{a.e. on } E. \tag{8.8}$$

To this end, note that, for almost all x, $c'(x) = \Sigma c_j(x)\bar{c}_j(t) + h_x(t)$, where (for x fixed) $h_x(t) \to 0$ as $t \to x$. Let δ be an open interval containing x and let $Q = \delta \cap E$. Then the Lebesgue density is 1 (that is, $|Q|\,|\delta|^{-1} \to 1$ as $|\delta| \to 0$) for almost all $x \in E$, and, in particular, $|Q| > 0$. Choose x to be such a point and for which $c'(x)$ exists. Then

$$c'(x) = |Q|^{-1} \int_Q c'(x)\, dt = |Q|^{-1} \int_Q \sum c_j(x)\bar{c}_j(t)\, dt + |Q|^{-1} \int_Q h_x(t)\, dt.$$

The last term tends to 0 as $|\delta| \to 0$ and so

$$c'(x) = \lim_{|\delta| \to 0} |Q|^{-1} \int_Q \sum c_j(x)\bar{c}_j(t)\, dt.$$

The Schwarz inequality and the boundedness of $\Sigma |c_j(x)|^2$ on E make it clear that the integral and limit signs may be moved inside the summation, so that

$$c'(x) = \sum c_j(x) \left(\lim_{|\delta| \to 0} |Q|^{-1} \int_Q \bar{c}_j(t)\, dt \right),$$

and (8.8) follows.

Let $c(x)$, as an operator on $L^2(E)$, have the spectral resolution

$$c = \int \lambda\, dE_\lambda. \tag{8.9}$$

Then, for any open interval Δ and any $f \in L^2(E)$, $E(\Delta)f = f(x)$ if $x \in M(\Delta)$, where $M(\Delta) = \{t \in E: c(t) \in \Delta\}$, and $E(\Delta)f = 0$ otherwise. In view of (8.3), it follows from [15, p. 42], that the operator c of (8.6) (and (8.9)) is absolutely continuous on $L^2(E)$ and hence $c'(x) > 0$ a.e. on E. Since E has Lebesgue density 1 a.e. on E then clearly

$$\text{both } c'(x) > 0 \text{ and } E \text{ has density 1 at } x \text{ hold a.e. on } E. \tag{8.10}$$

Let x satisfy (8.10) and let δ be any open interval containing x. By (8.10), $\operatorname{ess\,inf}_Q c(t) < \operatorname{ess\,sup}_Q c(t)$, where $Q = \delta \cap E$; let $\Delta = (\operatorname{ess\,inf}_Q c, \operatorname{ess\,sup}_Q c)$. Clearly, $0 < |Q| \to 0$ as $|\delta| \to 0$.

If $f(t) = |Q|^{-1/2}$ or 0 according as $t \in Q$ or $t \in E - Q$, then it is clear that

$$\frac{(G_\Delta f, f)}{|\Delta|} = \pi^{-1}|Q|\,|\Delta|^{-1} \sum_j \sum_k \left| |Q|^{-1} \int_Q b_j(t) c_k(t)\, dt \right|^2, \tag{8.11}$$

where $G_\Delta = E(\Delta)\, G E(\Delta)$. But $|Q|\,|\Delta|^{-1} = |Q|\,|\delta|^{-1}|\delta|\,|\Delta|^{-1} \to 1/c'(x) > 0$ as $|\delta| \to 0$, and hence

$$\limsup_{|\Delta| \to 0} \|G_\Delta\|/|\Delta| \geq \pi^{-1} \sum |b_j(x)|^2 \quad \text{for a.a. } x \in E, \tag{8.12}$$

where $c(x) \in \Delta$. An argument similar to that used in §3 yields

$$2B(x) \leq F(c(x)) \quad \text{a.e. on } E, \tag{8.13}$$

where

$$F(X) = \operatorname{meas}_1 \{y : X + iy \in \operatorname{sp}(S)\}, \qquad S = c + iL. \tag{8.14}$$

For any function $g(x)$ defined on E and any open interval Δ, let $g_\Delta = \operatorname{ess\,sup}_{M(\Delta)} g(x)$, where $M(\Delta) = \{x \in E: c(x) \in \Delta\}$. Then define $g^*(X)$ on the essential range, R, of c on E by $g^*(X) = \lim_{|\Delta| \to 0} g_\Delta$, $X \in \Delta$.

Next, let

$$F_0(X) = \operatorname{meas}_1 \{y : X + iy \in \operatorname{sp}(S_0)\}, \qquad S_0 = c + iL_0, \tag{8.15}$$

where L_0 is defined by (7.5). It follows from (7.6) applied to $E(\Delta)L_0\, E(\Delta)$ that $\|E(\Delta)\, L_0 E(\Delta)\| \leq B_\Delta$. It follows from the Lemma of §2 applied now to $S_0 = c + iL_0$ that

$$\{y : X + iy \in \operatorname{sp}(S_0)\} \subset [-B^*(X), B^*(X)]. \tag{8.16}$$

(Note (cf. (1.3)) that $\operatorname{Re}(\operatorname{sp}(S_0)) = \operatorname{sp}(c) = R$.)

By (8.13), with F replaced by F_0, we have $2B^*(X) \leq F_0^*(X)$ on R. Since $F_0(X)$ is upper semicontinuous, then $F_0^*(X) \leq F_0(X)$ for all X. Also, by (8.16), $F_0(X) \leq 2B^*(X)$ on R. Thus, $F_0(X) = B^*(X)$ on R, and (8.16) now implies that

$$\{y : X + iy \in \mathrm{sp}(S_0)\} = [-B^*(X), B^*(X)] \quad \text{for } X \in \mathrm{Re}(\mathrm{sp}(S_0)). \tag{8.17}$$

It follows from (1.3) that $\mathrm{sp}(L_0) = [-m, m]$, where $m = \operatorname{ess\,sup}_E B(x)$, so that Theorem 6 is proved in the special case when $L = L_0$. The proof for the general operator $L = L_0 - A$ of (7.4) is similar to the argument given in §4 following (4.1) and will be omitted.

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