ON THE NONSTANDARD REPRESENTATION
OF MEASURES

BY

C. WARD HENSON

ABSTRACT. In this paper it is shown that every finitely additive probability
measure $\mu$ on $S$ which assigns 0 to finite sets can be given a nonstandard
representation using the counting measure for some $^*\text{finite}$ subset $F$ of $^*S$.
Moreover, if $\mu$ is countably additive, then $F$ can be chosen so that
\[ \int f \, d\mu = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right) \]
for every $\mu$-integrable function $f$. An application is given of such representations.
Also, a simple nonstandard method for constructing invariant measures is presented.

Let $S$ be a set in some set theoretical structure $\mathbb{M}$ and let $^*S$ be the
corresponding set in an enlargement $^*\mathbb{M}$ of $\mathbb{M}$. Bernstein and Wattenberg have
noted [2] that if $F$ is a $^*$-finite subset of $^*S$, then a finitely additive probability
measure $\mu_F$ can be defined for all subsets $A$ of $S$ by
\[ \mu_F(A) = \text{st}(\|A \cap F\|/\|F\|). \]
They used this observation as the basis for a nonstandard proof of the theorem,
due to Banach [1], which states that Lebesgue measure on $[0, 1]$ can be extended
to a totally defined (finitely additive) measure which is invariant under trans-
lations (mod 1).

This paper concerns the representation of probability measures as non-
standard counting measures $\mu_F$. Let $\mu$ be any finitely additive probability
measure which is defined on an algebra $\mathcal{B}$ of subsets of $S$ and which satisfies
$\mu(A) = 0$ for each finite set $A$ in $\mathcal{B}$. In §1 it is shown that there exists a
$^*$-finite subset $F$ of $^*S$ which satisfies $\mu = \mu_F$ on $\mathcal{B}$. This has the consequence
that for any bounded, $\mu$-integrable function $f$,
\[ \int f \, d\mu = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right). \]

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Moreover, if \( B \) is a \( \sigma \)-algebra and \( \mu \) is countably additive, then \( F \) can be chosen so that (2) holds for every \( \mu \)-integrable function.

Closely related to these results is a nonstandard representation for bounded linear functionals on the space \( l_\infty \) of bounded sequences in \( R \), which was given by Robinson [7]. In \$2 a straightforward extension of Robinson's result is used to give a nonstandard proof of a convergence result (Theorem 3) for bounded linear functionals on \( C(X) \), where \( X \) is a compact, Hausdorff space.

Also, in \$3 a nonstandard construction of invariant measures is given which yields a particularly simple proof of Banach's extension result for Lebesgue measure.

Preliminaries. The given structure \( \mathfrak{M} \) is assumed to have the set \( R \) of real numbers as an element (thus also the set \( N \) of nonnegative integers). Moreover, the embedding \( x \mapsto x^* \) of \( M \) into \( \mathfrak{M} \) is taken to be the identity on \( R \). The standard part of a finite element \( p \) of \( R \) is denoted by \( \text{st}(p) \). If \( p, q \in R \), then \( p = q \) means that \( p - q \) is infinitesimal.

For each set \( S \) in \( \mathfrak{M} \) and each \( \ast \)-finite subset \( F \) of \( S \), \( \|F\| \) is the "cardinality" of \( F \), in the sense of \( \mathfrak{M} \). That is, if \( c \) is the function assigning to each finite subset \( A \) of \( S \) the cardinality of \( A \), then \( \|F\| = c(F) \). Alternately, \( \|F\| \) is the smallest element \( \omega \) of \( N \) for which there is an internal bijection between \( F \) and \( \{\omega' \mid \omega' \in *N \text{ and } \omega' < \omega \} \). (For an introduction to the methods of nonstandard analysis see [5], [6], or [8].)

Given a set \( S \), \( \mathcal{P}(S) \) is the algebra of all subsets of \( S \). Also, \( l_\infty(S) \) is the linear space of all bounded, real valued functions on \( S \), furnished with the sup norm. In this paper \( \mu \) is a measure on \( S \) if it is a nonnegative, finitely-additive set function defined on an algebra of subsets of \( S \). If \( \mu \) is normalized to satisfy \( \mu(S) = 1 \), then it is a probability measure. The notation \( A \Delta B \) will be used for the symmetric difference, \( (A \sim B) \cup (B \sim A) \), of two subsets of \( S \).

1. Nonstandard representations. Let \( \mu \) be a probability measure on \( \mathcal{P}(S) \) and let \( \phi \) be the linear functional on \( l_\infty(S) \) defined by integration with respect to \( \mu \). Then \( \phi \) is a positive linear functional of norm 1. Therefore, by the principal result of [7], there exist a \( \ast \)-finite subset \( F \) of \( S \) and an internal function \( \lambda \) from \( F \) to \( *R \) which satisfy

\[
\text{st} \left( \sum_{p \in F} |\lambda(p)| \right) = 1
\]

and, for each \( f \) in \( l_\infty(S) \),

\[
\phi(f) = \text{st} \left( \sum_{p \in F} \lambda(p) *f(p) \right).
\]
Robinson's result [7] only covers the case $S = N$ explicitly, but his argument is easily extended to cover the general case.) Therefore the measure $\mu$ has the representation

$$\mu(A) = \text{st} \left( \sum_{p \in A \cap F} \lambda(p) \right).$$

Theorem 1 below states that, if $\mu(\{s\}) = 0$ for every $s \in S$, then $F$ can be chosen so that $\mu$ is represented as in (3), but with every $X(p)$ equal to $1/\|F\|$. That is, $\mu(A) = \mu_F(A)$ for every $A \subset S$.

**Theorem 1.** If $\mu$ is a probability measure on $\mathcal{P}(S)$ which satisfies $\mu(\{s\}) = 0$ for each $s \in S$, then there is a $^*\mu$-finite set $F \subset ^*S$ for which $\mu = \mu_F$.

**Proof.** Since $^*\mathcal{M}$ is an enlargement of $\mathcal{M}$, there exists a $^*\mu$-finite subset $\overline{A}$ of $^*\mathcal{P}(S)$ which satisfies $^*\Lambda \in \overline{A}$ for each $A \subset S$. For each internal subset $F$ of $\overline{A}$, define

$$E(F) = \bigcap \{E \mid E \in F \cap \bigcap ^*S \sim E \mid E \in \overline{A} \sim F\},$$

so that the function taking $F$ to $E(F)$ is internal. Let $\overline{A}' = \{E(F)\} \cap F$ is an internal subset of $\overline{A}$, so that $\overline{A}'$ is a $^*\mu$-finite set. Moreover, $\overline{A}'$ is a partition of $^*S$, and each member of $\overline{A}'$ is the union of an internal subset of $\overline{A}$.

Let $\omega = \|\overline{A}'\|$ and choose $r \in ^*N$ so that $\omega^2/r$ is infinitesimal. For each $E$ in $\overline{A}'$ define $r(E)$ in $^*N$ by the inequalities

$$r(E)/r \leq ^*\mu(E) < (r(E) + 1)/r.$$

Then the function $E \mapsto r(E)$ on $\overline{A}'$ is internal. Moreover, if $E$ is a $^*\mu$-finite element of $\overline{A}'$, then $^*\mu(E) = 0$, from which it follows that $r(E) = 0$. Therefore there exists an internal function $f$ which is defined on $\overline{A}'$ and which satisfies: For each $E$ in $\overline{A}'$, $f(E)$ is a $^*\mu$-finite subset of $E$ and $\|f(E)\| = r(E)$.

It will be shown that the set $F$ defined by

$$F = \bigcup \{f(E) \mid E \in \overline{A}'\}$$

satisfies the condition $\mu = \mu_F$. Since the elements of $\overline{A}'$ are pairwise disjoint, the elements of $\{f(E) \mid E \in \overline{A}'\}$ have the same property, and therefore,

$$\|F\| = \sum_{E \in \overline{A}'} r(E).$$

Moreover, since the function $^*\mu$ is $^*\mu$-finitely additive,

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(1) The added condition on $\mu$ is only slightly more restrictive than necessary. Indeed, if $F$ is infinite and $s \in S$, then $\mu_F(\{s\}) \leq \text{st}(1/\|F\|) = 0$. If $F$ is finite, say with $k$ elements, then $\mu_F$ is of the form $\mu = k^{-1}(\mu_1 + \cdots + \mu_k)$, where each of the measures $\mu_j$ takes on as values only 0 and 1.
\[ 1 = *\mu(*S) = \sum_{E \in \mathcal{G}} *\mu(E). \]

Therefore, from the inequalities (4) follows
\[ \|F\|/r \leq 1 < \|F\|/r + \omega/r, \]
by summing over \( E \). That is, by the choice of \( r \), \( \omega(\|F\|/r - 1) \) is infinitesimal.

Now let \( A \) be any element of \( \mathcal{G} \) and let \( \mathcal{F} \) be the collection of \( E \) in \( \mathcal{G} \)
which are subsets of \( A \). Therefore \( A \) is the union of \( \mathcal{F} \), by the construction of \( \mathcal{G} \). It follows that
\[ \|A \cap F\| = \sum_{E \in \mathcal{F}} r(E), \quad \text{and} \quad *\mu(A) = \sum_{E \in \mathcal{F}} *\mu(E). \]

Therefore
\[ *\mu(A) - \frac{\|A \cap F\|}{\|F\|} = \sum_{E \in \mathcal{F}} \left( *\mu(E) - \frac{r(E)}{\|F\|} \right). \tag{5} \]

But for each \( E \) in \( \mathcal{G} \),
\[ |*\mu(E) - r(E)/\|F\|| \leq |*\mu(E) - r(E)/r| + |r(E)/r - r(E)/\|F\|| \]
\[ \leq 1/r + (r(E)/\|F\|)|\|F\|/r - 1| \leq 1/r + |\|F\|/r - 1|. \]
Thus (5) implies
\[ |*\mu(A) - \|A \cap F\|/\|F\|| \leq \omega/r + \omega|\|F\|/r - 1| \]
which is infinitesimal. In particular, for each \( A \subset S \),
\[ \mu(A) = *\mu(*A) = \text{st}(\|A \cap F\|/\|F\|) = \mu_F(A). \]
This completes the proof.

While Theorem 1, as stated, applies only to totally defined measures, it is
valid for any probability measure \( \mu \) which is defined on an algebra of subsets of \( S \) and which assigns measure 0 to any finite set in its domain. This is because
any such measure can be extended to a measure which satisfies the conditions
of Theorem 1.

A different nonstandard representation for measures, based on partitions of
\( *S \) rather than \( * \) finite subsets, has been developed and applied by Peter Loeb
[5, 6].

Lemma 1. Let \( E \) be any \( * \)-finite subset of \( *S \) and let \( F \) be an internal
subset of \( E \) which satisfies \( \|F\|/\|E\| = 1 \). Then \( \mu_F = \mu_E \) on \( \mathcal{P}(S) \) and
\[ \int f \, d\mu_E = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right) \]
for each \( f \) in \( l_{\infty}(S) \).
Proof. Let $A$ be any subset of $S$. Then

$$| \| *A \cap E \| / \| E \| - \| *A \cap F \| / \| E \| | \leq \| E \sim F \| / \| E \| = 0.$$ 

Therefore

$$\mu_E(A) = \text{st} \left( \| F \| / \| E \| \cdot \| *A \cap F \| / \| F \| \right) = \mu_F(A).$$

Now let $f$ be any element of $l_\infty(S)$, and define

$$\psi(f) = \text{st} \left( \sum_{p \in F} *f(p) \right).$$

Then $\psi$ is a bounded linear functional on $l_\infty(S)$. Also, if $V$ is the subspace of $l_\infty(S)$ generated by the characteristic functions, then $\psi$ agrees with the $\mu_E$-integral on $V$. The fact that $V$ is norm-dense in $l_\infty(S)$ implies that $\psi$ and the $\mu_E$-integral are equal on all of $l_\infty(S)$.

Now let $B$ be a $\sigma$-algebra of subsets of $S$ and let $\mu$ be a countably additive probability measure on $B$ which satisfies $\mu(A) = 0$ for each finite set $A$ in $B$. There exists an extension $\tilde{\mu}$ of $\mu$ to $\mathcal{P}(S)$ which satisfies $\tilde{\mu}(\{s\}) = 0$ for $s \in S$. By Theorem 1, there exists a $*S$-finite subset $F$ of $*S$ which satisfies $\tilde{\mu} = \mu_F$, and thus $\mu(A) = \mu_F(A)$ for every $A$ in $B$.

For any bounded, $\mu$-integrable function $f$, $\int f \, d\mu = \int f \, d\tilde{\mu}$. Therefore, by Lemma 1,

$$\int f \, d\mu = \text{st} \left( \sum_{p \in F} *f(p) \right).$$

However, for unbounded, $\mu$-integrable functions (6) may not be true. (Indeed, if $f$ is any unbounded function on $S$, then $F$ may be chosen satisfying $\mu = \mu_F$ on $B$, but such that the sum $\| F \|^{-1} \sum_{p \in F} *f(p)$ is infinite.) It is possible, nonetheless, to choose $F$ in such a way that (6) is true for every $\mu$-integrable function.

It is convenient to assume that $^*\mathbb{R}$ is $\kappa$-saturated (in the sense of [7]), where $\kappa$ is any cardinal number greater than the number of functions from $S$ to $R$. The remainder of this section is devoted to showing that, under this assumption, it is possible to represent $\mu$ on $B$ in such a way that (6) holds for every $\mu$-integrable function.

Given $n \in \mathbb{N}$ and a function $f$ from $S$ to $R$, define $f_n$ on $S$ by

$$f_n(x) = \begin{cases} 
 f(x) & \text{if } |f(x)| \leq n, \\
 0 & \text{otherwise}.
\end{cases}$$
Each $f_n$ is a bounded function, and it is measurable whenever $f$ is. Also, if $\omega \in \mathbb{N}$ and $p \in \mathbb{S}$, then

$$^{*}f_\omega(p) = \begin{cases} \frac{1}{\|f\|} \sum_{p \in F} \frac{1}{\|F\|} (p) & \text{if } ^{*}f(p) \leq \omega, \\ 0 & \text{otherwise}. \end{cases}$$

**Lemma 2.** Let $E$ be any $\mathbb{S}$-finite subset of $\mathbb{S}$ which satisfies $\mu = \mu_E$ on $\mathbb{B}$ and let $f$ be a nonnegative, $\mu$-integrable function. There exists an internal subset $F_f$ of $E$ which satisfies $\|F_f\|/\|E\| = 1$ and, for any internal subset $F$ of $F_f$,

$$\frac{\|F\|}{\|E\|} = 1 \rightarrow \int f \, d\mu = \text{st} \left( \frac{1}{\|F_f\|} \sum_{p \in F} ^{*}f(p) \right).$$

**Proof.** For each $n \in \mathbb{N}$, let $A_n = \{x \mid f(x) > n\}$. Then $\{A_n \mid n \in \mathbb{N}\}$ is a decreasing chain of sets in $\mathbb{B}$ and $\bigcap \{A_n \mid n \in \mathbb{N}\} = \emptyset$. Thus the sequence $\{\mu(A_n)\}$ decreases monotonically to 0. Since $\mu = \mu_E$ on $\mathbb{B}$, it follows that for each $\delta > 0$ in $\mathbb{R}$, there exists $n_0 \in \mathbb{N}$ which satisfies

$$\|M_n E\|/\|E\| < \delta.$$

If $\omega$ is an infinite member of $\mathbb{N}$, then $^{*}A_\omega \subset ^{*}A_n$, so $\|{^{*}A_\omega} \cap E\|/\|E\| < \delta.$

This shows that for every such $\omega$,

$$\|{^{*}A_\omega} \cap E\|/\|E\| = 0.$$

Since $f$ is nonnegative, the sequence of integrals $\int f \, d\mu$ is increasing. By the monotone convergence theorem, the supremum of this sequence is $\int f \, d\mu$. If $\int f \, d\mu = \int f_n \, d\mu$ for some $n \in \mathbb{N}$, then $\mu(A_n) = 0$ and hence

$$\|E \sim \|A_n\|/\|E\| = 1.$$

In this case let $F_f = E \sim \|A_n\|$. If $F \subset F_f$ and $\|F\|/\|E\| = 1$, then

$$\int f \, d\mu = \int f_n \, d\mu = \text{st} \left( \frac{1}{\|F_f\|} \sum_{p \in F} ^{*}f(p) \right)$$

since $^{*}f = ^{*}f_n$ on $F$ and $\mu_F = \mu_E$.

Therefore it may be assumed that $\int f_n \, d\mu < \int f \, d\mu$ for all $n \in \mathbb{N}$. Thus

$$\frac{1}{\|E\|} \sum_{p \in E} ^{*}f_\omega(p) < \int f \, d\mu,$$

for all $n \in \mathbb{N}$. It follows that there is an infinite $\omega$ in $\mathbb{N}$ which satisfies

$$\frac{1}{\|E\|} \sum_{p \in E} ^{*}f_\omega(p) < \int f \, d\mu.$$

In this case let $F_f = E \sim \|A_\omega\|$, so that $\|F_f\|/\|E\| = 1$ by (7). Suppose $F$ is any internal subset of $F_f$ which satisfies $\|F\|/\|E\| = 1$.

Then, for each $n \in \mathbb{N}$,
\[
\int f_n \, d\mu \leq \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right) \\
\leq \text{st} \left( \frac{1}{\|E\|} \sum_{p \in E} *f_\omega(p) \right) = \int f \, d\mu,
\]
using Lemma 1 and the fact that \(*f = *f_\omega\) on \(F\). By the monotone convergence theorem
\[
\text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right) = \int f \, d\mu,
\]
completing the proof.

**Theorem 2.** Let \(\mathcal{B}\) be an \(\sigma\)-algebra of subsets of \(S\) and let \(\mu\) be a countably additive probability measure on \(\mathcal{B}\) which satisfies \(\mu(A) = 0\) for each finite set \(A\) in \(\mathcal{B}\). There exists a *-finite subset \(F\) of *\(S\) which satisfies \(\mu = \mu_F\) on \(\mathcal{B}\) and
\[
\int f \, d\mu = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right)
\]
for every \(\mu\)-integrable function \(f\).

**Proof.** Let \(I\) be the set of nonnegative, \(\mu\)-integrable functions. Since each \(\mu\)-integrable function is the difference of two elements of \(I\), it suffices to find an \(F\) which satisfies the conditions of the theorem for every \(f\) in \(I\). By Theorem 1 (and the remarks following) there exists a *-finite subset \(E\) of *\(S\) which satisfies \(\mu = \mu_E\) on \(\mathcal{B}\). For each \(f \in I\), let \(F_f\) be a subset of \(E\) which satisfies the conditions of Lemma 2. Given \(n \in \mathbb{N}\) and \(f \in I\), define
\[
A(n, f) = \{F | F \text{ is an internal subset of } F_f \text{ and } \|F\|/\|E\| > n/(n+1)\}.
\]
This family of internal sets has cardinality \(\text{card}(\mathbb{N} \times I)\), which is less than \(\kappa\). Moreover, the family has the finite intersection property. \((F_{f_1} \cap \cdots \cap F_{f_n})\) is an element of \(A(m_1, f_1) \cap \cdots \cap A(m_n, f_n)\) whenever \(m_1, \ldots, m_n \in \mathbb{N}\) and \(f_1, \ldots, f_n \in I\). Since \(*\mathcal{M}\) is \(\kappa\)-saturated, there exists a *-finite set \(F\) which satisfies \(F \in A(n, f)\) for every \(n \in \mathbb{N}\) and \(f \in I\) (Theorem 2.7.12 of [5]). That is, \(F \subseteq F_f\) for every \(f \in I\), and \(\|F\|/\|E\| = 1\). It follows by Lemma 2 that \(F\) satisfies the conditions of the theorem.

**Remark.** Theorem 2 is true even if \(*\mathcal{M}\) is not \(\kappa\)-saturated, but the proof of that fact is somewhat more complicated. The proof given here proves the stronger result that \(F\) can be chosen as a subset of any given set \(E\) which satisfies \(\mu = \mu_E\) on \(\mathcal{B}\).
An application. The following standard result can be proved easily using the Riesz Representation Theorem. The nonstandard proof given here uses the extension to \( l_\infty(\mathcal{S}) \) of Robinson's representation result \([9]\) instead.

**Theorem 3.** Let \( X \) be a compact, Hausdorff space, \( \{f_n\} \) a sequence in \( C(X) \) and \( \phi \) a bounded linear functional on \( C(X) \). If \( \{f_n\} \) is uniformly bounded on \( X \) and converges to 0 pointwise, then \( \phi(f_n) \to 0 \).

**Proof.** Let \( \phi \) be any bounded linear functional on \( C(X) \). By the Hahn-Banach theorem, \( \phi \) may be extended to a bounded linear functional \( \tilde{\phi} \) on \( l_\infty(X) \). By the extension to \( l_\infty(X) \) of the principal result of \([9]\), there exist a \(*\)-finite subset of \(*X\) and an internal function \( \lambda \) from \( F \) into \(*R\) which satisfy

\[
\tilde{\phi}(f) = \text{st} \left( \sum_{p \in F} \lambda(p)^* \right)
\]

for every \( f \) in \( l_\infty(X) \), and \( \sum_{p \in F} |\lambda(p)| \) is finite.

Let \( \{f_n\} \) be a sequence in \( C(X) \) which is uniformly bounded on \( X \) by 1, and which converges to 0, pointwise. If \( \phi(f_n) \) does not converge to 0, then it may be assumed (by taking a subsequence) that for some \( \delta > 0 \) in \( R \), \( |\phi(f_n)| > \delta \) for every \( n \in N \). Let \( M = \text{st} (\sum_{p \in F} |\lambda(p)|) + 1 \). For \( n \in N \), define \( A_n = \{x | x \in X \text{ and } |f_n(x)| \geq \delta/2M\} \).

Therefore,

\[
\delta < \left| \sum_{p \in F} \lambda(p)^* f_n(p) \right| \leq \sum_{p \in *A_n \cap F} |\lambda(p)^* f_n(p)| + \sum_{p \in *A_n \cap F} |\lambda(p)^* f_n(p)| \leq \sum_{p \in *A_n \cap F} |\lambda(p)| + \frac{\delta}{2}.
\]

Thus, for each \( n \in N \), \( \sum_{p \in *A_n \cap F} |\lambda(p)| > \delta/2 \).

Now define \( \mu' \) on \( \mathcal{F}(X) \) by

\[
\mu'(A) = \text{st} \left( \sum_{p \in *A \cap F} |\lambda(p)| \right)
\]

for each \( A \subset X \). Then \( \mu' \) is a measure on \( \mathcal{F}(X) \), and \( \mu'(A_n) > \delta/2 \) for every \( n \in N \). It follows that there is an infinite subset \( K \) of \( N \) such that \( \{A_n | n \in K\} \) has the finite intersection property (see Lemma 17.9 of \([4]\)). Since \( *\mathbb{N} \) is an enlargement, there is an element \( p \) of \(*X\) which satisfies \( |\lambda(p)^*| \geq \delta/2M \) for all \( n \in K \). \( X \) is compact, so \( p \) is near-standard to some \( x \in X \). In particular, \( *f_n(p) = f_n(x) \) for every \( n \in N \). This implies \( |f_n(x)| \geq \delta/2M \) for every \( n \in K \),
which contradicts the assumption that \( f_n(x) \) converges to 0. Therefore \( \phi(f_n) \)
must converge to 0.

3. Constructing invariant measures. Let \( G \) be a group of permutations on \( S \),
and assume that \( G \) satisfies Følner’s condition:

For each \( a_1, \ldots, a_n \in G \) and \( k \in \mathbb{N} \), there exists a finite set \( A \subseteq G \)
which satisfies \( \|A \Delta Aa_j\|/\|A\| < 1/(k+1) \) for each \( j = 1, \ldots, n \).

To apply the corresponding statement in \( *\mathbb{N} \), let \( E \) be a \( *\)-finite subset of
\( *G \) which contains \( \{g| g \in G\} \) and let \( \omega \) be an infinite member of \( *\mathbb{N} \). Then
there is a \( *\)-finite set \( F \subseteq *G \) which satisfies \( \|F \Delta Fp\|/\|F\| < 1/\omega \) for every \( p \in E \). In particular,

\[
g \in G \rightarrow \|F \Delta F^*g\|/\|F\| = 0.
\]

If \( F \) satisfies (8), then \( \mu_F \) is a probability measure on \( \mathcal{P}(G) \) and \( \mu_F \)
is invariant under the action of \( G \) on itself by right multiplication. The principal
result of [3] is, essentially, that the converse holds: If there is such a measure on
\( \mathcal{P}(G) \), then \( G \) satisfies Følner’s condition.

Theorem 4. Let \( G \) be a group of permutations of \( S \) and let \( F \) be a \( *\)-finite
subset of \( *G \) which satisfies (8). Let \( \mu \) be any measure on \( \mathcal{P}(S) \) and define
\( \tilde{\mu} \) by

\[
\tilde{\mu}(A) = st \left( \frac{1}{\|F\|} \sum_{p \in F} *\mu(p^*A) \right)
\]

for \( A \subseteq S \). Then \( \tilde{\mu} \) is a \( G \)-invariant measure on \( \mathcal{P}(S) \). Moreover, if \( A \subseteq S \)
satisfies \( \mu(gA) = \mu(A) \) for every \( g \in G \), then \( \tilde{\mu}(A) = \mu(A) \).

Proof. Each element of \( *G \) is a permutation of \( *S \). Thus if \( A, B \) are dis-
joint subsets of \( S \), then \( p^*A, p^*B \) are disjoint subsets of \( *S \) for each \( p \in *G \).
Thus \( *\mu(p^*(A \cup B)) = *\mu(p^*A) + *\mu(p^*B) \). From this the finite additivity of \( \tilde{\mu} \)
is immediate.

Given \( A \) in \( \mathcal{P}(S) \) and \( g \) in \( G \),

\[
|\tilde{\mu}(gA) - \tilde{\mu}(A)| = \left| \frac{1}{\|F\|} \sum_{p \in F} (*\mu(p^*g^*A) - *\mu(p^*A)) \right|
\]

\[
\leq \frac{1}{\|F\|} \sum_{p \in F \Delta F^*g} *\mu(p^*A)
\]

\[
\leq \mu(S) \cdot \|F \triangle F^*g\|/\|F\| = 0.
\]

Therefore \( \tilde{\mu}(gA) = \tilde{\mu}(A) \), so that \( \tilde{\mu} \) is \( G \)-invariant.

Finally, suppose \( A \) is a subset of \( S \) which satisfies \( \mu(gA) = \mu(A) \) for every
\( g \in G \). Then \( *\mu(p^*A) = *\mu(A) \) for every \( p \in *G \). Therefore
To prove Banach’s extension result, let \( G \) be the group of all translations (mod 1) of \([0, 1] \), and let \( \mu \) be any extension of Lebesgue measure to \( \mathcal{P}([0, 1]) \). It is well known, and easy to prove using the decomposition theorem for finitely generated abelian groups, that every abelian group satisfies Föllner’s condition. Since \( G \) is abelian, Theorem 4 can be applied to obtain a \( G \)-invariant measure \( \tilde{\mu} \) on \( \mathcal{P}([0, 1]) \). If \( A \) is a Lebesgue measurable subset of \([0, 1] \), then \( \mu(gA) = \mu(A) \) for every \( g \in G \). Theorem 4 thus asserts that \( \tilde{\mu}(A) = \mu(A) \); that is, \( \tilde{\mu} \) is an extension of Lebesgue measure.

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