MULTIPLIERS ON MODULES OVER THE
FOURIER ALGEBRA

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ABSTRACT. Let $G$ be an infinite compact group and $\hat{G}$ its dual. For
$1 \leq p < \infty$, $L^p(\hat{G})$ is a module over $L^1(\hat{G}) \cong A(G)$, the Fourier algebra of $G$.
For $1 \leq p, q < \infty$, let $\mathcal{M}_{p,q} = \text{Hom}_{A(G)}(L^p(\hat{G}), L^q(\hat{G}))$. If $G$ is abelian, then
$\mathcal{M}_{p,p}$ is the space of $L^p(\hat{G})$-multipliers. For $1 \leq p < 2$ and $p'$ the conjugate
index of $p$,

$$A(G) \cong \mathcal{M}_{1,1} \subset \mathcal{M}_{p,p} = \mathcal{M}_{p',p'} \subset \mathcal{M}_{2,2} \cong L^\infty(G).$$

Further, the space $\mathcal{M}_{p,p}$ is the dual of a space called $\mathcal{A}_p$, a subspace of
$C_0(\hat{G})$. Using a method of J. F. Price we observe that

$$\bigcup \{ \mathcal{M}_{q,q} : 1 \leq q < p \} \subset \mathcal{M}_{p,p} \subset \bigcap \{ \mathcal{M}_{q,q} : p < q \leq 2 \}$$

(where $1 < p < 2$). Finally, $\mathcal{M}_{q,p} = \{0\}$ for $1 \leq p < q < \infty$.

1. Modules over the Fourier algebra. Throughout this paper $G$ will denote
an infinite compact group and $\hat{G}$ its dual (we use the notation from [1]).
Throughout, $1 \leq p, q, r < \infty$. Given $p$, the conjugate index will be denoted by
$p'$ ($1/p + 1/p' = 1$).

Definition. Let $\varphi \in \mathcal{C}_F(\hat{G})$ and so $\varphi = \hat{f}$ for $f$ a trigonometric polynomial
on $G$. We define $\tilde{\varphi}$ by the rule $\tilde{\varphi} = \hat{f}$ where $\hat{f}(x) = f(x^{-1})$, $x \in G$.

Proposition 1. The map $\varphi \mapsto \tilde{\varphi}$ from $\mathcal{C}_F(\hat{G})$ to $\mathcal{C}_F(\hat{G})$ extends to an
isometry of $L^p(\hat{G})$ ($1 \leq p < \infty$) and of $C_0(\hat{G})$.

Proof. For $f$ a trigonometric polynomial on $G$, we have that $(\hat{f})^* = (\hat{f})^*$ (see [1, p. 87]). Thus for $\varphi \in \mathcal{C}_F(\hat{G})$, $\|\tilde{\varphi}\|_p = \|\varphi\|_p$. □

Definition. Let $\varphi, \psi \in \mathcal{C}_F(\hat{G})$, we define $\varphi \ast \psi \in \mathcal{C}_F(\hat{G})$ by the rule

$$\varphi \ast \psi = \hat{\varphi} \hat{\psi} \hat{\varphi}$$

(\varphi denotes the inverse Fourier transform of \varphi [1, p. 97]). We note that $\|\varphi \ast \psi\|_1 \leq \|\varphi\|_1 \|\psi\|_1$. $\varphi, \psi \in \mathcal{C}_F(\hat{G})$ (see [1, p. 93]). We define the
pairing $\langle \varphi, \psi \rangle = \text{Tr}(\varphi \hat{\psi}) = (\varphi \ast (\hat{\psi}^*)^*)(e) = \int_G \varphi(x) \hat{\psi}(x) dm_G(x), \varphi, \psi \in \mathcal{C}_F(\hat{G})$.

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The map \((\phi, \psi) \mapsto \langle \phi, \psi \rangle\) extends to a pairing between \(\mathcal{L}^p(\hat{G})\) and \(\mathcal{L}^q(\hat{G})\) (1 \leq p < \infty), that is, \(\|\langle \phi, \psi \rangle\| \leq \|\phi\|_p \|\psi\|_q\), and \(\|\phi\|_p = \sup\|\langle \phi, \psi \rangle| : \|\psi\|_p \leq 1\), \(\phi, \psi \in \mathcal{C}_F(\hat{G})\) (see [1, p. 144]).

**Theorem 2.** For \(1/p + 1/q > 1\), the map \((\phi, \psi) \mapsto \phi \times \psi: \mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G}) \to \mathcal{C}_F(\hat{G})\) extends to a map of \(\mathcal{L}^p(\hat{G}) \times \mathcal{L}^q(\hat{G}) \to \mathcal{L}^r(\hat{G})\), \(1/r = 1/p + 1/q - 1\) (we replace \(\mathcal{L}^\infty(\hat{G})\) by \(\mathcal{C}_0(\hat{G})\)), such that \(\|\phi \times \psi\|_r \leq \|\phi\|_p \|\psi\|_q\), \(\phi \in \mathcal{L}^p(\hat{G})\), \(\psi \in \mathcal{L}^q(\hat{G})\).

**Proof.** For \(\phi, \psi, \theta \in \mathcal{C}_F(\hat{G})\) we define the form \(F\) on \(\mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G})\) by the rule \(F(\phi, \psi, \theta) = \langle \phi \times \psi, \theta \rangle = \int_G \phi(x) \psi(x) \theta(x) dm_G(x)\), and thus \(F\) is symmetric. Now \(\|F(\phi, \psi, \theta)\| \leq \|\phi \times \psi\|_1 \|\theta\|_\infty \leq \|\phi\|_1 \|\psi\|_1 \|\theta\|_\infty\), \(\phi, \psi, \theta \in \mathcal{C}_F(\hat{G})\). Let

\[M(a_{123}) = \sup\|F(\phi_1, \phi_2, \phi_3)|: \phi_j \in \mathcal{C}_F(\hat{G}), \|\phi_j\|_1/a_j \leq 1, 1 \leq j \leq 3\],

\[a_1, a_2, a_3 \in [0, 1]\]. By the Riesz-Thorin convexity theorem for integration algebras [1, p. 143], it follows that \(\log M\) is a convex function on \([0, 1] \times [0, 1] \times [0, 1]\). Since \(M(1, 0, 1), M(1, 1, 0), M(0, 1, 1) \leq 1\), it follows by interpolating that \(M(1/p, 1/q, 1/r') \leq 1\) where \(1/r = 1/p + 1/q - 1\). □

**Corollary 3.** For 1 \leq p < \infty, \(\mathcal{L}^1(\hat{G}) \times \mathcal{L}^p(\hat{G}) = \mathcal{L}^p(\hat{G})\) and so \(\mathcal{L}^p(\hat{G})\) is an \(\mathcal{L}^1(\hat{G})\)-module. Also \(\mathcal{L}^1(\hat{G}) \times \mathcal{C}_0(\hat{G}) = \mathcal{C}_0(\hat{G})\). For 1 \leq p < \infty, \(\mathcal{L}^p(\hat{G}) \times \mathcal{L}^q(\hat{G}) \subset \mathcal{C}_0(\hat{G})\). For \(1/p + 1/q > 1\), \(\mathcal{L}^p(\hat{G}) \times \mathcal{L}^q(\hat{G}) \subset \mathcal{L}^r(\hat{G})\), \(1/r = 1/p + 1/q - 1\).

**Theorem 4.** \(\mathcal{L}^2(\hat{G}) \times \mathcal{L}^2(\hat{G}) = L^1(\hat{G})\).

**Proof.** Let \(\phi, \psi \in \mathcal{L}^2(\hat{G})\) and choose \(\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty}\) sequences of trigonometric polynomials on \(G\) such that \(\int_G f_n \phi \psi dm_G = f \phi \psi \in \mathcal{L}^2(\hat{G})\). Then \(f_n g_n \in L^1(\hat{G})\), and we wish to show that \(\phi \times \psi = \lim_{n \to \infty} f_n \times \phi \psi = \lim_{n \to \infty} f_n g_n \in L^1(\hat{G})\). But this follows since \(L^1(\hat{G})\) is a Cauchy sequence in \(L^1(\hat{G})\).

Conversely, for \(b \in L^1(\hat{G})\), write \(b = f g, f, g \in L^2(\hat{G})\). Choose \(\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty}\) sequences from \(\mathcal{C}_F(\hat{G})\) such that \(\phi_n \to f, \psi_n \to g \in L^2(\hat{G})\). Now \(\phi_n \psi_n \to f g \in L^1(\hat{G})\) and so \(b = (f g) = (\lim_{n \to \infty} \phi_n \psi_n) = \lim_{n \to \infty} (\phi_n \psi_n) \times \lim_{n \to \infty} \phi_n \times \psi_n \in \mathcal{L}^2(\hat{G}) \times \mathcal{L}^2(\hat{G})\). □

2. Multipliers on modules over the Fourier algebra.

**Definition.** Let \(1/p + 1/q \geq 1\), \(\phi \in \mathcal{L}^p(\hat{G})\), \(\psi \in \mathcal{L}^q(\hat{G})\). We define \(\langle \phi, \psi \rangle = \phi \times \psi\). This is an extension of \(\langle \cdot, \cdot \rangle\) from \(\mathcal{C}_F(\hat{G}) \times \mathcal{C}_F(\hat{G})\).

**Definition.** Let 1 \leq p, q \leq \infty. We define \(\mathcal{M}_{p,q} = \text{Hom}_{\mathcal{L}^1(\hat{G})}(\mathcal{L}^p(\hat{G}), \mathcal{L}^q(\hat{G}))\), except that we replace \(\mathcal{L}^\infty(\hat{G})\) by \(\mathcal{C}_0(\hat{G})\). Note that \(\mathcal{L}^p(\hat{G})\) is an \(\mathcal{L}^1(\hat{G})\)-module (Corollary 3). (See Rieffel [7] for a more general setting.)
Proposition 5. Let $T : \mathcal{C}_F(G) \to \mathcal{C}_0(G)$ be a linear map. Define $\|T\|_{p,q} = \sup \{ |\langle T\phi, \psi \rangle| : \|\phi\|_p \leq 1, \|\psi\|_{q'} \leq 1, \phi, \psi \in \mathcal{C}_F(G) \}$. Then $\log \|T\|_{1/a_1,1/a_2}$ is a convex function for $(a_1, a_2) \in [0, 1] \times [0, 1]$.

Proof. Apply the Riesz-Thorin convexity theorem for integration algebras [1, p. 143].

Proposition 6. $\mathbb{M}_{2,2} \cong L^\infty(G)$.

Proof. By taking the inverse Fourier transform we see that $\mathbb{M}_{2,2}$ is isomorphic to the space of bounded maps $T$ from $L^2(G)$ to $L^2(G)$ which commute with multiplication by elements of $A(G)$, that is, $T : L^2(G) \to L^2(G)$, $T(g) = f(Tg)$, $f \in A(G)$, $g \in L^2(G)$. Thus $T$ is multiplication by an element of $L^\infty(G)$, that is, there exists $b \in L^\infty(G)$ such that $Tg = bg$, $g \in L^2(G)$ (let $b = T1$).

Theorem 7. Let $1 \leq p, q \leq \infty$. Then $\mathbb{M}_{p,q} = \mathbb{M}_{q',p'}$.

Proof. We first suppose $1 < p, q < \infty$. Let $T \in \mathbb{M}_{p,q}$. Thus $T : \mathcal{C}_F(G) \to \mathcal{C}_0(G)$, and $\|T\|_{p,q} < \infty$. Now $T(\phi \times \psi) = \phi \times (T\psi)$, $\phi, \psi \in \mathcal{C}_F(G)$. Define the adjoint of $T$, $S$ by $S : \mathcal{C}_F(G) \to \mathcal{C}_0(G)$ and $\langle T\phi, \psi \rangle = \langle \phi, S\psi \rangle$, $\phi, \psi \in \mathcal{C}_F(G)$. For $\phi, \psi \in \mathcal{C}_F(G)$, $\langle T\phi, \psi \rangle = (\langle T\phi, \psi \rangle) = (\langle T\phi \times \psi \rangle) = (T(\phi \times \psi)) = (T(\phi) \times \psi) = (\phi \times (T\psi))$. Thus $S$ and $T$ agree on $\mathcal{C}_F(G)$.

Now for $\phi, \psi \in \mathcal{C}_F(G)$, $\langle T\phi, \psi \rangle = \langle \phi, S\psi \rangle = \langle \phi, T\psi \rangle$, and so $\|T\|_{q',p'} = \|T\|_{q',p'}$. It follows that $T | \mathcal{C}_F(G)$ extends uniquely to an element of $\mathbb{M}_{q',p'}$, and so $\mathbb{M}_{p,q} \subset \mathbb{M}_{q',p'}$. By symmetry $\mathbb{M}_{q',p'} = \mathbb{M}_{p,q}$.

We consider now the exceptional cases. Since $\mathcal{O}^1(G)$ has an identity, we obtain $\mathbb{M}_{1,1} = \mathcal{O}^1(G)$ for $1 \leq p < \infty$ and $\mathbb{M}_{1,\infty} = \mathcal{C}_0(G)$. Further, applying the previous argument we see that $T \in \mathbb{M}_{p,\infty}$ implies $T \in \mathbb{M}_{1,1} = \mathcal{O}^1(G)$. But by Corollary 3, $\mathcal{O}^p(G) \subset \mathbb{M}_{p,\infty}$, so $\mathbb{M}_{p,\infty} = \mathbb{M}_{p,1}$. The other spaces $\mathbb{M}_{p,1}$ ($p > 1$) and $\mathbb{M}_{\infty,q}$ ($q < \infty$) will be shown to be trivial in Theorem 10.

Theorem 8. Let $1 < p < q < 2$. Then $A(G) \cong \mathcal{O}^1(G) = \mathbb{M}_{1,1} \subset \mathbb{M}_{p,p} \subset \mathbb{M}_{q,q} \subset \mathbb{M}_{2,2} \cong L^\infty(G)$.

Proof. That $\mathcal{O}^1(G) = \mathbb{M}_{1,1}$ follows since $A(G)$ has an identity.

Since $\mathcal{O}^p(G)$ is an $\mathcal{O}^1(G)$-module, $\mathbb{M}_{1,1} \subset \mathbb{M}_{p,p}$ (recall Theorem 2).

Let $T \in \mathbb{M}_{q,q}$. Then $\|T\|_{q,q} = \|T\|_{q',q'} < \infty$. Since $\log \|T\|_{1/a_1,1/a_2}$ is a convex function of $(a_1, a_2) \in [0, 1] \times [0, 1]$, $\|T\|_{2,2} \leq \|T\|_{q,q}$. Thus $\mathbb{M}_{q,q} \subset \mathbb{M}_{2,2}$.

Now for $T \in \mathbb{M}_{p,p}$, $\|T\|_{p,p} < \infty$. Also $\|T\|_{2,2} \leq \|T\|_{p,p} < \infty$. Now since $1/2 < 1/q < 1/p$, we can interpolate to get $\|T\|_{q,q} \leq \|T\|_{p,p} < \infty$. Thus $\mathbb{M}_{p,p} \subset \mathbb{M}_{q,q}$.

Theorem 9. Let $1 \leq p < 2$. Then $\mathbb{M}_{p,p} \not\cong \mathbb{M}_{2,2}$. 

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Proof. By way of contradiction, suppose $\mathcal{M}_{p,p} = \mathcal{M}_{1/2} = L^\infty(G)^\sim$. Then $L^\infty(G)^\sim \subset L^p(G)$ (since $\hat{f} \in L^p(\hat{G})$), and so $\|f\|_p \leq C\|f\|_\infty$, $f \in L^\infty(G)$, $C < \infty$. In particular, $f \mapsto \hat{f}$ maps $C(G)$ into $L^p(\hat{G})$, and its adjoint $\mathcal{T}$ maps $L^p(\hat{G})$ into $M(G)$. Further $\mathcal{T}: L^p(\hat{G}) \to L^1(G)$ (since $\mathcal{T}(\mathcal{C}_p(\hat{G})) \subset L^1(G)$ and $L^1(G)$ closed). Let $\phi \in \mathcal{L}^p(\hat{G})$ and $\omega \in \mathcal{L}^\infty(\hat{G})$. Then $\phi \omega \in \mathcal{L}^p(\hat{G})$ and $(\phi \omega)^\sim \in L^1(G)$, that is, the map $\omega \mapsto (\phi \omega)^\sim$ takes $\mathcal{L}^\infty(\hat{G})$ into $L^1(G)$. It follows now from a theorem of S. Helgason [5, p. 785] that $\phi \in \mathcal{L}^2(\hat{G})$. Thus $\mathcal{L}^p(\hat{G}) \subset \mathcal{L}^2(\hat{G})$, a contradiction. 

Theorem 10. Let $1 < p < q < \infty$, then $\mathcal{M}_{q,p} = \{0\}$.

Proof. First, let $1 < p' < 2 < p$. We show that $\mathcal{M}_{p,p'} = \{0\}$. For if $T \in \mathcal{M}_{p,p'}$, $T \neq 0$, then there exists $b \in L^\infty(G)$, $b \neq 0$, such that $f \mapsto bf$ is a bounded linear operator from $L^p(G) \to L^{p'}(G)$ (consider the maps: $L^p(G) \to \mathcal{L}^p(G)$ and $\mathcal{L}^p(G) \to L^1(G)$, see [1, p. 144]). Thus there exists $C < \infty$ such that $\|bf\|_p \leq C\|f\|_{p'}$, $f \in L^p(G)$. Let $\epsilon > 0$ be such that $\{x: |f(x)| \geq \epsilon\}$ contains a measurable set $E$ with $m(E) > 0$, and let $\chi_E$ denote the characteristic function of $E$. Then

$$
\epsilon^p m_G(E) \leq \|b\chi_E\|_p^p \leq C^p \|\chi_E\|_{p'}^p = C^p (m_G(E))^{p/p'},
$$

and so $0 < \epsilon^p/C^p \leq (m_G(E))^{p/p' - 1}$. But let $m_G(E)$ tend to $0$ for the required contradiction. Thus we have established $\mathcal{M}_{p,p'} = \{0\}, 1 < p' < 2 < p$.

Now let $T \in \mathcal{M}_{q,p}$, $T \neq 0$, $1 < p < q < \infty$, excepting the case $\mathcal{M}_{\infty,1}$. Thus, $\|T\|_{q,p'} = \|T\|_{q,p} < \infty$. The Riesz-Thorin convexity theorem implies for $1/r = 1/2 - 1/2p + 1/2q$ that $\mathcal{M}_{r,q'} \neq \{0\}$, a contradiction. Finally, $\mathcal{M}_{\infty,1} \subset \mathcal{M}_{2,1} = \{0\}. \square$

Remark. The proof of the above theorem was suggested to us by our colleague John Fournier.

3. Multipliers as dual spaces. For $G$ abelian, $\mathcal{M}_{p,p}$ is the space of $L^p(\hat{G})$-multipliers, and A. Figà-Talamanca [4] (also M. Rieffel [7]) has shown it to be a dual space. We now will exhibit this result for the case of $G$ nonabelian (compact). For $p = 1$, $\mathcal{M}_{1,1}$ is clearly a dual space; indeed, $\mathcal{M}_{1,1} = \mathcal{L}^1(\hat{G}) = \mathcal{C}_0(\hat{G})^*$ (see [1, p. 88]).

Definition. Let $1 < p \leq 2$. For $\phi \in \mathcal{C}_0(\hat{G})$, we define

$$
\|\phi\|_p = \inf \left\{ \sum_{n=1}^\infty \|\phi_n\|_p \psi_n\|_p : \phi = \sum_{n=1}^\infty \phi_n \times \psi_n \text{ (convergence in } \mathcal{C}_0(\hat{G})) \right\},
$$

We use the convention that $\inf \emptyset = \infty$. The subspace of $\mathcal{C}_0(\hat{G})$ consisting of all $\phi$ with $\|\phi\|_p < \infty$ is denoted by $\mathcal{A}_p$. 
Remark. By Theorem 4, $\mathcal{A}_2 = L^1(G)$.  

Proposition 11. For $1 < p \leq 2$, $\mathcal{A}_p$ is a Banach space.

Proof. It is easy to show $\| \cdot \|_p$ is a norm. We wish now to show that $\mathcal{A}_p$ is complete with respect to $\| \cdot \|_p$. Let $\{ \phi_n \}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{A}_p$. We may assume that $\| \phi_n - \phi_{n+1} \|_p < 1/2^{n+1}$. Let $\psi_n = \phi_{n+1} - \phi_n \in \mathcal{A}_p$, and so write $\psi_n$ as $\sum_{m=1}^{\infty} \theta_{nm} \omega_{nm} \phi_{n+1}^{m} \phi_{n}^{m'}$. Let $\phi = \phi_1 + \sum_{m=1}^{\infty} \psi_n$, and $\| \phi \|_p \leq \| \phi_1 \|_p + \sum_{m=1}^{\infty} \| \theta_{nm} \|_p \| \omega_{nm} \phi_{n+1}^{m} \phi_{n}^{m'} \|_p < 1/2^n$, and so $\phi \in \mathcal{A}_p$. Also $\| \phi_n - \phi \|_p = \| \sum_{m=1}^{\infty} \psi_n \|_p < \sum_{m=1}^{\infty} \| \psi_n \|_p < \sum_{n=m+1}^{\infty} 1/2^n$, which is small for large enough $m$. $\square$  

Theorem 12. Let $\xi \in \mathcal{A}_p^*$ $(1 < p < 2)$. Then there exists $T \in M_{p,p}$ such that $\| T \|_{p,p} \leq \| \xi \|_\infty$ and $\langle T \phi, \psi \rangle = \xi(\phi \times \psi), \phi, \psi \in \mathcal{C}_p(G)$.  

Proof. For $\phi, \psi \in \mathcal{C}_p(G)$, $|\xi(\phi \times \psi)| \leq \| \phi \|_p \| \psi \|_p \| \xi \|_\infty \leq \| \phi \|_p \| \psi \|_p \| \xi \|_\infty.$ Thus, for each $\phi \in \mathcal{C}_p(G)$, the map $\psi \mapsto \xi(\phi \times \psi)$ extends to a bounded linear functional on $\mathcal{L}^p(G)$. Let $\omega \in \mathcal{L}^p(G) = (\mathcal{L}^p(G))^*$ be such that $\langle \omega, \psi \rangle = \xi(\phi \times \psi)$. Define $T \phi = \omega(\phi \times \psi)$. Now $T : \mathcal{C}_p(G) \to \mathcal{L}^p(G)$ and $\| T \|_{p,p} \leq \| \xi \|_\infty$, so we may extend $T$ to all of $\mathcal{L}^p(G)$. Finally, to see that $T \in M_{p,p}$ we note that $\langle T(\phi_1 \times \phi_2), \psi \rangle = \xi((\phi_1 \times \phi_2) \times \psi) = \xi(\phi_1 \times (\phi_2 \times \psi)) = \langle (T \phi_1) \times (\phi_2 \times \psi), \phi_1, \phi_2, \psi \in \mathcal{C}_p(G)$. Thus $T(\phi_1 \times \phi_2) = (T \phi_1) \times (\phi_2 \times \phi_1, \phi_2 \in \mathcal{C}_p(G))$. Thus $T \in M_{p,p}$. $\square$  

Proposition 13. Let $\phi \in \mathcal{L}^p(G)$ $(1 < p < \infty)$ or $\phi \in \mathcal{C}_0(G)$ $(p = \infty)$ and $\varepsilon > 0$. Then there exists a sequence $\{ \phi_n \}_{n=1}^{\infty} \subset \mathcal{C}_p(G)$ such that $\| \phi_n - \phi \|_p < \| \phi \|_p + \varepsilon$ and $\Sigma_{m=1}^{\infty} \| \phi_m \|_p < \| \phi \|_p + \varepsilon$ and $\Sigma_{m=1}^{\infty} \| \phi_m \|_p < \| \phi \|_p + \varepsilon$.  

Proof. For $n = 1, 2, \ldots$, let $\psi_n \in \mathcal{C}_p(G)$ be such that $\| \psi_n - \phi \|_p < \varepsilon/2^n$. Let $\phi_1 = \psi_1$ and, for $n = 2, 3, \ldots$, let $\phi_n = \psi_{n+1} - \psi_n$. Then $\{ \phi_n \}_{n=1}^{\infty} \subset \mathcal{C}_p(G), \Sigma_{m=1}^{\infty} \| \phi_m \|_p < \| \phi \|_p + \varepsilon$, and $\Sigma_{n=1}^{\infty} \phi_n = \psi \in \mathcal{C}_p(G)$. $\square$  

Proposition 14. Let $\phi \in \mathcal{L}^p(G), \psi \in \mathcal{L}^p(G)$, and $\varepsilon > 0$ $(1 < p \leq 2)$. Then there exist sequences $\{ \phi_n \}_{n=1}^{\infty}, \{ \psi_n \}_{n=1}^{\infty} \subset \mathcal{C}_p(G)$ such that $\Sigma_{n=1}^{\infty} \| \phi_n \|_p \| \psi_n \|_p < \| \phi \|_p \| \psi \|_p + \varepsilon$, and $\Sigma_{n=1}^{\infty} \phi_n \times \psi_n = \phi \times \psi$ (convergence in $\mathcal{C}_0(G)$).  

Proof. Let $\varepsilon', \varepsilon'' > 0$ be chosen in a way to be specified later. By Proposition 13, there exist sequences $\{ \phi_n \}_{n=1}^{\infty}, \{ \psi_n \}_{n=1}^{\infty} \subset \mathcal{C}_p(G)$ such that $\Sigma_{n=1}^{\infty} \phi_n = \phi, \Sigma_{n=1}^{\infty} \psi_n = \psi, \Sigma_{n=1}^{\infty} \| \phi_n \|_p < \| \phi \|_p + \varepsilon'$, and $\Sigma_{n=1}^{\infty} \| \psi_n \|_p < \| \psi \|_p + \varepsilon''$. Let $\phi_n = \Sigma_{m=1}^{n} \phi_k$ and $\psi_n = \Sigma_{m=1}^{n} \psi_k$. Now $\phi_n \times \psi_n \to \phi \times \psi$ in $\mathcal{C}_0(G)$ (by joint continuity). Now $\Sigma_{n=1}^{\infty} \phi_k \times \psi_k \to \phi \times \psi$. Also, $\Sigma_{n=1}^{\infty} \| \phi_k \|_p \| \psi_k \|_p < (\| \phi \|_p + \varepsilon')(\| \psi \|_p + \varepsilon'') < \| \phi \|_p \| \psi \|_p + \varepsilon$ for the
appropriate choice of $\epsilon'$, $\epsilon''$. Finally, note that $\phi'_n \times \psi'_n = \sum_{k,l=1}^{n} \phi_k \times \psi_l$. \hfill $\square$

**Proposition 15.** Let $\omega \in \hat{G}$, $(1 < p \leq 2)$, and $\epsilon > 0$. Then there exist sequences $\{\phi_n\}_{n=1}^{\infty}, \{\psi_n\}_{n=1}^{\infty} \subset \mathbb{C}_F(\hat{G})$ such that $\omega = \sum_{n=1}^{\infty} \phi_n \times \psi_n$ (convergence in $\mathbb{C}_0(\hat{G})$) and $\sum_{n=1}^{\infty} \|\phi_n\|_p \|\psi_n\|_p < \|\omega\|_p + \epsilon$.

**Proof.** There exist sequences $\{\phi_n\}_{n=1}^{\infty} \subset \mathbb{L}^p(\hat{G})$ and $\{\psi_n\}_{n=1}^{\infty} \subset \mathbb{L}^p(\hat{G})$ such that $\omega = \sum_{n=1}^{\infty} \phi_n \times \psi_n$ and $\sum_{n=1}^{\infty} \|\phi_n\|_p \|\psi_n\|_p < \|\omega\|_p + \epsilon/2$. For each $n = 1, 2, \ldots$, there exist sequences $\{\phi_{nm}\}_{m=1}^{\infty}, \{\psi_{nm}\}_{m=1}^{\infty} \subset \mathbb{C}_F(\hat{G})$ such that $\phi_n \times \psi_n = \sum_{m=1}^{\infty} \phi_{nm} \times \psi_{nm}$ and $\sum_{m=1}^{\infty} \|\phi_{nm}\|_p \|\psi_{nm}\|_p < \|\phi'_n\|_p \|\psi'_n\|_p + \epsilon/2^{n+1}$. Now $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \|\phi_{nm}\|_p \|\psi_{nm}\|_p < \sum_{m=1}^{\infty} \|\phi_{nm}\|_p \|\psi_{nm}\|_p < \|\omega\|_p + \epsilon$ and $\sum_{m=1}^{\infty} \|\phi_{nm}\|_p \|\psi_{nm}\|_p < \omega_p + \epsilon$. \hfill $\square$

**Proposition 16.** Let $\delta > 0$ and let $X_\delta = \{\omega \in \mathbb{C}_F(\hat{G}) : \omega = \sum_{n=1}^{N} \phi_n \times \psi_n, \phi_n, \psi_n \in \mathbb{C}_F(\hat{G}), \|\omega\|_p < \|\omega\|_p < \sum_{n=1}^{N} \|\phi_n\|_p \|\psi_n\|_p, \text{ some } N = 1, 2, \ldots \}$. Then each $X_\delta$ is dense in $\mathbb{C}_p(1 \leq p < 2)$.

**Proof.** Fix $\delta > 0$, $\xi \in \mathbb{G}_p$, and $0 < \epsilon < \delta/2$. By Proposition 15, there exist sequences $\{\phi_n\}_{n=1}^{\infty}, \{\psi_n\}_{n=1}^{\infty} \subset \mathbb{C}_F(\hat{G})$ such that $\xi = \sum_{n=1}^{\infty} \phi_n \times \psi_n$ and $\sum_{n=1}^{\infty} \|\phi_n\|_p \|\psi_n\|_p < \|\xi\|_p + \epsilon$. Choose $N$ such that $\sum_{n=1}^{N+\infty} \|\phi_n\|_p \|\psi_n\|_p < \epsilon$ and let $\omega = \sum_{n=1}^{N} \phi_n \times \psi_n$. Then $\|\omega - \xi\|_p \leq \|\sum_{n=1}^{N+\infty} \phi_n \times \psi_n\|_p < \epsilon$ and $\sum_{n=1}^{N} \|\phi_n\|_p \|\psi_n\|_p < \sum_{n=1}^{N} \|\phi_n\|_p \|\psi_n\|_p < \|\xi\|_p + \epsilon \leq \|\omega\|_p + 2 \epsilon$. Thus $\omega \in X_\delta$ and $\|\omega - \xi\|_p < \epsilon$. \hfill $\square$

**Theorem 17.** Let $1 < p \leq 2$ and $T \in M_{p,p}$. Then $T$ extends to a bounded linear map from $\mathbb{C}_p(\hat{G}) \to \mathbb{C}_p$, and the linear functional $T^*: \mathbb{C}_p \to \mathbb{C}$ given by $T^*(\omega) = (T\omega)_t$ is in $\mathbb{C}^*_p$ with $\|T^*\| \leq \|T\|$. Thus $\mathbb{G}_p \approx \mathbb{M}_{p,p}$.

**Proof.** Let $\delta > 0$ and $\omega \in X_\delta \subset \mathbb{C}_F(\hat{G}) \subset \mathbb{L}^p(\hat{G})$. Write $\omega = \sum_{n=1}^{N} \phi_n \times \psi_n$, $\phi_n, \psi_n \in \mathbb{C}_F(\hat{G})$, where $\|\omega\|_p < \|\omega\|_p < \sum_{n=1}^{N} \|\phi_n\|_p \|\psi_n\|_p$. Now $T\omega = \sum_{n=1}^{N} T(\phi_n \times \psi_n) = \sum_{n=1}^{N} (T\phi_n) \times \psi_n$, and $\|T\omega\|_p \leq \sum_{n=1}^{N} \|T\phi_n\|_p \|\psi_n\|_p \leq \|T\|_{p,p} \|\omega\|_p + \delta$. But $X_\delta$ is dense in $\mathbb{G}_p$ and so $T$ extends to $\mathbb{G}_p$ with norm less than or equal to $\|T\|_{p,p}(1 + \delta)$. But $\delta > 0$ is arbitrary and so $\|T\|_{p,p} \leq \|T\|_{p,p} \|\omega\|_p$. \hfill $\square$

**Corollary 18.** For $1 \leq r < 2$, $M_{r,r} \subset \bigcap_{s,s'} \{M_{s,s'} : r < s < 2\}$, and for $1 < r < 2$, $\bigcup_{s,s'} \{M_{s,s'} : 1 < s < r \} \subset M_{r,r}$. \hfill $\square$

**Proof.** J. F. Price [6, pp. 326–330] has given a general argument based on the Riesz-Thorin convexity theorem which yields the corollary using only the facts that $M_{q,q'} \not= M_{2,2}$ ($q < 2$) (see Theorem 9), that $M_{q,q'}$ is the dual space of $\mathbb{G}_q$, and that $\mathbb{G}_q$ contains $\mathbb{L}^1(\hat{G})$ as a dense subspace (see Proposition 16). \hfill $\square$

**Definition.** Let $1 \leq p, q < \infty$, $1/p + 1/q \geq 1$, and $1/r = 1/p + 1/q - 1$. We define for $\phi \in \mathbb{L}(\hat{G})$, \hfill $\square$
The subspace of $\mathcal{L}^r(\hat{G})$ consisting of all $\phi$ with $\|\phi\|_{p,q} < \infty$ is denoted by $C^p_{p,q}$.

Remark. For $1 < p < \infty$, observe that $C^p_{p,p'} = C^p_p$, and indeed, for $1 \leq p < q \leq \infty$, one can show that $C^p_{p,q} \cong H^p_{\infty}$, by appropriately modifying the preceding proofs. (Note for $p > q$ that $H^p_{\infty} = \mathbb{R}^n$, and for $1 < p < q < \infty$ that $1/p + 1/q' > 1$.)

Definition. Let $W_0$ denote the weak operator topology on $C^p_{p,p}$, and let $W^*$ denote the weak-* topology on $C^p_{p,p}$ $(1 < p \leq 2)$ from the pairing of $C^p_p$ with $C^p_{p,p}$. Thus $T^* a \to T \in C^p_{p,p}$ in $W_0$ if and only if $(T^* a \phi, \psi) \to (T \phi, \psi)$, $\phi \in \mathcal{L}^p(\hat{G})$, $\psi \in \mathcal{L}^q(\hat{G})$; and $T^* a \to T$ in $W^*$ if and only if $T^* a \phi \to T \phi$, for each $\phi \in C^p_p$.

Theorem 19. In $C^p_{p,p}$ $(1 < p \leq 2)$, $W_0 \subset W^*$.

Proof. Let $T^* a \to T$ in $C^p_{p,p}$ with $T^* a \to a \to T$ in $W_0$. Thus $T^* \phi \to T \phi \to a \to T \phi$ for all $\phi \in C^p_p$. Extend $T_0 \to T$ to operators from $C^p_p$ to $C^p_p$ (as in Theorem 17) such that $T_0 \phi = T \phi$, $\phi \in C^p_p$. Let $\phi \in \mathcal{L}^p(\hat{G})$, $\psi \in \mathcal{L}^q(\hat{G})$.

We wish to show that $T_0 \phi \to T \phi \to a \to T \phi$. It suffices to show that $S(\phi \times \psi) = (S\phi) \times \psi$, $S \in C^p_{p,p}$, for then $T_0 \phi \to T \phi \to a \to T \phi$.

Now let $\phi_n \to \phi$ in $C^p_p$, $\psi_n \to \psi$ in $C^p_p$. Then for $S \in C^p_{p,p}$, we have that $T^* \phi \to T \phi$ in $C^p_p$ and so $S(\phi \times \psi) = \lim_{n \to \infty} S(\phi_n \times \psi_n) = \lim_{n \to \infty} (S\phi_n) \times \psi_n = (S\phi) \times \psi$. \qed

Corollary 20. On bounded subsets of $C^p_{p,p}$ $(1 < p \leq 2)$, $W^* = W_0$.

Proof. Bounded closed subsets of $C^p_{p,p}$ are $W_0$-compact. \qed

Theorem 21. Let $\Phi$ denote the $w^*$-closure of $C^p_F(\hat{G})$ or $C^1_F(\hat{G})$ in $C^p_{p,p}$.

1 < p < \infty. Then $\Phi = C^p_{p,p}$.

Proof. Suppose $\Phi \neq C^p_{p,p}$, then there exists $\omega \in C^p_p$ such that $\omega \neq 0$ and $T^* \omega = 0$ for all $T \in C^p_F(\hat{G}) \subset C^p_{p,p}$. But if $T \in C^p_F(\hat{G})$, considered as a subspace of $C^p_{p,p}$, then there exists a $\phi \in C^p_F(\hat{G})$ such that $T \psi = \phi \times \psi$ for all $\psi \in \mathcal{L}^p(\hat{G})$. Thus $T^* \omega = (T \phi, \omega) = (\phi, \omega) = 0$, for all $\phi \in C^p_F(\hat{G})$. But $\omega \in C^p_p \subset C^p_p$, so $\omega = 0$. \qed

Corollary 22. For $1 < p < \infty$, $C^p_F(\hat{G})$ is $W_0$-dense in $C^p_{p,p}$.

Remark. An invariant mean on $\mathcal{L}^\infty(\hat{G})$ is a bounded linear functional $\mu_0$ on
$\ell^\infty(\hat{G})$ such that (1) $p(\phi) \geq 0$ whenever $\phi \geq 0$, (2) $p(l) = 1$ ($l$ is the identity in $\ell^\infty(\hat{G})$), and (3) $p(f \times \phi) = f(e)p(\phi)$, $f \in A(G)$, $\phi \in \ell^\infty(\hat{G})$. In [2] we showed that invariant means exist on $\ell^\infty(\hat{G})$.

Let $p$ be an invariant mean on $\ell^\infty(\hat{G})$. Define $T: \ell^\infty(\hat{G}) \to \ell^\infty(\hat{G})$ by $(\psi, T\phi) = p(\psi \times \phi)$, $\psi \in \ell^1(\hat{G})$, $\phi \in \ell^\infty(\hat{G})$; and so $T\phi = p(\phi)l$. Thus $T \in \text{Hom}_{\ell^1(\hat{G})}(\ell^\infty(\hat{G}), \ell^\infty(\hat{G}))$. Also $Tf = 0$ for $f \in L^1(G)^\wedge$, and it follows that $T$ annihilates $C_0(\hat{G}) = c1(L^1(G)^\wedge)$ (closure in $C_0(\hat{G})$): since for $\mu \in M(G)$, $p(\mu) = \mu(\{e\})$ (see [3]).

**BIBLIOGRAPHY**


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