A PROOF THAT $\mathcal{C}^2$ AND $\mathcal{J}^2$ ARE DISTINCT MEASURES \(^{(1)}\)

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ABSTRACT. We prove that there exists a nonempty family $X$ of subsets of $\mathbb{R}^3$ such that the two-dimensional Carathéodory measure of each member of $X$ is less than its two-dimensional $\mathcal{J}$ measure. Every member of $X$ is the Cartesian product of 3 copies of a suitable Cantor type subset of $\mathbb{R}$.

1. Introduction. To any positive integers $m$, $n$ with $m \leq n$ there correspond several $m$-dimensional measures over $\mathbb{R}^n$. These measures are studied extensively in [3]. We consider two of them, the $m$-dimensional Carathéodory measure, denoted by $\mathcal{C}^m$, and the $m$-dimensional $\mathcal{J}$ measure, denoted by $\mathcal{J}^m$. It is known that $\mathcal{C}^m(S) < \mathcal{J}^m(S)$ for all $S \subset \mathbb{R}^n$ [3, 2.10.34], and $\mathcal{C}^m(S) = \mathcal{J}^m(S)$ if $m = 1$, $m = n$, or $S$ is $m$ rectifiable [3, 2.10.35, 3.2.26].

In this paper we prove (Theorem 3.4) that there exists a nonempty family $X$ of subsets of $\mathbb{R}^3$ such that $\mathcal{C}^2(S) < \mathcal{J}^2(S)$ for all $S \in X$. A precise definition of $X$ is given in §2, using the method of [3, 2.10.28], but roughly each member of $X$ is the Cartesian product of 3 copies of a suitable Cantor type subset of $\mathbb{R}$. We obtain Theorem 3.4 directly from Theorems 3.2, 3.3. A key step in the proof of Theorem 3.2 depends in turn on Lemma 3.1.

2. Preliminaries. In general we adopt in this paper the notation and terminology of [3]. Presented in this section are modifications and additional definitions that we use.

For $S \subset \mathbb{R}^n$ let $S - S = \{x - y : x, y \in S\}$.

For $a, b \in \mathbb{R}^n$ define $[a, b]$ to be the closed line segment with endpoints $a$, $b$.

For $\emptyset \neq S \subset \mathbb{R}^n$ let

$$c^2(S) = \sup \{\xi^2[p(S)] : p \in O^*(n, 2)\},$$

$$t^2(S) = (\pi/4) \sup \{|(a_1 - b_1) \wedge (a_2 - b_2)| : a_1, b_1, a_2, b_2 \in S\}.$$
These are the gauge functions used in defining $C^2$ and $T^2$ respectively [3, 2.10.1, 2.10.3, 2.10.4].

The following series of definitions culminate in the definition of $X$ and $E$. For any sequence $v = (v_1, v_2, v_3, \ldots)$ of integers greater than 1 we denote

$$H_0(v) = \lfloor 0, 1 \rfloor,$$

$$H_k(v) = \bigcup \{\Phi(j, v_k) : j \in H_{k-1}(v)\} \text{ for } k \geq 1,$$

where

$$\Phi(j, v_k) = \{[\inf j + (i-1)p, \inf j + (i-1)p + q] : i = 1, \ldots, v_k\}$$

with $p = (1 - v_k^{-3/2})(v_k - 1)^{-1} \text{diam } j$, $q = v_k^{-3/2} \text{diam } j$; then we define

$$A(v) = \bigcap_{k=0}^{\infty} \bigcup H_k(v).$$

Finally, we let $X = \{A(v) \times A(v) \times A(v) : v \text{ is a bounded sequence}\}$ and $E = A(v) \times A(v) \times A(v)$ be any fixed element of $X$.

3. Principal results. We proceed to prove that $C^2(E) < T^2(E)$.

3.1. Lemma. If $B$ is a compact convex subset of $\mathbb{R}^2$, $Y = \text{Bdry } B$, $d = \text{diam } B$, and there exists a one-dimensional vector subspace $L$ of $\mathbb{R}^2$ and a positive real number $k$ such that $H^1(L \cap Y) \geq kd$, then

$$Q^2(B) \leq t^2(B)/(1 + 2^{-10}k^4).$$

Proof. We can assume that $Q^2(B) > 0$, since (1) clearly holds when $Q^2(B) = 0$.

Choose a one-dimensional vector subspace $V$ of $\mathbb{R}^2$ perpendicular to $L$, and let $B'$ be the subset of $\mathbb{R}^2$ obtained by applying Steiner symmetrization [3, 2.10.30] to $B$ with respect to $V$. Let $Y' = \text{Bdry } B'$. Then by [3, 2.10.30, proof in 2.10.32] $B'$ is a compact convex set, $Q^2(B') = Q^2(B)$, $t^2(B') \leq t^2(B)$ and $H^1(L \cap Y') \geq kd$.

We now proceed to prove the existence of $G \subset \mathbb{R}^2$ satisfying

$$Q^2(B') \leq Q^2(G)/(1 + 2^{-10}k^4),$$

$$t^2(G) \leq t^2(B').$$

This will establish the lemma, since (2) and (3) combined with the relations $Q^2(B) = Q^2(B')$, $Q^2(G) \leq t^2(G)$ [3, 2.10.32], and $t^2(B') \leq t^2(B)$ yield (1).

Let $a_1$ be the midpoint of $L \cap Y'$. Choose orthonormal basis vectors $e_1, e_2$ for $\mathbb{R}^2$ so that $Re_2 = L$, and $(x - a_1) \cdot e_1 \geq 0$ for all $x \in B'$. Let $m_1, m_2 \in L \cap Y'$ be such that $m_2 - a_1 = a_1 - m_1 = kd/4$. Choose $b_1, b_2 \in Y'$ satisfying
(a_1 - b_1) \cdot e_2 = (b_2 - a_1) \cdot e_2 = \sup \{ (x - a_1) \cdot e_2 : x \in B' \}
and (b_2 - b_1) \cdot e_1 = 0. \ Let \ z = \text{diam} (B' \cap \{ x: (x - b_1) \cdot e_2 = 2^{-6}k^2d \}).

Choose \ m_3 \in \mathbb{R}^2 \ with \ a_1 - m_3 = kze_1/4. \ Then \ let

F = B' \cap \{ x: (x - b_1) \cdot e_2 \geq 2^{-6}k^2d \ and \ (b_2 - x) \cdot e_2 \geq 2^{-6}k^2d \}
and \ G be the convex hull of \ F \cup \{ m_3 \}.

We now verify (2). \ Since \ B' \sim G \ is contained in the union of two rectangles with dimensions \ z \ and \ 2^{-6}k^2d, \ \mathcal{L}^2(B') \leq 2^{-5}k^2dz, \ while, \ since \ the \ interior \ of \ the \ convex \ hull \ of \ \{ m_1, m_2, m_3 \} \ is contained in \ G \sim B', \ \mathcal{L}^2(G \sim B') \geq 2^{-4}k^2dz; \ hence

\mathcal{L}^2(G) - \mathcal{L}^2(B') \geq 2^{-5}k^2dz.

Choose \ a_2 \in Y' with (a_2 - a_1) \cdot e_2 = 0, a_2 \neq a_1. \ Then \ B' \ is contained in a rectangle of side lengths \ |a_2 - a_1| \ and \ |b_2 - b_1|; \ consequently,

\mathcal{L}^2(B') \leq |a_2 - a_1| \cdot |b_2 - b_1|.

Take \ i = 1, 2. \ Let \ w_i = [a_i, b_i] \cap \{ x: (x - b_i) \cdot e_2 = 2^{-6}k^2d \}, \ s = (a_1 - b_1)
\cdot e_2/[w_i - b_1) \cdot e_2]. \ We see from our construction that \ |s| \leq 2^5k^{-2}, \ since

|(a_i - b_i) \cdot e_2| \leq d/2 \ and \ (w_i - b_i) \cdot e_2 = 2^{-6}k^2d. \ Furthermore, \ (a_1 - b_1) = s(w_2 - w_1), \ and \ by \ subtraction \ a_2 - a_1 = s(w_2 - w_1). \ Therefore, \ |a_2 - a_1| \leq 2^5k^{-2}|w_2 - w_1|. \ We \ combine \ this \ result \ with \ the \ inequalities (4), \ |w_2 - w_1| \leq z, \ |b_2 - b_1| \leq d \ and \ (5) \ to \ obtain

\mathcal{L}^2(G) - \mathcal{L}^2(B') \geq 2^{-5}k^2dz \geq 2^{-10}k^4|a_2 - a_1| \cdot |b_2 - b_1| \geq 2^{-10}k^4\mathcal{L}^2(B')

and then by addition (2).

To establish (3) we need only show that

\mathcal{L}^2(F \cup \{ m_3 \}) \leq \mathcal{L}^2(B'),

since \ \mathcal{L}^2(F \cup \{ m_3 \}) = \mathcal{L}^2(G) \ by \ [3, \ 2.10.3]. \ We \ now \ proceed \ to \ prove \ (6) \ by \ the \ following \ method:

Let

Q = [(F \cup \{ m_3 \}) - (F \cup \{ m_3 \})] \times [(F \cup \{ m_3 \}) - (F \cup \{ m_3 \})].
To \ each \ ordered \ pair \ (v_1, v_2) \in Q, \ v_1 = p_1 e_1 + p_2 e_2, \ v_2 = q_1 e_1 + q_2 e_2, \ we \ associate \ (v_1^*, v_2^*) \in Q \ by \ means \ of \ a \ map \ f \ such \ that \ (v_1^*, v_2^*) = f(v_1, v_2) \ satisfies \ the \ three \ conditions, \ v_1^* \in F - F, \ v_2 \in F - F \ implies \ v_2^* \in F - F, \ and \ |v_1^* \land v_2^*| \geq |v_1 \land v_2|. \ The \ existence \ of \ such \ a \ map \ f \ will \ prove \ (6), \ since \ (v_2^*, v_1^*) = 
\( f(v_2^*, v_1^*) \) will then satisfy \( v_2^{**}, v_1^{**} \in F - F \subset B' - B' \) and \( |v_2^{**} \wedge v_1^{**}| \geq |v_1 \wedge v_2| \). To define \( f \) and show the required conditions are satisfied we will consider the following cases and subcases:

**Case I.** \( v_1 \in F - F \).

Let \( v_1^* = v_1 \) and \( v_2^* = v_2 \).

**Case II.** \( v_1 \notin F - F \) and \( p_1 \geq 0 \).

We note that \( v_1 = x - m_3 \) for a unique \( x \) in \( F \). Let \( r = z/d, u = kd/4 \). We then consider four subcases:

**Case II.A.** \( |q_2| \geq r |q_2| \) and \( q_1(p_1 q_2 - p_2 q_1) \geq 0 \).

Let \( v_1^* = x - m_2, v_2^* = v_2 \). Then, since \( x - m_2 = (x - m_3) + (m_3 - m_2) = (p_1 - ru)e_1 + (p_2 - u)e_2 \), we deduce that

\[
|v_1^* \wedge v_2^*| = |p_1 q_2 - p_2 q_1 + u q_1 - r u q_2| \geq |p_1 q_2 - p_2 q_1| = |v_1 \wedge v_2|.
\]

**Case II.B.** \( |q_1| \geq r |q_2| \) and \( q_1(p_1 q_2 - p_2 q_1) \leq 0 \).

Let \( v_1^* = x - m_1 = (x - m_3) + (m_3 - m_1), v_2^* = v_2 \), and proceed as in Case II.A.

**Case II.C.** \( |q_1| \leq r |q_2| \) and \( p_2 \geq 0 \).

Let \( v_1^* = x - m_1, v_2^* = r q_2 e_1 - q_2 e_2 \). Note that \( v_2^* \in F - F \) by the construction of \( F \) and the definition of \( r \). Furthermore,

\[
|v_1^* \wedge v_2^*| = |p_1 q_2 + r p_2 q_2| \geq |p_1 q_2 - p_2 q_1| = |v_1 \wedge v_2|.
\]

**Case II.D.** \( |q_1| \leq r |q_2| \) and \( p_2 \leq 0 \).

Let \( v_1^* = x - m_2, v_2^* = r q_2 e_1 + q_2 e_2 \), and proceed as in Case II.C.

**Case III.** \( v_1 \notin F - F \) and \( p_1 \leq 0 \).

Let \( f(v_1, v_2) = f(-v_1, v_2) \).

Thus the existence of the required map \( f \) has been shown, (6) has been established, and the proof of the lemma is complete.

3.2. Theorem. **There exists** \( s < 1 \) **such that if** \( K \) **is a closed subset of** \( E \) **and** \( M \) **is the convex hull of** \( K \), **then** \( c^2(M) \leq s t^2(M) \).

**Proof.** Choose any \( \theta \in O^*(3, 2) \). Let \( \theta(M) = B, d = \text{diam } B \). We can assume by excluding the trivial case when \( B \) is a single point that \( d > 0 \). Denote by \( Y \) the boundary of \( B \) in \( \theta(R^3) \). Let \( S = M \cap \theta^{-1}(Y) \). Note that \( S \) is a continuum.

The first main step in our proof will be to show that there exists a closed line segment \( [a, b] \subset S \) such that \( |a - b| = rd \), where \( r \) is a number depending only on the sequence \( \nu \). (The construction involved in establishing this is basically a generalization of a procedure in [4].) Choose \( \alpha, \beta \in S \) with \( |\alpha - \beta| \geq d \). Let \( i \) be such that \( |\alpha_i - \beta_i| \geq |\alpha_j - \beta_j| \) for \( j = 1, 2, 3 \), where \( \alpha_j \) is the \( j \)th coordinate of \( \alpha \). Then \( |\alpha_i - \beta_i| \geq d/3^{1/2} > d/2 \).
Let $Q(k) = \bigcap_{j=1}^{k} \nu_j^{-3/2}$. Then $H_k(\nu)$ is a disjointed family consisting of $Q(k)^{-2/3}$ closed intervals of length $Q(k)$, and $[0, 1] \sim \bigcup H_k(\nu)$ is the union of $Q(k)^{-2/3} - 1$ open intervals of length $[(1 - \nu_j^{-1/2})/\nu_j^{1/2})]Q(j - 1)$, where $j$ ranges from 1 to $k$. Furthermore, since $1 - \nu_j^{-1/2} \geq 1 - 2^{-1/2} > 1/4$, $\nu_j - 1 < \nu_j^{3/2}$, it follows that 

$$[(1 - \nu_j^{-1/2})/(\nu_j - 1)]Q(j - 1) > \nu_j^{-3/2}Q(j - 1)/4 = Q(j)/4 \geq Q(k)/4.$$

Therefore, if $J \subset [0, 1]$ is a closed interval such that $\text{diam } J > 3Q(k)/2$, then there exists an open interval $U \subset J$ with

$$U \subset [0, 1] \sim \bigcup H_k(\nu) \subset [0, 1] \sim A(\nu)$$

and $\text{diam } U > Q(k)/4$. Consequently, if we choose $k$ satisfying $d > 3Q(k)$ and $d \leq 3Q(k - 1)$, then, since $|\alpha_i - \beta_i| > d/2$, there exists an open interval $I \subset [\alpha_i, \beta_i]$ such that $I \subset [0, 1] \sim A(\nu)$ and

$$\text{diam } I > Q(k)/4 = \nu_k^{-3/2}Q(k - 1)/4 \geq \nu_k^{-3/2}d/12 \geq \xi^{-3/2}d/12,$$

where $\xi$ is the least upper bound of the sequence $\nu$. Let $r = \xi^{-3/2}/24$ and let $G$ be the set of all $x$ for which $x_i$ is the midpoint of $I$. Observe that $S \cap G \neq \emptyset$, since $\alpha, \beta$ are on opposite sides of $G$, and $S$ is connected. Choose $a \in S \cap G$. Then distance $(a, K) > rd$, since $K \cap \{x: x_i \in I\} = \emptyset$. Let $N$ be a supporting line of $a$ at $\theta(a)$ and $D = \theta^{-1}(N)$. Then $D$ is a supporting plane of $M$ at $a$. Since $M$ is the convex hull of $K$, $D \cap S$ is convex and distance $(a, K) > rd$, it follows that there exists $b \in D \cap S$ with $[a, b] \subset D \cap S \subset S$, $|a - b| = rd$.

At this point we will divide the proof into cases and subcases in each of which it will be shown that there exists a number less than 1, depending only on $\nu$, which multiplied by $t^2(M)$ is greater than or equal to $Q^2(\beta)$. We will then let $s$ be the largest of these numbers among all cases.

We first divide the remainder of the proof into two cases:

**Case I.** $|\theta(a) - \theta(b)| > 2^{-9}r^3d$.

We use Lemma 3.1 with $k, L \cap Y$ replaced by $2^{-9}r^3, [\theta(a), \theta(b)]$ to obtain that $Q^2(B) \leq t^2(B)/(1 + 2^{-46}r^{12})$. Furthermore, we note that $t^2(B) \leq t^2(M)$, since $\|\wedge_2 \theta\| = 1$. We then conclude that

$$Q^2(B) \leq t^2(M)/(1 + 2^{-46}r^{12}).$$

**Case II.** $|\theta(a) - \theta(b)| \leq 2^{-9}r^3d$.

Let $\lambda = b - a$. Choose orthonormal basis vectors $e_1, e_2, e_3$ for $\mathbb{R}^3$ so that kernel$(\theta) = Re_3$ and $\lambda \cdot e_1 = 0$. As a result of this choice and the fact that $r < 1$ it follows that $\lambda = p_2e_2 + p_3e_3$ with $p_2, p_3$ satisfying $|p_2| \leq 2^{-9}r^3d$, $|p_3| > (1 - 2^{-18})^{1/2}rd > rd/2$, $|p_2/p_3| < 2^{-8}$. Let $m$ be the midpoint of $[\theta(a), \theta(b)]$. Choose $w \in S - S$, $w = q_1e_1 + q_2e_2 + q_3e_3$, satisfying $|\theta(w)| = d$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
We consider now four subcases of Case II:

**Case II.A.** \(|q_3| > d, d > d.\)

Choose \(z \in S - S\) satisfying \(\theta(z) \cdot \theta(w) = 0\) and

\[
|\theta(z)| = \sup \{|v|: v \in Y - Y \text{ and } v \cdot \theta(w) = 0|.
\]

Using [5, 1.15(7)] we obtain that

\[
4t^2(M)/\pi \geq |w \wedge z| \geq |[w]/|\theta(w)|]|\theta(w) \wedge \theta(z)|
\]

\[
> 2^{1/2}|\theta(w) \wedge \theta(z)| = 2^{1/2}|\theta(w)| \cdot |\theta(z)| \geq 2^{1/2}L^2(B),
\]

hence

\[
(8) \text{ Case II.A implies } L^2(B) \leq 4t^2(M)/(2^{1/2}\pi).
\]

**Case II.B.** \(|q_3| \leq d\) and \(L^2(B) < rd^2/4.\)

We deduce that

\[
4t^2(M)/\pi \geq |\lambda \wedge w| = ((p_2q_1)^2 + (p_3q_2)^2 + (p_2q_3 - p_3q_2)^2)^{1/2}
\]

\[
> [(p_3q_1)^2 + (p_3q_2)^2 - 2p_2p_3q_2q_3]^ {1/2} = |p_3|^2 + q_2^2 - 2(p_2/p_3)q_2q_3
\]

\[
> (rd/2)(d^2 - 2d^2)^{1/2} > 3rd^2/8 > 3L^2(B)/2,
\]

where the fifth relation in this chain follows from the conditions \(|p_3| > rd/2, q_1^2 + q_2^2 = d^2, |p_2/p_3| < 2^{-8}, |q_2| \leq d, |q_3| \leq d.\) Therefore,

\[
(9) \text{ Case II.B implies } L^2(B) \leq 8t^2(M)/(3\pi).
\]

**Case II.C.** \(L^2(B) > rd^2/4,\) and \(|(x - m) \wedge v| \leq 4(1 - 2^{-8}r^2)t^2(B)/\pi\) for all \(x \in B, v \in B - B.\)

Let \(\rho = 2^{-12}r^3d, W = B \cup B(m, \rho).\) We take any \(u_1, u_2 \in W - W\) and consider two possibilities:

First, if \(u_1, u_2 \in B - B,\) then clearly \((\pi/4)|u_1 \wedge u_2| \leq t^2(B).\)

On the other hand, suppose at least one of \(u_1, u_2,\) say \(u_1,\) for the sake of argument, is not in \(B - B.\) Then \(u_1 = u_3 + u_4, u_2 = u_5 + u_6,\) where \(u_3 = x - m\) for some \(x \in B, |u_4| \leq \rho, u_5 \in B - B, |u_6| \leq 2\rho.\) We also note that \(r < 1, u_3 \wedge u_5 \leq 4(1 - 2^{-8}r^2)t^2(B)/\pi, rd^2/4 \leq L^2(B) \leq t^2(B)\) by [3, 2.10.32], and then obtain

\[
|u_1 \wedge u_2| \leq |u_3 \wedge u_5| + |u_3| \cdot |u_6| + |u_4| \cdot |u_5| + |u_4| \cdot |u_6|
\]

\[
\leq |u_3 \wedge u_5| + 3\rho d + 2\rho^2 < |u_3 \wedge u_5| + 4\rho d < 4t^2(B)/\pi.
\]

Consequently, \(t^2(W) \leq t^2(B).\) Furthermore,

\[
L^2(W) \geq L^2(B) + \pi \rho^2/2 \geq (1 + 2^{-25}r^6)L^2(B),
\]

since \(L^2(B) \leq rd^2\) by [3, 2.10.33]. In addition, \(L^2(W) \leq t^2(W), t^2(B) \leq t^2(M).\)

We combine all these inequalities and conclude that
Case II.C implies $\mathcal{O}^2(\mathcal{B}) \leq t^2(M)/(1 + 2^{-8}r^2)$.

Case II.D. $\mathcal{O}^2(\mathcal{B}) \geq rd/4$, and there exists $y \in \mathcal{Y}$, $\nu_1, \nu_2 \in \mathcal{Y} - \mathcal{Y}$ with $\nu_1 = y - m$, such that $|\nu_1 \wedge \nu_2| > 4(1 - 2^{-8}r^2)t^2(B)/\pi$.

Take any $r \in S \cap \theta^{-1}[y]$. Then choose $\zeta \in [a - r, b - r]$, $\zeta = k_1e_1 + k_2e_2 + k_3e_3$, satisfying $|k_3| \geq rd/4$, and $\eta \in S - S$ satisfying $\theta(\eta) = \nu_2$. Let $\nu_3 = \theta(\zeta)$. We observe that $|\nu_1 - \nu_3| = |p_2|/2 \leq 2^{-10}\rho^2d$, $t^2(\mathcal{B}) \geq rd/4$, and then deduce

$$|\nu_3 \wedge \nu_2| \geq |\nu_1 \wedge \nu_2| - |\nu_1 - \nu_3| \cdot |\nu_2|$$

$$\geq 4(1 - 2^{-8}r^2)t^2(B)/\pi - 2^{-10}r^3d^2 > 4(1 - 2^{-7}r^2)t^2(B)/\pi.$$

Furthermore, $|\zeta/\nu_3| > 1 + 2^{-6}r^2$, since $|k_3| \geq rd/4$. These last two results combined with [5, 1.15(7)] and the inequalities $r < 1$, $\mathcal{O}^2(\mathcal{B}) \leq t^2(\mathcal{B})$ yield

$$t^2(M) \geq (\pi/4)|\zeta \wedge \eta|$$

$$\geq (\pi/4)|\zeta/\nu_3||\nu_3 \wedge \nu_2| > (1 + 2^{-8}r^2)\mathcal{O}^2(\mathcal{B}).$$

Therefore,

$$t^2(M) \geq (\pi/4)|\zeta \wedge \eta| > (1 + 2^{-8}r^2)\mathcal{O}^2(\mathcal{B}).$$

and hence $\mathcal{O}^2(\mathcal{B}) \leq t^2(M)/(1 + 2^{-8}r^2)$.

We now finish the proof of Theorem 3.2 by letting $s = 1/(1 + 2^{-46}r^{12})$ and then using (7), (8), (9), (10), (11) to conclude that $c^2(M) \leq st^2(M)$.

**3.3. Theorem.** $0 < \mathcal{I}_2(E) < \infty$.

**Proof.** From [3, 2.10.28] we see that $H^{2/3}[A(\nu)] = \alpha(2/3)2^{-2/3}$; consequently, repeated application of [3, 2.10.27] yields

$$H^2(E) \geq \alpha(2)[a(4/3)a(2/3)]^{-1}H^{2/3}[A(\nu) \times A(\nu)]H^{2/3}[A(\nu)]$$

$$\geq \alpha(2)a(2/3)^{-3}H^{2/3}[A(\nu)] \times H^{2/3}[A(\nu)] \times H^{2/3}[A(\nu)] = \pi/4.$$

Furthermore, $\mathcal{I}^2(E) \geq H^2(E)/6$ by [3, 2.10.39, 2.10.6]. Therefore, $\mathcal{I}^2(E) \geq \pi/24$.

Let $P(j) = \Pi_{i=1}^{n} i_{\nu_3}$. Given any $\delta > 0$ choose $k$ so that $3^{1/2}P(k)^{-1/2} < \delta$. $H_k(\nu) \times H_k(\nu) \times H_k(\nu)$ covers $E$ and consists of $P(k)$ cubes $D_j$ of diameter $3^{1/2}P(k)^{-1/2}$. Therefore,

$$\sum_{j=1}^{P(k)} t^2(D_j) \leq (\pi/4) \sum_{j=1}^{P(k)} (\text{diam } D_j)^2 = (\pi/4) \sum_{j=1}^{P(k)} 3P(k)^{-1} = 3\pi/4$$

and hence $\mathcal{I}^2(E) \leq 3\pi/4$.

**3.4. Main Theorem.** $\mathcal{C}^2(S) < \mathcal{I}^2(S)$ for all $S$ in $X$.

**Proof.** This follows directly from Theorems 3.2, 3.3 and the definitions of $\mathcal{C}^2$ and $\mathcal{I}^2$. 

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