TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 173, November 1972

ACTIONS OF GROUPS OF ORDER pq

BY

CONNOR LAZAROV (1)

ABSTRACT. We study the bordism group of stably complex G-manifolds in the case where G is a metacyclic group of order pq and p and q are distinct primes. This bordism group is a module over the complex bordism ring and we compute the projective dimension of this module. We develop some techniques necessary for the study of this module in case G is a more general metacyclic group.

The purpose of this paper is to study the bordism theory of actions of a metacyclic group on stably complex manifolds in a special case. More general and complete results will appear in [8]. However, many of the techniques used can be isolated and simplified in this special case. We will be concerned with actions of a group G which is the semidirect product of two cyclic groups of prime order.

By a family F of subgroups of G we will mean a subset of the subgroups of G with the property that if K is an element of F every subgroup and every conjugate of K is in F. $\Omega_*(G; F)$ is the bordism group of actions of G on stably complex manifolds such that every isotropy group lies in the family F. Ω_* will denote the complex bordism ring.

Theorem A. *Let F be the family consisting of all subgroups of G. Then $\Omega_*(G; F)$ is a free Ω_* module on even dimensional generators.*

Theorem B. *Let F be any family of subgroups of G. Then $\Omega_+(G; F) = \bigoplus \Omega_{2i}(G; F)$ is a free Ω_* module. And $\Omega_-(G; F) = \bigoplus \Omega_{2i+1}(G; F)$ has projective dimension one over Ω_*.*

Basic material on the groups $\Omega_*(G; F)$ can be found in [3], [10]. Our analysis largely follows the lines of [2] but was influenced by [10] and [6].

In §1 we discuss general facts about equivariant bordism groups. In §2 we discuss the cohomology of a cyclic group acting on a polynomial ring. In §3 we discuss $\Omega_*(G; F, F')$ for adjacent families F, F' in G and in §4 we present the proofs of the main theorems.

Received by the editors September 13, 1971.

AMS (MOS) subject classifications (1970). Primary 57D85; Secondary 55B20, 57E25.

Key words and phrases. Bordism theory of actions, actions of metacyclic groups, adjacent families of subgroups, projective dimension of bordism modules, equivariant bordism, cohomology of groups, actions of cyclic groups, induced representations.

(1) Research partially supported by National Science Foundation grant GP-12639.

1. **Basic facts.** Let p and q be distinct primes. Let f be a homomorphism of Z_p into the automorphism group of Z_q. We denote the resulting semidirect product by $Z_p \times_f Z_q = G$. G is generated by two elements a and b with relations $b^p = a^q = 1$, $bab^{-1} = a^r$ where r is a nonzero element of Z_q. f, of course, is the homomorphism which takes b to the automorphism $a \to a^r$.

Lemma (1.1). *$H_{2k-1}(BG) = Z_p$ if k is not a multiple of p and $H_{2k-1}(BG) = Z_p \oplus Z_q$ if k is a multiple of p.*

Proof. Consider the Hochschild-Serre spectral sequence

$$E^2_{i,j} = H_i(Z_p; H_j(Z_q)).$$

$E^2_{i,j} = 0$ unless i or j is zero. $E^2_{i,0} = H_i(Z_p; Z) = 0$ or Z_p depending upon whether i is even or odd. To compute $E^2_{0,j} = H_0(Z_p; H_j(Z_q))$ we use the resolution

$$Z[Z_p] \xrightarrow{N} Z[Z_p] \xrightarrow{D} Z[Z_p] \tag{1.2}$$

where $D = 1 - b$, $N = 1 + b + \cdots + b^{p-1}$ of [9, p. 121]. Tensoring on the right with $H_j(Z_q)$, we find that $E^2_{0,j} = \text{cokernel}(D \otimes 1)$. A simple computation shows the generator b of Z_p acts on $H_{2j-1}(Z_q) = Z_q$ by multiplication by r^j and $1 - r^j$ is a unit in Z_q unless $j \equiv 0 \bmod p$. It is clear that the inclusion $Z_p \subset G$ induces an injection $H_*(BZ_p) \to H_*(BG)$ onto the p-torsion, and the inclusion $Z_q \subset G$ induces a map $H_*(BZ_q) \to H_*(BG)$ which is onto the q-torsion.

Lemma (1.3). *$\Omega_*(BG)$ has projective dimension one over Ω_*. The left G-manifolds $G \times_{Z_p} S^{2k-1}$ generate the p-torsion and the left G-manifolds $G \times_{Z_q} S^{2kp-1}$ generate the q-torsion where Z_p acts on S^{2k-1} by multiplication of each coordinate in C^k by $e^{2\pi i/p}$ and Z_q acts on S^{2kp-1} by multiplication by $e^{2\pi i/q}$. Let V be the representation of G induced from the one-dimensional representation $e^{2\pi i/q}$ of Z_q (see [5, p. 333]). $V = Z[G] \otimes_{Z[Z_q]} C^*$. Then the left G-manifolds $S(kV)$ also serve as generators for the q-torsion.*

Proof. The statement about dimension follows directly from [7] since $H_*(BG)$ is concentrated in odd dimensions. The manifolds $[Z_p, S^{2k-1}]$ and $[Z_q, S^{2kp-1}]$ give rise, via the Thom map, to generators of $H_{2k-1}(BZ_p)$ and $H_{2kp-1}(BZ_q)$ and so give rise to generators of $H_*(BG)$. Thus the G extensions of these manifolds yield generators of $\Omega_*(BG)$ by standard arguments [1, p. 49]. It is well known that if l is an integer, $\xi_1, \cdots, \xi_l$ are primitive qth roots of unity, and if Z_q acts on C^l by $a(x_1, \cdots, x_l) = (\xi_1 x_1, \cdots, \xi_l x_l)$, then the fundamental class of S^{2l-1}/Z_q is a generator of $H_{2l-1}(BZ_q)$. If $C^{kp} = C^k \times \cdots \times C^k$ and Z_q acts on the jth factor by ξ^{r-j} then S^{2kp-1} gives rise to a generator of $H_{2kp-1}(BZ_q)$. V has $1 \otimes 1, b \otimes 1, \cdots, b^{p-1} \otimes 1$ as basis and

$a(b^j \otimes 1) = \xi^{r-j}(b^j \otimes 1)$. Now we claim that $G \times_{Z_q} S^{2kp-1}$ and $Z_p \times S(kV)$ with the diagonal action of G are G diffeomorphic. It is enough to take the case $k = 1$. Let e_j be a basis for the jth factor in $C \times \cdots \times C$. Send $[b^i, e_j]$ in $G \times_{Z_q} C^p$ to $(b^i, b^{i+j} \otimes 1)$ in $Z_p \times V$. Thus the manifolds $Z_p \times S(kV)$ generate the q-torsion in $\Omega_*(BG)$. Let $\mu: \Omega_*(BG) \to H_*(BG)$ be the Thom map. Then $\{\mu(Z_p \times_G S(kV))\}$ are generators for the q-torsion in $H_*(BG)$ and $\mu(Z_p \times_G S(kV)) = p\mu(S(kV)/G)$. Since p and q are relatively prime, $\{\mu(S(kV)/G)\}$ are generators for the q-torsion in $H_*(BG)$ and so $S(kV)/G$ give generators for the q-torsion in $\Omega_*(BG)$.

In [10] and [2] the groups $\Omega_*(G; F, F')$, where G is a finite group and F and F' are families of subgroups of G, are studied. If K is a subgroup of G we let $F(K)$ denote the family consisting of all conjugates of subgroups of K and we let $F_0(K)$ denote the family consisting of all conjugates of proper subgroups of K. We call F and F' strictly adjacent families if there is a subgroup K of G such that $F - F'$ consists of the conjugates of K.

Let F and F' be strictly adjacent families of a finite group G which differ by the conjugates of K. We have well-known isomorphisms

$$\Omega_*(G; F, F') \cong \Omega_*(G; F(K), F_0(K)) \cong \Omega_*(N(K); F(K), F_0(K)) \tag{1.4}$$

where $N(K)$ is the normalizer of K in G. We enunciate these isomorphisms. Let W be a stably complex left G-manifold with boundary. Let M be the points in W with isotropy group conjugate to K and N a closed G-invariant tubular neighborhood of K. Then the class of W in $\Omega_*(G; F, F')$ is sent to the class of N in $\Omega_*(G; F(K), F_0(K))$. If we let ν be the normal bundle of M in W, M_0 the points in M with isotropy group equal to K, ν_0 the restriction of ν to M_0, then the disk bundle $D(\nu_0)$ is an $N(K)$ manifold with boundary which represents an element in $\Omega_*(N(K); F(K), F_0(K))$ as we assign the class of $D(\nu_0)$ to the class of N.

The analysis of these groups is carried out in [11] for unoriented bordism and in more detail in a special case in [2]. We present a treatment of $\Omega_*(G; F(K), F_0(K))$ which we will later explicitly need. Let I_x denote inner automorphism by x. Let ρ and ρ' be n-dimensional complex representations of K. We will say that ρ and ρ' are G-equivalent if there are elements g in G and A in $U(n)$ such that $\rho' = I_A \rho I_g$.

Theorem (1.5). *Let K be normal in G. Then*

$$\Omega_m(G; F(K), F_0(K)) \cong \sum \Omega_{m-2n}(B(N(\rho)/\Gamma(\rho)))$$

where n runs from 0 to $[m/2]$, ρ runs over a set of representitives of G-equivalent n-dimensional representations of K, $\Gamma(\rho)$ is the graph of ρ in $G \times U(n)$, and $N(\rho)$ is the normalizer of $\Gamma(\rho)$ in $G \times U(n)$.

Proof. Let W be a manifold with boundary representing an element in $\Omega_m(G; F(K), F_0(K))$, M the set of points with isotropy group equal to K, ν the normal bundle, M_n the union of the $(m-2n)$-dimensional components of M, ν_n the restriction of ν to M_n. Then W and $\Sigma D(\nu_n)$ are bordant and the bordism relation respects this decomposition. Thus $\Omega_m(G; F(K), F_0(K)) \cong \Sigma \Psi_n$, where Ψ_n is the bordism group of complex n-dimensional left G vector bundles E over compact $(m-2n)$-dimensional left G-manifolds M such that every point on the zero section of E has isotropy group equal to K and every point off the zero section has isotropy group properly contained in K. Let P be the principal $U(n)$ bundle corresponding to E. As in [2], P can be regarded as a right $G \times U(n)$ space such that the isotropy group of a point e in P is $\Gamma(\rho_e)$ for some n-dimensional complex representation ρ_e of K. $H = G \times U(n)$ acts differentiably on P so there is a neighborhood V of e in P such that for e' in V there is a (g, A) in H such that $H_{e'} \subset (g^{-1}, A^{-1})H_e(g, A)$. From this it follows that $\rho_{e'} = I_{A^{-1}}\rho_e I_x$. Further, if $e' = eA$ for some A in $U(n)$, then $\rho_{e'} = I_{A^{-1}}\rho_e$. Thus for each x in M there is a neighborhood U of x such that, for any two points e and e' in $P \mid U$, ρ_e and $\rho_{e'}$ are G-equivalent and so H_e and $H_{e'}$ are conjugate in H. Thus $\Psi_n \cong \Sigma \Psi_{n,\rho}$ where $\Psi_{n,\rho}$ is the bordism group of compact right $G \times U(n)$ spaces P such that $U(n)$ acts freely on P, $P/U(n) = M$ is a stably complex G-manifold, every point in P has isotropy group conjugate, in H, to $\Gamma(\rho)$, and ρ runs over a set of representatives of n-dimensional representations of K under G-equivalence. So $M/G/K$ is a stably complex manifold. Now let $P \to M$ represent an element in $\Psi_{n,\rho}$ and let $Q \subset P$ be the points in P with isotropy group equal to $\Gamma(\rho)$. Q is a principal right $N(\rho)/\Gamma(\rho)$ bundle, $Q/N(\rho)/\Gamma(\rho) = M/G/K$, and Q and P determine each other since $Q \times_{N(\rho)} H = P$. Thus

$$\Psi_{n,\rho} \cong \Omega_{m-2n}(B(N(\rho)/\Gamma(\rho))). \tag{1.6}$$

(1.7) We should make a few remarks about extensions of actions. If H is a subgroup of G and F a family of subgroups of H, then we let F also denote that family of subgroups of G consisting of all conjugates of elements of F. We have a map $E: \Omega_*(H; F) \to \Omega_*(G; F)$ which takes the class of a stably complex H-manifold M to the class of $G \times_H M$. Note $G \times_H M$ has a well-defined G-invariant stable complex structure. In a similar manner we have $E: \Omega_*(H; F, F') \to \Omega_*(G; F, F')$. Note that if H_1 and H_2 are conjugate in G, then the image of $\Omega_*(H_1; F_0(H_1))$ is equal, in $\Omega_*(G; F)$, to the image of $\Omega_*(H_2; F_0(H_2))$ where F is any family of G containing $F_0(H_1) = F_0(H_2)$.

2. **Actions of cyclic groups on polynomial rings.** We will develop some algebra which will be relevant to the analysis of $\Omega_*(G; F, F')$. Let $P_1, \cdots, P_t$

be polynomial rings over the integers, $P_j^{(p)}$ the p-fold tensor product. Z_p acts on $P_j^{(p)}$ by letting the generator b of Z_p act by $b(x_0 \otimes \cdots \otimes x_{p-1}) = x_{p-1} \otimes x_0 \otimes \cdots \otimes x_{p-2}$. Let $P = P_1^{(p)} \otimes \cdots \otimes P_t^{(p)}$. P is a polynomial ring and Z_p acts on P via the tensor product of the actions on $P_j^{(p)}$. $\Omega_* \otimes P$ is a polynomial ring over Ω_* on which Z_p acts. It follows directly from the definition of homology via resolutions that $H_*(Z_p; \Omega_* \otimes P) \cong \Omega_* \otimes H_*(Z_p; P)$.

Lemma (2.1). *$H_i(Z_p; P) = 0$ for i even.*

Proof. We use the resolution of (1.2) for Z over $Z[Z_p]$. We must calculate kernel(N)/image(D), where N and D act on P. Let X be a monomial in P. Let $Z\{X\}$ be the subgroup of P which has basis consisting of the distinct images of X under Z_p. $P = \bigoplus Z\{X\}$. It is enough to show $H_i(Z_p; Z\{X\}) = 0$. Either $bX = X$ or $X, bX, \cdots, b^{p-1}X$ are distinct. In the first case, the kernel of N is clearly zero. In the second case

$$N\left(\sum_{j=0}^{p-1} m_j(b^jX)\right) = \sum_{j=0}^{p-1} m_j N(X) = 0 \quad \text{implies} \quad \sum_{j=0}^{p-1} m_j = 0.$$

Now for any integer k, $X - b^kX = \sum_{j=0}^{k-1} (b^jX - b^{j+1}X)$. So $X \equiv b^kX$ mod image(D). Thus $\sum_{j=0}^{p-1} m_j(b^jX) \equiv \sum_{j=0}^{p-1} m_jX$ mod image(D).

Lemma (2.2). *$H_0(Z_p; P)$ is a free Z module.*

Proof. $H_0(Z_p; P) = P/\text{image}(D)$. As in (2.1) it is enough to compute $Z\{X\}/\text{image}(D)$. If $bX = X$, image$(D) = 0$ and $Z\{X\}$ is a free module with basis X. In the other case, $Z\{X\}$ has, as basis $X, (1-b)X, \ldots, (1-b)b^{p-2}X$. From the formula $(1-b)b^{p-1}X = -(1-b)X - (1-b)bX - \cdots - (1-b)b^{p-2}X$ it follows that $(1-b)X, \ldots, (1-b)b^{p-2}X$ is a basis for image(D) and so $Z\{X\}/\text{image}(D)$ is a free Z module with basis $X + \text{image}(D)$.

Now let us consider the map $\phi_j: P_j \to P_j^{(p)}$

$$\phi_j(X) = \sum X_0 \otimes \cdots \otimes X_{p-1} \tag{2.3}$$

where the sum is taken over all $X_0 \otimes \cdots \otimes X_{p-1}$ such that $X_0X_1 \cdots X_{p-1} = X$ in P_j. We then let $\overline{P} = P_1 \otimes \cdots \otimes P_t$ and let

$$\Phi: \overline{P} \to P \tag{2.4}$$

be the tensor product of the $\phi_j: P_j \to P_j^{(p)}$. $\overline{P}$ is a trivial Z_p module. Φ is a map of Z_p modules.

Theorem (2.5). *$\Phi_*: H_{2i-1}(Z_p; \overline{P}) = \overline{P} \to H_{2i-1}(Z_p; P)$ is surjective. Let $X = X_1 \otimes \cdots \otimes X_t$ be a monomial in $\overline{P}$ and let $\overline{P}_1$ be the subgroup with basis consisting of those X such that each monomial X_j is a pth power. Let $\overline{P}_2$ be*

the submodule with basis consisting of the remaining monomials. Then $P_2 =$ kernel(Φ_*) *and* $\Phi_*: \overline{P}_1 \xrightarrow{\cong} H_{2i-1}(Z_p; P)$.

Proof. Let $H_-(Z_p; P)$ denote $H_{2i-1}(Z_p; P)$. Consider $\Phi(X) = \phi(X_1) \otimes \cdots \otimes \phi(X_t)$ where $\phi_j(X_j) = \Sigma X_{j0} \otimes \cdots \otimes X_{jp-1}$. X_j is a pth power in P_j if and only if for one term in this sum $X_{j0} = \cdots = X_{jp-1}$. Call this common value Y_j. Thus $\Phi(X)$ has a term

$$\underbrace{Y_1 \otimes \cdots \otimes Y_1}_{p} \otimes \cdots \otimes \underbrace{Y_t \otimes \cdots \otimes Y_t}_{p} = Y$$

if and only if each X_j is a pth power. Furthermore, a monomial in P is invariant under Z_p if and only if it is such a Y. Thus $\Phi(X) = Y + \Sigma T$ where $bY = Y$ and $bT \neq T$ if each X_j is a pth power, and $\Phi(X) = \Sigma T$ where $bT \neq T$ if some X_j is not a pth power.

Now if $bY = Y$, then $D(Y) = 0$ and $N(Y) = pY$ and so $H_-(Z_p; Z\{Y\}) = Y \bmod pY$. If $bT \neq T$, then the kernel of D restricted to $Z\{T\}$ is the subgroup generated by $T + bT + \cdots + b^{p-1}T = N(T)$ and so $H_-(Z_p; Z\{T\}) = 0$. Thus $H_-(Z_p; P) = \bigoplus H_-(Z_p; Z\{Y\}) = \bigoplus Y \bmod pY$ for those Y such that $bY = Y$. If $Y = Y_1 \otimes \cdots \otimes Y_1 \otimes \cdots \otimes Y_t \otimes \cdots \otimes Y_t$, let $X_j = Y_j^p$ and $X = X_1 \otimes \cdots \otimes X_t$. Then $\Phi_*(X) = \Phi(X) \bmod$ image $N = Y \bmod$ image N. And if $\Phi(X) = \Sigma T$, then $\Phi(X) = 0 \bmod$ image N.

3. $\Omega_*(G; F, F')$. Throughout this section $G = Z_p \times_f Z_q$, $\eta = e^{2\pi i/p}$, $\xi = e^{2\pi i/q}$. Let F_α be the family of all subgroups of G, F_p be $F_0(Z_p)$, F_q be $F_0(Z_q)$, F_0 be the family consisting of all proper subgroups, and F_1 the family consisting of the identity subgroup

$$\Omega_*(G; F_0, F_q) \cong \Omega_*(G; F_p, F_1) \cong \Omega_*(Z_p; F(Z_p), F_0(Z_p)),$$

$$\Omega_*(G; F_0, F_p) \cong \Omega_*(G; F(Z_q), F_0(Z_q)) \quad \text{by (1.4)}.$$

First we study $\Omega_*(Z_p; F(Z_p), F_0(Z_p))$. The first lemma is well known.

Lemma (3.1). *Let* ρ *be an* n*-dimensional representation of* Z_p. *Let* $N(\rho)$ *be the normalizer of* $\Gamma(\rho)$ *in* $Z_p \times U(n)$. *Then*

$$\Omega_*(B(N(\rho)/\Gamma(\rho))) \cong \Omega_*(BU(k_1) \times \cdots \times BU(k_{p-1})).$$

The isomorphism $\Omega_*(Z_p; F(Z_p), F_0(Z_p)) \cong \Omega_*(G; F_0, F_q)$ *of* (1.4) *is induced by extension.*

Proof. Let $\rho(b) = k_1\eta \oplus \cdots \oplus k_{p-1}\eta^{p-1}$. Then $N(\rho) = Z_p \times C(\rho)$, where

$C(\rho)$ is the centralizer of ρ in $U(n)$. By Schur's lemma, $C(\rho) = U(k_1) \times \cdots \times U(k_{p-1})$. $N(\rho)/\Gamma(\rho) \cong U(k_1) \times \cdots \times U(k_{p-1})$, an explicit isomorphism is obtained by sending $(b^j, A_1, \cdots, A_{p-1})$ to $(\eta^{-1}A_1, \cdots, \eta^{-(p-1)}A_{p-1})$. Now let W represent an element of $\Omega_*(Z_p; F(Z_p), F_0(Z_p))$. $G \times_{Z_p} W$ is the extension. Let M be the fixed point set of Z_p in W and D an invariant tubular neighborhood. Then $G \times_{Z_p} M = G/Z_p \times M$ is the points with isotropy group conjugate to Z_p, $G \times_{Z_p} D = D'$ is an invariant tubular neighborhood of $G \times_{Z_p} M$. The points with isotropy group equal to Z_p is $Z_p \times_{Z_p} M = M$ and D' restricted to M is D. So, the last statement follows.

Now we study $\Omega_*(G; F(Z_q), F_0(Z_q))$. Let σ be an n-dimensional representation of Z_q. Suppose $\sigma(a) = k_1\xi \oplus \cdots \oplus k_{q-1}\xi^{q-1}$. Then σI_{b^j}, $j = 0, 1, \cdots, p-1$, are all G-equivalent to σ. Let $C(\sigma)$ be the centralizer of σ in $U(n)$. Then $[\Gamma(\sigma), 1 \times C(\sigma)] \subset N(\sigma)$. (Here [ ] denotes the group generated by .)

Lemma (3.2). *(b, B) is in $N(\sigma)$ for some B in $U(n)$ if and only if $\sigma I_b = I_B\sigma$, and this holds if and only if $k_j = k_{jr^s}$ for all j and all $s = 0, 1, \cdots, p-1$.*

Proof. $(b, B)\Gamma(\sigma)(b^{-1}, B^{-1}) = \Gamma(\sigma)$ means that $B\sigma(a)B^{-1} = \sigma(a^r) = \sigma I_b(a)$. So $\sigma I_b = I_B\sigma$. Let $V = C^n$ be a representation space for σ, and V_j the ξ^j eigenspace for $\sigma(a)$. Suppose $\sigma(a)B^{-1} = B^{-1}\sigma(a^r)$. Then x is in V_j if and only if $\sigma(a)B^{-1}(x) = B^{-1}\sigma(a)x = \xi^{jr}B^{-1}(x)$. Thus $B^{-1}: V_j \cong V_{jr}$, and this can hold only if $k_j = k_{jr^s}$. On the other hand, if $k_j = k_{jr}$ for all j, choose a basis for V by choosing a basis for each V_j. Let B^{-1} be the permutation matrix relative to this basis which takes the basis in V_j to the basis in V_{jr}. Then (b, B) is in $N(\sigma)$.

Lemma (3.3). *If $\sigma I_b = I_B\sigma$ for some B, then*

$$N(\sigma) = [(b, B), \Gamma(\sigma), 1 \times C(\sigma)].$$

If not, $N(\sigma) = [\Gamma(\sigma), 1 \times C(\sigma)]$. $C(\sigma) \cong U(k_1) \times \cdots \times U(k_{q-1})$. If $A = (A_1, \cdots, A_{q-1})$ is the matrix of an element in $C(\sigma)$ relative to the basis indicated in (3.2), *we can choose B to be the permutation matrix of* (3.2) *and then*

$$(b, B)(1, A_1, \cdots, A_{q-1})(b^{-1}, B^{-1}) = (1, A'_1, \cdots, A'_{q-1})$$

where $A'_j = A_{jr}$.

Proof. It is clear that the B of (3.2) gives us an element (b, B) in $N(\sigma)$. If (b, B') is in $N(\sigma)$, then $B^{-1}B'$ is in $C(\sigma)$ and so $(b, B') = (b, B)(1, B^{-1}B')$.

Let Z_q^* be the multiplicative group of nonzero elements of Z_q, and $[r]$ the subgroup generated by r. Let $\{s_j\}$, $j = 1, \cdots, (q-1)/p$, be representatives for $Z_q^*/[r]$. Then

$$\prod_{j=1}^{q-1} U(k_j) \qquad \prod_{j=1}^{q-1/p} \prod_{t=0}^{p-1} U(k_{s_j r} t).$$

Inner automorphism by B in (3.3) sends $\Pi_{t=0}^{p-1} U(k_{s_j r} t)$ to itself and takes $(A_0, \cdots, A_{p-1})$ to $(A_1, \cdots, A_{p-1}, A_0)$. Thus

Lemma (3.4). *If $\sigma l_b = l_B \sigma$, then*

$$\frac{N(\sigma)}{\Gamma(\sigma)} \cong Z_p \times_f \prod_{j=1}^{q-1/p} \prod_{0}^{p-1} U(l_j)$$

where l_j is the common value of $k_{s_j r} t$, and inner automorphism by the generator b of Z_p preserves $\Pi_0^{p-1} U(l_j)$ and takes $(A_0, \cdots, A_{p-1})$ to $(A_1, \cdots, A_{p-1}, A_0)$.

Now we will analyze $\Omega_*(B(N(\sigma)/\Gamma(\sigma)))$ for such σ. Let

$$U = \prod_{j=1}^{q-1/p} \prod_{0}^{p-1} U(l_j). \tag{3.5}$$

$\Omega^*(BU)$ is a power series ring in Chern classes.

$$\Omega_*(BU) = \mathrm{Hom}_{\Omega^*}(\Omega^*(BU); \Omega^*).$$

For each monomial C_α in the Chern classes, let X_α be the element in $\Omega_*(BU)$ such that $\langle X_\alpha, C_\beta \rangle = \delta_\alpha^\beta$. Then $\Omega_*(BU)$ is a free Ω_* module with basis $\{X_\alpha\}$. We can make $\Omega_*(BU)$ into a polynomial ring by defining $X_\alpha X_\beta = X_\gamma$ where X_γ is dual to $C_\alpha C_\beta$. And $\Omega_*(BU) \cong \Omega_* \otimes P$ where P is the polynomial ring over the integers with monomials X_α. Consider the diagonal map $U(l_j) \to \Pi_0^{p-1} U(l_j)$. Let

$$\overline{U} = \prod_{j=1}^{q-1/p} U(l_j) \tag{3.6}$$

and let

$$\Psi : \overline{U} \to U \tag{3.7}$$

be the product of the diagonal maps. Let Ψ also denote the induced map $B\overline{U} \to BU$. Let $t = (q-1)/p$.

Lemma (3.8). *The map $\Omega_*(BU(l_j)) \to \bigotimes_1^{p-1} \Omega_*(BU(l_j))$ induced by the diagonal takes a monomial X in $\Omega_*(BU(l_j))$ to $\Sigma X_0 \otimes \cdots \otimes X_{p-1}$ where the sum is taken over all $X_1 \otimes \cdots \otimes X_t$ such that $X_0 \cdots X_{p-1} = X$ in $\Omega_*(BU(l_j))$. The map $\Omega_*(B\overline{U}) \to \Omega_*(BU)$ is the tensor product of these maps.*

Proof. Consider the diagonal map $\Delta: BU(l) \to \Pi_0^{p-1} BU(l)$. Let X be a monomial in $\Omega_*(BU(l))$. Then $\Delta_*(x) = \Sigma M(X_0 \otimes \cdots \otimes X_{p-1})$ where X_i is in

$\Omega_*(BU(l))$ and M is in Ω_*. Consider $C_{\alpha_0} \otimes \cdots \otimes C_{\alpha_{p-1}}$ in $\Omega^*(\Pi_0^{p-1} BU(l))$.

$$\langle \Delta_*(X), C_{\alpha_0} \otimes \cdots \otimes C_{\alpha_{p-1}} \rangle = \langle X, \Delta^*(C_{\alpha_0}) \cdots \Delta^*(C_{\alpha_{p-1}}) \rangle$$

$$= \langle X, C_{\alpha_0} \cdots C_{\alpha_{p-1}} \rangle = 1$$

if X is dual to $C_{\alpha_0} \cdots C_{\alpha_{p-1}}$ and zero otherwise. A term $M\langle X_0 \otimes \cdots \otimes X_{p-1}, C_{\alpha_0} \otimes \cdots \otimes C_{\alpha_{p-1}} \rangle$ is one only if $M = 1$ and X_j is dual to C_{α_j}. Then by definition, $X_0 \cdots X_{p-1}$ is dual to $C_{\alpha_0} \cdots C_{\alpha_{p-1}}$ and so equals X. Now the lemma follows.

Remark. Let $P_j = \Omega_*(BU(l_j))$, $\Omega_* \otimes P_j^{(p)} = \Omega_*(\Pi_0^{p-1} BU(l_j))$, and $\Omega_* \otimes P_1^{(p)} \otimes \cdots \otimes P_t^{(p)} = \Omega_* \otimes P = \Omega_*(BU)$.

$$\Omega_* \otimes P_1 \otimes \cdots \otimes P_t = \Omega_* \otimes \bar{P} = \Omega_*(B\bar{U}).$$

And the map $\Omega_*(B\bar{U}) \to \Omega_*(BU)$ is precisely the map $1 \otimes \Phi$ where Φ is the map of (2.4).

The map (3.7) induces a group homomorphism $Z_p \times \bar{U} \to Z_p \times_f U$ and so a map

$$BZ_p \times B\bar{U} \to B(Z_p \times_f U). \tag{3.9}$$

Lemma (3.10). *The maps* $\Omega_-(BZ_p \times B\bar{U}) \to \Omega_-(B(Z_p \times_f U))$ *and* $H_-(BZ_p \times B\bar{U}) \to H_-(B(Z_p \times_f U))$ *are surjective.*

Proof. Let us consider the two Atiyah spectral sequences

$$\bar{E}^2_{i,*} = H_i(Z_p; \Omega_*(B\bar{U})) \Rightarrow \Omega_*(B(Z_p \times \bar{U})),$$

$$E^2_{i,*} = H_i(Z_p; \Omega_*(BU)) \Rightarrow \Omega_*(B(Z_p \times_f U)).$$

The map (3.9) induces a map of these spectral sequences $\bar{E}_{*,*} \to E_{*,*}$. In the remark following (3.8) we noted that $\Omega_*(B\bar{U}) = \Omega_* \otimes \bar{P}$, $\Omega_*(BU) = \Omega_* \otimes P$ and the map $H_i(Z_p; \Omega_* \otimes \bar{P}) = \bar{E}^2_{i,*} \to E^2_{i,*} = H_i(Z_p; \Omega_* \otimes P)$ is the map $1 \otimes \Phi_*$. $\bar{E}_{i,j} = E_{i,j} = 0$ if i is even, $i \neq 0$ by (2.1). $\bar{E}_{i,j} = E_{i,j} = 0$ for i odd, j odd, since $\Omega_*(B\bar{U})$, $\Omega_*(BU)$ is concentrated in even dimensions. $\bar{E}_{i,j}$, $E_{i,j}$ are p-torsion and $\bar{E}_{i,j} \to E_{i,j}$ is surjective for i odd, j even, by (2.5). $\bar{E}_{0,j} = \Omega_j(BU)$ is free abelian for j even, and $E_{0,j} = H_0(Z_p; \Omega_* \otimes P)$ is free abelian by (2.2). For j odd $\bar{E}_{0,j} = E_{0,j} = 0$. Thus no differential could possibly be nonzero and so both spectral sequences collapse. Consider the filtrations $\bar{F}_{*,*}$ on $\Omega_*(BZ_p \times B\bar{U})$ and $F_{*,*}$ on $\Omega_*(B(Z_p \times_f U))$.

$$\bar{F}_{i,j}/\bar{F}_{i-1,j+1} = F_{i,j}/F_{i-1,j+1} = 0$$

for $i + j$ even and $i \neq 0$. For m odd,

$$\Omega_m(BZ_p \times B\overline{U}) = \overline{F}_{m,0} \supset \cdots \supset \overline{F}_{1,m-1} \supset 0,$$

$$\Omega_m(B(Z_p \times_f U)) = F_{m,0} \supset F_{m-2,2} \supset \cdots \supset F_{1,m-1} \supset 0.$$

And

$$\overline{F}_{i,j}/\overline{F}_{i-2,j+2} = H_i(Z_p; \Omega_j(B\overline{U})), \qquad F_{i,j}/F_{i-2,j+2} = H_i(Z_p; \Omega_j(BU))$$

for $i + j$ odd. We have maps

$$\begin{array}{ccccccccc} 0 & \to & \overline{F}_{i-2,j+2} & \to & F_{i,j} & \to & H_i(Z_p; \Omega_j(B\overline{U})) & \to & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \to & F_{i-2,j+2} & \to & F_{i,j} & \to & H_i(Z_p; \Omega_j(BU)) & \to & 0 \end{array}$$

For $i = 1$, $j = m - 1$, $\overline{F}_{1,m-1} = H_1(Z_p; \Omega_{m-1}(B\overline{U})) \to H_1(Z_p; \Omega_{m-1}(BU)) = F_{1,m-1}$ is onto. By induction α' is surjective, and α'' is surjective by computation. Hence α is surjective. Thus by induction $\overline{F}_{m,0} \to F_{m,0}$ is surjective. The same argument applied to $H_*(BZ_p \times B\overline{U})$ and $H_*(B(Z_p \times_f U))$ gives the corresponding conclusion for homology.

Lemma (3.11). *$\Omega_+(BU) \to \Omega_+(B(Z_p \times_f U))$ and $H_+(BU) \to H_+(B(Z_p \times_f U))$ are surjective.*

Proof. In the spectral sequence $E_{*,*}$ of (3.10), $E^2_{0,*} = H_0(Z_p; \Omega_*(BU))$ and $E^2_{i,j} = 0$ for $i + j$ even, $i \neq 0$. Thus $\Omega_+(B(Z_p \times_f U)) = H_0(Z_p; \Omega_*(BU))$ and $\Omega_*(BU) \to H_0(Z_p; \Omega_*(BU))$ is surjective by definition of H_0. Similarly for homology.

Theorem (3.12). *$\Omega_-(B(Z_p \times_f U))$ has projective dimension one over Ω_*.*

Proof. First consider

$$\begin{array}{ccc} \Omega_+(BU) & \to & \Omega_+(B(Z_p \times_f U)) \\ \downarrow{\scriptstyle \bar{\mu}} & & \downarrow{\scriptstyle \mu} \\ H_+(BU) & \to & H_+(B(Z_p \times_f U)) \end{array}$$

where $\bar{\mu}$ and μ are the Thom maps. $\bar{\mu}$ is surjective since $\Omega_*(BU)$ is a free Ω_* module, and the two horizontal maps are surjective. Hence μ is surjective. Now consider

$$\begin{array}{ccc} \Omega_-(BZ_p \times BU) & \to & \Omega_-(B(Z_p \times_f U)) \\ \downarrow{\scriptstyle \bar{\mu}} & & \downarrow{\scriptstyle \mu} \\ H_-(BZ_p \times BU) & \to & H_-(B(Z_p \times_f U)). \end{array}$$

Now $\Omega_-(BZ_p \times B\bar{U}) \cong \bigoplus \Omega_-(BZ_p)$ and so has projective dimension one. Then from [4], [7], $\bar{\mu}$ is onto, and since the horizontal maps are onto, μ is onto. Thus $\mu: \Omega_*(B(Z_p \times_f U)) \to H_*(B(Z_p \times_f U))$ is onto and so again by [4], [7], $\Omega_*(B(Z_p \times_f U))$ and $\Omega_-(B(Z_p \times_f U))$ have projective dimension one over Ω_*.

Corollary (3.12). *$\Omega_-(G; F(Z_q), F_0(Z_q))$ has projective dimension one over Ω_*.*

Proof. From (1.5), $\Omega_-(G; F(Z_q), F_0(Z_q)) \cong \bigoplus \Omega_-(B(N(\sigma)/\Gamma(\sigma)))$. From (3.3), $\Omega_-(B(N(\sigma)/\Gamma(\sigma))) \cong \Omega_-(BU) = 0$ or $\Omega_-(B(Z_p \times_f U))$ which has dimension one over Ω_*.

As an indication of a technique necessary in the case of the more general metacyclic group we give more information on the module $\Omega_-(B(Z_p \times_f U))$ of (3.12). As in the remark following (3.8), $\Omega_*(B\bar{U}) = \Omega_* \otimes \bar{P}$, and as in (2.5) let $\bar{P}_1$ be the free Z-module with basis consisting of $X_1 \otimes \cdots \otimes X_t$ where each X_j is a pth power in $\Omega_*(BU(l_j))$.

Theorem (3.13). $\Omega_-(BZ_p) \otimes \Omega_* \otimes \bar{P}_1 \cong \Omega_-(B(Z_p \times_f U))$.

Proof. Let $\Lambda = \Omega_* \otimes \bar{P}_1$. Let $h'_*(X)$ be the homology theory $\Omega_*(X) \otimes \Lambda$, $h'''_*(X)$ the homology theory $\Omega_*(X) \otimes \Omega_*(B\bar{U})$, and $h''_*(X) = \Omega_*(X \times B\bar{U})$. We have a natural transformation $h'_*(X) \to h'''_*(X)$ induced by $\Lambda \subset \Omega_*(B\bar{U})$, and $h'''_*(X) \to h''_*(X)$ induced by $[M, f] \otimes [W, g] \to [M \times W, f \times g]$. So we have induced maps on the skeleton filtrations

$$(Fh')_{*,*}(X) \to (Fh''')_{*,*}(X) \to (Fh'')_{*,*}(X).$$

The second natural transformation is a natural isomorphism. Now consider the fibration $B\bar{U} \to X \times B\bar{U} \to X$. The spectral sequence of this fibration is $\bar{E}^2(X) = H_*(X; \Omega_*(B\bar{U})) \Rightarrow \Omega_*(X \times B\bar{U})$ and the associated filtration on $\Omega_*(X \times B\bar{U})$ is precisely $(Fh'')_{*,*}(X)$. Now let $X = BZ_p$ and consider the map of fibrations of (3.9).

$$\begin{array}{ccc} B\bar{U} & \longrightarrow & BU \\ \downarrow & & \downarrow \\ BZ_p \times B\bar{U} & \longrightarrow & B(Z_p \times_f U) \\ \downarrow & & \downarrow \\ BZ_p & \longrightarrow & BZ_p \end{array}$$

gives rise to a map $\bar{E}^2(BZ_p) = \bar{E}^2 \to E^2 = H_*(Z_p; \Omega_*(BU))$ and we know from (3.10) that both spectral sequences collapse. Let

$$E'^2 = H_*(BZ_p;\, h'_*), \qquad E'''^2 = H_*(BZ_p;\, h'''_*), \qquad E''^2 = H_*(BZ_p;\, h''_*).$$

All of these spectral sequences collapse. We have a map of spectral sequences $E'^2 \to E'''^2 \to E''^2 = \overline{E}^2 \to E^2$ where the first two maps are induced by the natural transformations. Thus we have a map $E'^2_{i,j} \to E^2_{i,j}$. Now $E'^2_{i,j} = E^2_{i,j} = 0$ if i is even and $i \neq 0$, or if j is odd. If i is odd and j is even, this map is an isomorby (2.5). Thus $E'^2_{i,j} = E^2_{i,j}$ unless $i = 0$ and j is even. Let $F'_{*,*}$ be the filtration on $h'_*(BZ_p)$ and $F_{*,*}$ on $\Omega_*(B(Z_p \times_f U))$. Then for m odd,

$$\begin{array}{ccccccccc}
0 \to & F'_{m-(2k+2),*} & \to & F'_{m-2k,*} & \to & H_{m-2k}(BZ_p;\, h'_*) = & E'^2_{m-2k,*} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & F_{m-(2k+2),*} & \to & F_{m-2k,*} & \to & H_{m-2k}(Z_p;\, \Omega_*(BU)) = & E^2_{m-2k,*} & \to 0
\end{array}$$

The left most map is an isomorphism by induction, and the right most by computation. Thus, the middle map is an isomorphism and so

$$h'_-(BZ_p) \cong \Omega_-(B(Z_p \times_f U)).$$

We need a description of the image of $\Omega_-(B(Z_p \times_f U))$ in $\Omega_-(G;\, F(Z_q),\, F_0(Z_q))$. In view of (3.10) we need only describe the image of $\Omega_-(BZ_p \times BU(l_1) \times \cdots \times BU(l_t))$.

Lemma (3.14). *Let P_0 be a left principal Z_p bundle, and P_j a right principal $U(l_j)$ bundle. Let $E_j = P_j \times_{U(l_j)} C^{l_j}$. Then $P_0 \times P_1 \times \cdots \times P_t$ in $\Omega_-(BZ_p \times B\overline{U})$ goes under the map*

$$\Omega_-(BZ_p \times B\overline{U}) \to \Omega_-(B(Z_p \times_f U)) \to \Omega_-(G;\, F(Z_q),\, F_0(Z_q))$$

to the G vector bundle $P_0 \times \bigoplus_0^{p-1} E_1 \times \cdots \times \bigoplus_0^{p-1} E_t \to P_0$ where G acts on P_0 by $G \to G/Z_q = Z_p$ acting on P_0, and G acts on $\bigoplus_0^{p-1} E_j$ by $b(x_0, \ldots, x_{p-1}) = (x_1, x_2, \ldots, x_{p-1}, x_0)$, $a(x_0, \ldots, x_{p-1}) = (\xi^{s_j} x_0, \ldots, \xi^{s_j r^{p-1}} x_{p-1})$.

Proof. Let $P = P_0 \times P_1 \times \cdots \times P_t$. P is a right $Z_p \times \overline{U}$ bundle. Let $Q = P \times_{Z_p \times \overline{U}} Z_p \times_f U$. It is easy to see that $Q \cong P_0 \times \prod (P_j \times_{U(l_j)} U(l_j)^p)$, where Z_p acts diagonally on the right and on $P_j \times_{U(l_j)} U(l_j)^p$, Z_p acts by $[e;\, A_0, \ldots, A_{p-1}]b = [e;\, A_{p-1}, A_0, \ldots, A_{p-2}]$. Now $N(\sigma)/\Gamma(\sigma) \cong Z_p \times_f U$, the isomorphism takes $(b, B)\Gamma(\sigma)$ to b and $(1 \times U)\Gamma(\sigma)$ to $1 \times U$. Thus Q becomes a principal $N(\sigma)/\Gamma(\sigma)$ bundle. Then $Q \times_{N(\sigma)} (G \times U(n)) \times_{U(n)} C^n$ is a left G vector bundle, and from (1.5) is the element in $\Omega_-(G;\, F(Z_q),\, F_0(Z_q))$ coming from P. By a series of obvious isomorphisms,

$$Q \times_{N(\sigma)} (G \times C^n) \cong Q \times_U C^n \cong P_0 \times \prod_{j=1}^{t} P_j \times_{U(l_j)} C^{pl_j}$$

where, following the left action of G along, we find G acting on $P_0 \times \Pi P_j \times_{U(l_j)} C^{pl_j}$ by acting diagonally. G acts on P_0 by $G \to G/Z_q = Z_p$ and on $P_j \times_{U(l_j)} C^{pl_j}$ by

$$b[e; x_0, \cdots, x_{p-1}] = [e; x_1, x_2, \cdots, x_{p-1}, x_0],$$

$$a[e; x_0, \cdots, x_{p-1}] = [e; \xi^{s_j} x_0, \cdots, \xi^{s_j r^{p-1}} x_{p-1}].$$

Finally $P_j \times_{U(l_j)} C^{pl_j} = \bigoplus_0^{p-1} E_j$ with the indicated action.

Corollary (3.15). *$\Omega_-(G; F(Z_q), F_0(Z_q))$ is generated by the G vector bundles $S^{2k-1} \times \bigoplus_0^{p-1} E_1 \times \cdots \times \bigoplus_0^{p-1} E_t \to S^{2k-1}$ where G acts on S^{2k-1} by $G \to Z_p$ and Z_p acts by $b(x_1, \cdots, x_k) = (\eta x_1, \cdots, \eta x_k)$ and G acts on $\bigoplus_0^{p-1} E_j$ as in* (3.14).

Proof. This follows from the fact that S^{2k-1} with the indicated action of Z_p for $k = 1, 2, \cdots$ are generators for $\Omega_-(BZ_p)$.

Remark.(3.16). If G is any finite group, F the family of all subgroups, F_0 the family of all proper subgroups, then it is well known that $\Omega_*(G; F, F_0)$ is the bordism group of G vector bundles E over trivial G manifolds. $E \cong \bigoplus_\pi V_\pi \otimes \mathrm{Hom}_G(\pi, E)$ where V_π runs over the irreducible representation of G. So $\Omega_*(G; F, F_0) \cong \bigoplus \Omega_*(BU(k_1) \times \cdots \times BU(k_t))$ is a free Ω_*-module.

4. **Proofs of Theorems A and B.** We start with the families F_p and F_1 which are adjacent:

$$\Omega_+(G; F_1) = \Omega_+, \qquad \Omega_-(G; F_p, F_1) = 0.$$

The long exact sequence

$$\to \Omega_*(G; F') \to \Omega_*(G; F) \to \Omega_*(G; F, F') \to \Omega_*(G; F') \to \tag{4.1}$$

for any pair of families breaks up as

$$0 \to \Omega_+ \to \Omega_+(G; F_p) \xrightarrow{\gamma} \Omega_+(G; F_p, F_1) \to \Omega_-(G; F_1) \to \Omega_-(G; F_p) \to 0.$$

Now we have, from (1.3) and (3.1),

$$\begin{array}{ccc} \Omega_+(G; F_p, F_1) & \to & \Omega_-(G; F_1) = \Omega_-(Z_p; F_0) \oplus \mathrm{Image}(\Omega_-(Z_q; F_0)) \\ \uparrow \cong & & \uparrow \\ \Omega_+(Z_p; F(Z_p), F_0(Z_p)) & \to & \Omega_-(Z_p; F_0). \end{array}$$

The map $\Omega_+(Z_p; F(Z_p), F_0(Z_p)) \to \Omega_-(Z_p; F_0)$ is well known to be onto. Thus we get short exact sequences

$$0 \to \Omega_+ \to \Omega_+(G;\, F_p) \xrightarrow{\gamma} \Omega_+(G;\, F_p,\, F_1) \to \Omega_-(Z_p;\, F_0) = \Omega_-(BZ_p) \to 0,$$

$$\text{Image}\,(\Omega_-(Z_q;\, F_0)) \cong \Omega_-(G;\, F_p).$$

Now $\text{Image}(\Omega_-(Z_q;\, F_0))$ being a summand of $\Omega_-(BG)$ has projective dimension one. Hence so does $\Omega_-(G;\, F_p)$. Now $\Omega_-(BZ_p)$ has dimension one and $\Omega_+(G;\, F_p,\, F_1)$ is free by (3.1), hence $\text{Image}(\gamma)$ is projective over Ω_* and hence free [4, 3.2]. Thus $0 \to \Omega_+ \to \Omega_+(G;\, F_p) \to \text{Image}(\gamma) \to 0$ and so $\Omega_+(G;\, F_p)$ is free.

Next consider the families F_q, F_1 and apply (4.1). $\Omega_-(G;\, F_q,\, F_1)$ is all torsion by (3.10). Thus

$$0 \to \Omega_+ \to \Omega_+(G;\, F_q) \xrightarrow{\gamma} \Omega_+(G;\, F_q,\, F_1) \to \Omega_-(G;\, F_1) \to \Omega_-(G;\, F_q) \to \Omega_-(G;\, F_q,\, F_1) \to 0.$$

Consider

$$\begin{array}{ccc} \Omega_+(G;\, F_q,\, F_1) & \to & \Omega_-(G;\, F_1) = \text{Image}\,(\Omega_-(Z_q;\, F_0)) + \Omega_-(Z_p;\, F_0) \\ \uparrow E & & \uparrow \\ \Omega_+(Z_q;\, F_q,\, F_0) & \to & \Omega_-(Z_q;\, F_0) \to 0. \end{array}$$

From (3.1) and (3.11) it follows that the map E is onto, and we also know, from (3.11), that $\Omega_+(G;\, F_q,\, F_1)$ is free. Thus we get

$$0 \to \Omega_+ \to \Omega_+(G;\, F_q) \xrightarrow{\gamma} \Omega_+(G;\, F_q,\, F_1) \to \text{Image}\,(\Omega_-(Z_q;\, F_0)) \to 0$$

$$0 \to \Omega_-(Z_p;\, F_0) \to \Omega_-(G;\, F_q) \to \Omega_-(G;\, F_q,\, F_1) \to 0.$$

Now $\Omega_-(Z_p;\, F_0)$ has dimension one and, by (3.12), $\Omega_-(G;\, F_q,\, F_1)$ has dimension one and thus $\Omega_-(G;\, F_q)$ has dimension one. By the same argument as for F_p, F_1, $\text{Image}(\gamma)$ is free and hence $\Omega_+(G;\, F_q)$ is free.

Now consider the families F_0, F_p. By (1.4), $\Omega_*(G;\, F_0,\, F_p) \cong \Omega_*(G;\, F_q,\, F_1)$. Thus we get the sequence of (4.1)

$$\begin{array}{c} 0 \to \Omega_+(G;\, F_p) \to \Omega_+(G;\, F_0) \xrightarrow{\gamma} \Omega_+(G;\, F_q,\, F_1) \to \Omega_-(G;\, F_p) \to \Omega_-(G;\, F_0) \to \Omega_-(G;\, F_q,\, F_1) \to 0 \\ \uparrow \text{onto} \qquad\qquad \uparrow \\ \Omega_+(Z_q;\, F_q,\, F_1) \to \Omega_-(Z_q;\, F_0) \to 0. \end{array}$$

Since we have previously shown $\Omega_-(G;\, F_p) \cong \text{Image}(\Omega_-(Z_q;\, F_0))$ we get $\Omega_-(G;\, F_0) \cong \Omega_-(G;\, F_q,\, F_1)$ and

$$0 \to \Omega_+(G;\, F_p) \to \Omega_+(G;\, F_0) \xrightarrow{\gamma} \Omega_+(G;\, F_q,\, F_1) \to \text{Image}\,(\Omega_+(BZ_q)) \to 0.$$

From the first isomorphism and (3.12), $\Omega_-(G; F_0)$ has projective dimension one. The isomorphism takes W in $\Omega_-(G; F_0)$ to the G vector bundle $E \to M$ where M is the set of points in W with isotropy group Z_q, and E is the normal bundle of M in W. From the second sequence we conclude, since $\Omega_+(G; F_q, F_1)$ is free and Image$(\Omega_-(BZ_q))$ has dimension one, that Image(γ) is free. Now

$$0 \to \Omega_+(G; F_p) \to \Omega_+(G; F_0) \to \text{Image}(\gamma) \to 0.$$

Since $\Omega_+(G; F_p)$ and Image(γ) are free, it follows that $\Omega_+(G; F_0)$ is free.

Finally consider the families F_α and F_0 of G:

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \xrightarrow{\gamma} \Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0) \to \Omega_-(G; F_\alpha) \to 0.$$

We want to show $\Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0) \cong \Omega_-(G; F_q, F_1)$ is onto. (3.15) tells us generators $S^{2k-1} \times \oplus E_1 \times \cdots \times \oplus E_t \to S^{2k-1}$ for $\Omega_-(G; F_q, F_1)$. Recall $\{s_j\}$ were representatives for $Z_q^*/[r]$, $j = 1, \cdots, t$. Let V_j be the representation of G induced from the one dimensional representation $Z_q \to C^*$ sending a to ξ^{s_j}. So

$$V_j = Z[G] \otimes_{Z[Z_q]} C^* \quad \text{(see [5, p. 333])}.$$

Let G act on C^k by $G \to G/Z_q = Z_p$ and Z_p acts by $b(x_1, \cdots, x_k) = (\eta x_1, \cdots, \eta x_k)$. Then the G vector bundle $C^k \times V_1 \otimes E_1 \times \cdots \times V_t \otimes E_t = E$ represents an element in $\Omega_+(G; F_\alpha, F_0)$. $S(E)$ represents an element in $\Omega_-(G; F_0)$. The set of points with isotropy group Z_q is precisely $S(C^k) = S^{2k-1}$ and the normal bundle to S^{2k-1} is $S^{2k-1} \times V_1 \otimes E_1 \times \cdots \times V_t \otimes E_t$. To check that this is the generator $S^{2k-1} \times \oplus E_1 \times \cdots \times \oplus E_t$ consider $V_j \otimes E_j$. We can take as a basis of V_j, $1 \otimes 1, b^{-1} \otimes 1, \cdots, b^{-(p-1)} \otimes 1$ where $a(b^{-i} \otimes 1) = \xi^{s_j r^i}(b^{-i} \otimes 1)$. The general element of $V_j \otimes E_j = (x_0, \cdots, x_{p-1}) = 1 \otimes 1 \otimes x_0 + \cdots + 1 \otimes b^{-(p-1)} \otimes x_{p-1}$, $b(x_0, \cdots, x_{p-1}) = 1 \otimes 1 \otimes x_1 + b^{-1} \otimes 1 \otimes x_2 + \cdots + 1 \otimes b^{-(p-2)} \otimes x_{p-1} + 1 \otimes b^{-(p-1)} \otimes x_0 = (x_1, x_2, \cdots, x_{p-1}, x_0)$. And $a(x_0, \cdots, x_{p-1}) = \sum \xi^{s_j r^i}(1 \otimes b^{-i} \otimes x_i) = (\xi^{s_j} x_0, \cdots, \xi^{s_j r^{p-1}} x_{p-1})$. Thus $\Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0)$ is onto and so $\Omega_-(G; F_\alpha) = 0$. Our exact sequence becomes

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \xrightarrow{\gamma} \Omega_+(G; F_\alpha, F_0) \to \Omega_-(G, F_0) \to 0.$$

We know $\Omega_-(G; F_0)$ has projective dimension one and so Image(γ) is free. We get the exact sequence

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \to \text{Image}(\gamma) \to 0.$$

$\Omega_+(G; F_0)$ and Image(γ) are free, hence $\Omega_+(G; F_\alpha)$ is free. This concludes the proof of Theorems A and B.

BIBLIOGRAPHY

1. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33, Academic Press, New York; Springer-Verlag, Berlin, 1964. MR 31 #750.

2. ———, *Maps of odd period*, Ann. of Math. (2) 84 (1966), 132–156. MR 34 #3587.

3. ———, *Periodic maps which preserve a complex structure*, Bull. Amer. Math. Soc. 70 (1964), 574–579. MR 29 #1653.

4. P. E. Conner and Larry Smith, *On the complex bordism of finite complexes*, Inst. Hautes Études Sci. Publ. Math. No. 37 (1969), 117–221. MR 42 #2473.

5. Charles W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. 11, Interscience, New York, 1962. MR 26 #2519.

6. Peter Landweber, *Equivariant bordism of cyclic groups*, Proc. Amer. Math. Soc. 31 (1972), 564–570.

7. ———, *Cobordism and classifying spaces*, Proc. Sympos. Pure Math., vol. 22, Amer. Math. Soc., Providence, R. I., 1971, pp. 125–129.

8. Connor Lazarov, *Actions of metacyclic groups* (to appear).

9. S. Mac Lane, *Homology*, Die Grundlehren der math. Wissenschaften, Band 114, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR 28 #122.

10. R. E. Stong, *Complex and oriented equivariant bordism*, Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), Markham, Chicago, Ill., 1970, pp. 291–316. MR 42 #8521.

11. Arthur Wasserman, *Cobordism of group actions*, Bull. Amer. Math. Soc. 72 (1966), 866–869. MR 37 #6943.

DEPARTMENT OF MATHEMATICS, HERBERT LEHMAN COLLEGE, CITY UNIVERSITY OF NEW YORK, BRONX, NEW YORK 10468