

CLOSED SUBGROUPS OF LATTICE-ORDERED PERMUTATION GROUPS

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ABSTRACT. Let G be an l -subgroup of the lattice-ordered group $A(\Omega)$ of order-preserving permutations of a chain Ω ; and in this abstract, assume for convenience that G is transitive. Let $\bar{\Omega}$ denote the completion by Dedekind cuts of Ω . The stabilizer subgroups $G_{\bar{\omega}} = \{g \in G \mid \bar{\omega}g = \bar{\omega}\}$, $\bar{\omega} \in \bar{\Omega}$, will be used to characterize certain subgroups of G which are *closed* (under arbitrary suprema which exist in G). If Δ is an *o-block* of G (a nonempty convex subset such that for any $g \in G$, either $\Delta g = \Delta$ or $\Delta g \cap \Delta$ is empty), and if $\bar{\omega} = \sup \Delta$, G_{Δ} will denote $\{g \in G \mid \Delta g = \Delta\} = G_{\bar{\omega}}$; and the *o-block system* $\tilde{\Delta}$ consisting of the translates Δg of Δ will be called *closed* if G_{Δ} is closed. When the collection of *o-block systems* is totally ordered (by inclusion, viewing the systems as congruences), there is a smallest closed system \mathcal{C} , and all systems above \mathcal{C} are closed. \mathcal{C} is the *trivial system* (of singletons) iff G is *complete* (in $A(\Omega)$). $G_{\bar{\omega}}$ is closed iff $\bar{\omega}$ is a *cut* in \mathcal{C} , i.e., $\bar{\omega}$ is not in the interior of any $\Delta \in \mathcal{C}$. Every closed convex l -subgroup of G is an intersection of stabilizers of cuts in \mathcal{C} . Every closed prime subgroup $\neq G$ is either a stabilizer of a cut in \mathcal{C} , or else is minimal and is the intersection of a tower of such stabilizers. $L(\mathcal{C}) = \bigcap \{G_{\Delta} \mid \Delta \in \mathcal{C}\}$ is the distributive radical of G , so that G acts faithfully (and completely) on \mathcal{C} iff G is completely distributive. Every closed l -ideal of G is $L(\mathfrak{D})$ for some system \mathfrak{D} . A group G in which every nontrivial *o-block* supports some $1 \neq g \in G$ (e.g., a generalized ordered wreath product) fails to be complete iff G has a smallest nontrivial system $\tilde{\Delta}$ and the restriction $G_{\Delta} \upharpoonright \Delta$ is *o-2-transitive* and lacks elements $\neq 1$ of bounded support.

These results about permutation groups are used to show that if H is an abstract l -group having a representing subgroup, its closed l -ideals form a tower under inclusion; and that if $\{K_{\lambda}\}$ is a Holland kernel of a completely distributive abstract l -group H , then so is the set of closures $\{K_{\lambda}^*\}$, so that if H has a transitive representation as a permutation group, it has a complete transitive representation.

1. Introduction. A permutation f of a chain Ω is said to *preserve order* if $\beta \leq \gamma$ implies $\beta f \leq \gamma f$ for all $\beta, \gamma \in \Omega$. The group $A(\Omega)$ of all order-preserving permutations becomes a lattice-ordered group (l -group) when ordered pointwise. We shall treat l -subgroups G of $A(\Omega)$, i.e., subgroups which are also sublattices; these will be known as *l-permutation groups*. G will be assumed transitive only

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in §§4 and 5. A convex l -subgroup P of an l -group G is *prime* if $g_1 \wedge g_2 \in P$ implies $g_1 \in P$ or $g_2 \in P$. If (G, Ω) is an l -permutation group, each $g \in G$ can be uniquely extended to an order-preserving permutation of $\bar{\Omega}$ which we shall identify with g . For the extended group we shall write $(G, \bar{\Omega})$. Stabilizer subgroups $G_{\bar{\omega}}$, $\bar{\omega} \in \bar{\Omega}$, are prime.

Lloyd [7, Theorem 2] showed that in a rather restricted class of (possibly intransitive) l -permutation groups, the stabilizers G_{α} , $\alpha \in \Omega$, are closed. In [9], the present author showed that for groups in that class, the stabilizers $G_{\bar{\omega}}$, $\bar{\omega} \in \bar{\Omega}$, constitute all closed primes $\neq G$. In preparation for the description in Theorem 9 of various closed subgroups of G , the present paper extends the information in [9], and the discussion in [8] of the conditions (1) All sups in G are pointwise, (2) G is complete, (3) Stabilizers G_{α} are closed, and (4) G is completely distributive. However, dependence on these papers is kept to a minimum.

Several examples are given at the end of the paper, including one of a transitive group (necessarily with an infinite number of o -block systems) with a closed prime which is not a stabilizer.

2. Prime subgroups of an l -group. If P is a prime subgroup of an l -group H , then $\{Q \mid P \subseteq Q, Q \text{ prime in } H\}$ is totally ordered by inclusion [3, Lemma 3], and this tower contains all convex subgroups of H which contain P .

Lemma 1 (Byrd and Lloyd [2, Lemma 3.3]). *If P and Q are prime subgroups of an l -group H , with P closed and $P \subseteq Q$, then Q is also closed.*

Another hypothesis that guarantees closure of a prime subgroup is

Proposition 2. *Let P be a prime subgroup of an l -group H and let Q be a convex l -subgroup of H , $Q \not\subseteq P$. Then if $P \cap Q$ is closed in H , so is P .*

Proof. Suppose $s = \sup\{s_i \mid i \in I\}$, with $1 \leq s_i \in P$ for all i . Pick $1 < q \in Q \setminus P$. Then $q \wedge s = \sup\{q \wedge s_i\}$, and each $q \wedge s_i \in P \cap Q$, so that if $P \cap Q$ is closed, $q \wedge s \in P \cap Q \subseteq P$; and thus $s \in P$ since P is prime and $q \notin P$.

Corollary 3. *Let P be a meet-irreducible set of prime subgroups of an l -group H (i.e., $\bigcap \mathcal{P} \subset \bigcap \mathcal{P} \setminus \{P\}$) for every $P \in \mathcal{P}$). If $\bigcap \mathcal{P}$ is closed, so is every $P \in \mathcal{P}$.*

We mention that the special case $\bigcap \mathcal{P} = \{1\}$ is a trivial consequence of a much stronger statement due to Lloyd [6, Theorem 2.3].

A convex subset $\Delta \neq \square$ of Ω is an o -block of (G, Ω) if, for every $g \in G$, either $\Delta g = \Delta$ or $\Delta g \cap \Delta = \square$. The *convexification* $\text{Conv}(\bar{\Lambda})$ of $\bar{\Lambda} \subseteq \bar{\Omega}$ means $\{\omega \in \Omega \mid \bar{\lambda}_1 \leq \omega \leq \bar{\lambda}_2 \text{ for some } \bar{\lambda}_1, \bar{\lambda}_2 \in \bar{\Lambda}\} \subseteq \Omega$. An o -block Δ of G will be called *extensive* if $\text{Conv}(\bar{\delta}G_{\Delta}) = \Delta$ for one (hence every) $\bar{\delta} \in \bar{\Delta}$. The special case of the following theorem in which G is transitive (so that all o -blocks are exten-

sive) and $\bar{\pi} \in \Omega$ was established by the author in [10, Theorem 11].

Theorem 4. *Let (G, Ω) be an l -permutation group, and let $\bar{\pi} \in \bar{\Omega}$. An o -correspondence between the tower of nontrivial extensive o -blocks Δ of (G, Ω) for which $\bar{\pi} \in \bar{\Delta}$ and the tower of prime subgroups C of G for which $G_{\bar{\pi}} \subset C$ is given by $\Delta \rightarrow G_{\Delta}$ and $C \rightarrow \text{Conv}(\bar{\pi}C)$.*

Proof. [5, Theorem 3] states that nondisjoint o -blocks of transitive groups are comparable under inclusion, and the same proof works in the intransitive case for extensive o -blocks, so the lattice of o -blocks Δ for which $\bar{\pi} \in \bar{\Delta}$ is indeed totally ordered. If $\bar{\pi} \in \bar{\Delta}$, Δ an extensive o -block, then G_{Δ} is clearly a prime subgroup of G containing $G_{\bar{\pi}}$, and $\text{Conv}(\bar{\pi}G) = \Delta$ since Δ is extensive. If $G_{\bar{\pi}} \subset C$, C prime, then $\Delta = \text{Conv}(\bar{\pi}C)$ is an o -block of G and $G_{\Delta} = C$. For if $\bar{\pi}c_1 \leq \bar{\pi}c_2g \leq \bar{\pi}c_3$, then $c_2g \leq c_2g \vee c_3 \in C$ since $(c_2g \vee c_3)c_3^{-1} \in G_{\bar{\pi}} \subseteq C$, and similarly $c_2g \geq c_2g \wedge c_1 \in C$. Since C is convex, $c_2g \in C$, so that $g \in C$ and $\Delta g = \Delta$, as required. Clearly Δ is extensive.

3. Closed subgroups of an l -permutation group. The author showed in [8, Theorem 7] that for a transitive l -subgroup G of $A(\Omega)$, the following are equivalent:

(1) *Sups in G are pointwise*, i.e., if in G , $g = \bigvee_{i \in I} g_i$, then, for each $\beta \in \Omega$, βg is the sup in Ω of $\{\beta g_i \mid i \in I\}$.

(2) G is a *complete* subgroup of $A(\Omega)$, i.e., if in G , $g = \bigvee_{i \in I} g_i$, then $g = \bigvee_{i \in I} g_i$, also in $A(\Omega)$.

(3) G_{α} is a *closed* subgroup of G for all $\alpha \in \Omega$. (Of course, if any $G_{\bar{\omega}}$, $\bar{\omega} \in \bar{\Omega}$, is closed, so is $G_{\bar{\sigma}}$ for every $\bar{\sigma} \in \bar{\omega}G$.)

Moreover, it was shown in [8, Corollary 15] that in the presence of these conditions, we have

(4) G is a *completely distributive* l -group, i.e., $\bigwedge_{i \in I} \bigvee_{k \in K} g_{ik} = \bigvee_{f \in K^I} \bigwedge_{i \in I} g_{if(i)}$ for any collection $\{g_{ik} \mid i \in I, k \in K\}$ of elements of G for which the indicated sups and infs exist.

Here we discuss these conditions without assuming transitivity.

Lemma 5. *If $s = \sup \{s_i\}$ and $\sup \{\bar{\omega}s_i\} < \bar{\omega}s$ for some $\bar{\omega} \in \bar{\Omega}$, this sup also fails to be pointwise for every $\bar{\tau}$ in the interval $((\sup \{\bar{\omega}s_i\})s^{-1}, \bar{\omega})$ of $\bar{\Omega}$.*

Proof. $\sup \{\bar{\tau}s_i\} \leq \sup \{\bar{\omega}s_i\} < \bar{\tau}s$.

The *support* of $g \in A(\Omega)$ means $\{\omega \in \Omega \mid \omega g \neq \omega\}$, and any $\Delta \subseteq \Omega$ containing this set is said to support g . We let $\bar{\Omega}_c = \{\bar{\omega} \in \bar{\Omega} \mid G_{\bar{\omega}} \text{ closed}\}$.

Lemma 6. *If $\{s_i\} \subseteq G_{\bar{\omega}}$, $\bar{\omega} \in \bar{\Omega}$, and if $\sup \{s_i\} = s \notin G_{\bar{\omega}}$, then $G_{\bar{\tau}}$ is not closed for any $\bar{\tau} \in [\bar{\omega}, \bar{\omega}s)$, and this interval does not support any $1 \neq g \in G$. Hence $\bar{\Omega}_c$ is closed (in the order topology on $\bar{\Omega}$).*

Proof. We may assume without loss of generality that the s_i 's are positive. Suppose by way of contradiction that $G_{\bar{\tau}}$ is closed for some $\bar{\tau} \in [\bar{\omega}, \bar{\omega}s)$, and let $\bar{\sigma} = \bar{\tau}s^{-1} < \bar{\omega}$. Then $G_{\bar{\sigma}}$ is closed, and $\bar{\sigma}s < \bar{\sigma}s^2$, so that $\bar{\sigma} < \bar{\sigma}s$. Thus $\bar{\sigma} < \bar{\sigma}s_j < \bar{\omega}$ for some s_j , and letting $\bar{\rho} = \bar{\sigma}s_j$, $G_{\bar{\rho}}$ is closed. But $s_i s_i^{-1} s_j \vee 1 \in G_{\bar{\rho}}$ for each i since $s_i \in G_{\bar{\omega}}$ guarantees that $\bar{\rho}s_i s_i^{-1} s_j < \bar{\omega}s^{-1} s_j \leq \bar{\tau}s^{-1} s_j = \bar{\sigma}s_j = \bar{\rho}$; whereas $\sup\{s_i s_i^{-1} s_j \vee 1\} = s_j \notin G_{\bar{\rho}}$, a contradiction. Hence $G_{\bar{\tau}}$ is not closed for any $\bar{\tau} \in [\bar{\omega}, \bar{\omega}s)$. But then $G_{\bar{\tau}}$ is not closed for any $\bar{\tau} \in [\bar{\omega}, \bar{\omega}s)s^{-1} = [\bar{\omega}s^{-1}, \bar{\omega})$. Therefore, $\bar{\Omega} \setminus \bar{\Omega}_c$ is open in the order topology on $\bar{\Omega}$. Also, if $[\bar{\omega}, \bar{\omega}s)$ supports some $1 \neq g \in G$ (which can be assumed positive), then $s_i \leq sg^{-1} < s$ for each i (since $\bar{\omega}s_i = \bar{\omega}$), a contradiction.

Theorem 7. For any l -permutation group, each condition (1)–(4) implies the next. Moreover,

- (a) If sups in G are pointwise (at points of Ω), they are also pointwise at cuts of Ω .
- (b) If G is complete in $A(\Omega)$ and transitive on Ω , then G is also complete in $A(\bar{\Omega})$.
- (c) If stabilizers of points are closed, so are stabilizers of cuts.

Proof. The proof of the first statement is exactly as in [8]. (None of the implications can be reversed; a counterexample for the last one can be found in § 7.) (a) follows from Lemma 5, and (c) from Lemma 6. When (G, Ω) is transitive, (2) for (G, Ω) implies (1) for (G, Ω) , which by (a) implies (1) for $(G, \bar{\Omega})$, which in turn implies (2) for $(G, \bar{\Omega})$, yielding (b); and (b) does in fact fail without transitivity.

A G -congruence \mathfrak{D} is said to be *convex* if its congruence classes are convex. The corresponding partition of Ω into o -blocks of G will also be denoted by \mathfrak{D} . The set of o -blocks associated with a given \mathfrak{D} inherits a natural total order from Ω , namely $\Delta_1 \leq \Delta_2$ iff $\delta_1 \leq \delta_2$ for some $\delta_1 \in \Delta_1, \delta_2 \in \Delta_2$. If G is transitive, the o -blocks associated with \mathfrak{D} are all translates of each other. In this case the partition \mathfrak{D} will be called an *o -block system* of G , and if Δ is an o -block, the system it determines will be denoted by $\tilde{\Delta}$.

We shall say that $\bar{\omega} \in \bar{\Omega}$ is a *cut in \mathfrak{D}* if $\bar{\omega}$ does not lie in the interior of $\bar{\Delta}$ for any $\Delta \in \mathfrak{D}$. If G is transitive and \mathfrak{D} nontrivial, then $\Delta \in \mathfrak{D}$ lacks end points, so the same is true of $\bar{\Delta}$, and thus the condition reads “ $\bar{\omega} \notin \bar{\Delta}$ for any $\Delta \in \mathfrak{D}$ ”.

We shall say that a convex G -congruence \mathfrak{D} is *closed* if for each \mathfrak{D} -class Δ, G_{Δ} ($= G_{\sup \Delta} = G_{\inf \Delta}$) is closed. If $\bar{\omega}$ is a cut in a closed \mathfrak{D} , then $G_{\bar{\omega}}$ is closed (by definition if $\bar{\omega}$ is an end point in $\bar{\Omega}$ of some $\Delta \in \mathfrak{D}$, and then in general by Lemma 6). If G is transitive, and Δ is an o -block of G , then if G_{Δ} is closed, $\tilde{\Delta}$ is closed, for if $\Gamma \in \tilde{\Delta}, G_{\Gamma}$ is a conjugate of G_{Δ} .

We partially ordered the collection of convex G -congruences by inclusion. In the transitive case, this gives a total order [5, Theorem 3].

Now we define an extremely important convex G -congruence \mathcal{C} as follows: $\sigma \mathcal{C} \tau$ iff either $\sigma = \tau$ or no $\bar{\delta} \in \bar{\Omega}$ lying between σ and τ (inclusive) has a closed stabilizer $G_{\bar{\delta}}$. Since $\bar{\Omega}_c = \{\bar{\omega} \in \bar{\Omega} \mid G_{\bar{\omega}} \text{ is closed}\}$ satisfies $\bar{\Omega}_c g = \bar{\Omega}_c$ for every $g \in G$, \mathcal{C} is indeed a (convex) G -congruence.

Theorem 8. *Let (G, Ω) be an l -permutation group. Then*

- (1) \mathcal{C} is the smallest closed convex G -congruence, and a convex G -congruence \mathcal{D} is closed iff $\mathcal{D} \geq \mathcal{C}$.
- (2) \mathcal{C} is the trivial system (of singletons) iff all stabilizers of points in Ω are closed.
- (3) The \mathcal{C} -classes are extensive o -blocks of G .
- (4) $G_{\bar{\omega}}$ is closed iff $\bar{\omega}$ is a cut in \mathcal{C} , i.e., $\bar{\Omega}_c$ is the set of cuts in \mathcal{C} .
- (5) If $\bar{\omega} \in \bar{\Delta}$, $\Delta \in \mathcal{C}$, then $G_{\bar{\omega}}^* = G_{\Delta}$.
- (6) If $\bar{\omega}$ is a cut in \mathcal{C} and $G_{\bar{\omega}}$ fixes $\bar{\tau}$, then $\bar{\tau}$ is also a cut in \mathcal{C} .

Proof. If Δ is a \mathcal{C} -class, then every interval $[\sup \Delta, \omega)$ meets $\bar{\Omega}_c$, so by Lemma 6, $G_{\sup \Delta}$ is closed. Hence \mathcal{C} is closed. (4) now follows immediately from the construction of \mathcal{C} . Hence if $\mathcal{D} \geq \mathcal{C}$, \mathcal{D} is closed; and conversely, if \mathcal{D} is closed, then for any $\Delta \in \mathcal{D}$, $G_{\sup \Delta}$ and $G_{\inf \Delta}$ are closed, so that $\sup \Delta$ and $\inf \Delta$ are cuts in \mathcal{C} , and thus $\mathcal{D} \geq \mathcal{C}$. This proves (1).

\mathcal{C} is trivial iff $\bar{\Omega}_c$ is dense in $\bar{\Omega}$ iff $\bar{\Omega}_c = \bar{\Omega}$ (since $\bar{\Omega}_c$ is closed by Lemma 6) iff all point stabilizers are closed (by (a) of Theorem 7). If $\delta \in \Delta \in \mathcal{C}$, then $\text{Conv}(\delta G_{\Delta})$ has the same (closed) stabilizer as Δ , so its end points cannot lie in $\bar{\Delta}$, proving that Δ is extensive.

If $\bar{\omega}$ is not a cut in \mathcal{C} , so that $\bar{\omega} \in \bar{\Delta}$, $\Delta \in \mathcal{C}$, then by Theorem 4, the tower of primes properly containing $G_{\bar{\omega}}$ is $\{G_{\Gamma} \mid \bar{\omega} \in \bar{\Gamma}, \Gamma \text{ an extensive } o\text{-block of } G\}$, and by (4), G_{Δ} is the smallest closed member of this tower, so that $G_{\bar{\omega}}^* = G_{\Delta}$. For (6), if $G_{\bar{\omega}}$ fixes $\bar{\tau}$, then $G_{\bar{\omega}} \subseteq G_{\bar{\tau}}$. Now if $\bar{\omega}$ is a cut in \mathcal{C} , so that $G_{\bar{\omega}}$ is closed, then $G_{\bar{\tau}}$ is also closed by Lemma 1, so that $\bar{\tau}$ is also a cut in \mathcal{C} .

Our main theorem extends a line of thought begun in [9]. For transitive groups, it will be sharpened somewhat in Theorem 12. The closure of a convex l -subgroup C will be denoted by C^* , and $\{\bar{\omega} \in \bar{\Omega} \mid \bar{\omega}C = \bar{\omega}\}$ by $\bar{F}xC$ (written without the bar in [9]). If $\bar{\Gamma} \subseteq \bar{\Omega}$, $G(\bar{\Gamma})$ will denote $\{g \in G \mid \bar{\gamma}g = \bar{\gamma} \text{ for each } \bar{\gamma} \in \bar{\Gamma}\} = \bigcap \{G_{\bar{\gamma}} \mid \bar{\gamma} \in \bar{\Gamma}\}$. If \mathcal{D} is a convex G -congruence, we let $L(\mathcal{D})$ be the l -ideal $\{g \in G \mid \Delta g = \Delta \text{ for all } \Delta \in \mathcal{D}\} = \bigcap \{G_{\bar{\omega}} \mid \bar{\omega} \text{ is a cut in } \mathcal{D}\}$. If \mathcal{D} is closed, $L(\mathcal{D})$ is closed. If $\mathcal{D} < \mathcal{E}$, then $L(\mathcal{D}) \subseteq L(\mathcal{E})$. We let G act on \mathcal{D} by defining $\Delta g \in \mathcal{D}$ to be $\Delta g \subseteq \Omega$. This action is faithful iff $L(\mathcal{D}) = \{1\}$. The *distributive radical* $D(G)$ of G is the intersection of all the closed primes of G , and is $\{1\}$ iff G is completely distributive [2, Theorem 3.4 and Corollary 3.8].

Main Theorem 9. *Let (G, Ω) be an l -permutation group.*

- (1) *If C is any convex l -subgroup of G , then $C^* \supseteq G(\overline{FxC})$.*
- (2) *If C is any closed convex l -subgroup of G , $C = G(\overline{FxC}) = \bigcap \{G_{\overline{\delta}} \mid C \subseteq G_{\overline{\delta}}\}$; and if C is the intersection of a finite collection of stabilizers, all stabilizers containing C are closed.*
- (3) *If the stabilizers of points of Ω are closed, the closed convex l -subgroups of G are precisely the $G(\overline{\Delta})$'s, $\overline{\Delta} \subseteq \overline{\Omega}$, and thus are intersections of closed primes.*
- (4) *If P is any prime subgroup of G , $\overline{Fxp}^* = \overline{Fxp} \cap \overline{\Omega}_c$ and $P^* = G(\overline{Fxp} \cap \overline{\Omega}_c)$.*
- (5) *If P is any closed prime subgroup of G , the tower of primes Q such that $P \subset Q \subset G$ consists entirely of closed stabilizers $G_{\overline{\omega}}$, and P is a minimal closed prime unless $G_{\overline{\gamma}} \subset P$ for some closed $G_{\overline{\gamma}}$.*
- (6) *In (5), if $P \neq G$ is not itself a stabilizer, then P has no cover in the tower of primes which contain it, and thus is an intersection of closed stabilizers; and P is a minimal closed prime.*
- (7) *The distributive radical $D(G)$ is $L(\mathcal{C})$.*
- (8) *Every closed l -ideal of G is $L(\mathcal{D})$ for some \mathcal{D} .*

Proof. The proof of [9, Theorem 3] proves (1), which gives the first part of (2), which, in view of Theorem 7, gives (3). For the second part of (2), if C is the intersection of a finite collection of stabilizers, it is the intersection of a (finite) meet-irreducible subcollection $\{G_{\overline{\gamma}} \mid \overline{\gamma} \in \overline{\Gamma}\}$, and the $G_{\overline{\gamma}}$'s are closed by Corollary 3. Now let $C \subseteq G_{\overline{\delta}}$. If $G_{\overline{\delta}}$ is not closed, then for each $\overline{\gamma}$, $G_{\overline{\gamma}} \not\subseteq G_{\overline{\delta}}$ by Lemma 1, so there exists $1 < g_{\overline{\gamma}} \in G_{\overline{\gamma}}$ such that $\overline{\delta g} > \overline{\delta}$. Now $\bigwedge g_{\overline{\gamma}} \in \bigcap G_{\overline{\gamma}} = C$, but $\bigwedge g_{\overline{\gamma}} \notin G_{\overline{\delta}}$, a contradiction.

If P is any prime subgroup of G , then $P^* \subseteq \bigcap \{G_{\overline{\omega}} \mid P \subseteq G_{\overline{\omega}}, \overline{\omega} \in \overline{\Omega}_c\}$ since the latter is closed, so that $\overline{Fxp}^* \supseteq \overline{Fxp} \cap \overline{\Omega}_c$. But of course $\overline{Fxp}^* \subseteq \overline{Fxp}$, and since $P^* \subseteq G_{\overline{\omega}}$ and P^* closed implies $G_{\overline{\omega}}$ closed by Lemma 1, we have $\overline{Fxp}^* \subseteq \overline{\Omega}_c$, so that $\overline{Fxp}^* = \overline{Fxp} \cap \overline{\Omega}_c$. By (1), $P^* = G(\overline{Fxp}^*) = G(\overline{Fxp} \cap \overline{\Omega}_c)$.

Now let $P \neq G$ be a closed prime and let \mathcal{J} be the tower of primes Q (automatically closed by Lemma 1) such that $P \subset Q \subset G$. Let $C \in \mathcal{J}$. If P is not a stabilizer, then since P is the intersection of the stabilizers which contain it (by (2)), \mathcal{J} has no smallest member (proving part of (6)) and the subset of \mathcal{J} consisting of stabilizers is cointial with \mathcal{J} . Thus in any case, C contains a stabilizer $G_{\overline{\pi}}$. By Theorem 4, $\Delta = \text{Conv}(\overline{\pi}C)$ is an o -block of G and $G_{\Delta} = C$. Hence $C = G_{\text{sup}\Delta}$. This establishes the first part of (5), which, together with the part of (6) already proved, yields the rest of (5) and (6).

The distributive radical $D(G)$ is the intersection of the closed primes of G , and by (6), every closed prime is an intersection of closed stabilizers, so that $D(G) = \bigcap \{G_{\overline{\omega}} \mid \overline{\omega} \in \overline{\Omega}_c\} = L(\mathcal{C})$.

Finally, let C be any closed l -ideal of G . Since C is normal in G , \overline{FxC} is a union of orbits of $(G, \overline{\Omega})$. Hence we obtain a convex G -congruence \mathfrak{D} by setting $\sigma \mathfrak{D} \tau$ iff either $\sigma = \tau$ or no $\overline{\delta}$ between σ and τ (inclusive) lies in \overline{FxC} . By (2), $C = G(\overline{FxC}) = L(\mathfrak{D})$. This concludes the proof.

Theorem 10. *Let (G, Ω) be an l -permutation group. The following are equivalent:*

- (1) G is completely distributive.
- (2) G acts faithfully on \mathcal{C} .
- (3) $C^* = G(\overline{FxC} \cap \overline{\Omega}_c)$ for every convex l -subgroup C of G .
- (4) Every closed convex l -subgroup C of G is the intersection of the closed stabilizers which contain it.

Proof. G is completely distributive iff $D(G) = \{1\}$, and $D(G) = L(C)$ by Theorem 9, so (1) and (2) are equivalent. In any completely distributive l -group, every closed convex l -subgroup is the intersection of the closed primes which contain it [9, Corollary 5], and hence, by (6) of Theorem 9, of the closed stabilizers which contain it. Thus (1) implies (4). Now suppose (4) holds, and let C be a convex l -subgroup of G . Then if $G_{\overline{\omega}}$ is closed, $C^* \subseteq G_{\overline{\omega}}$ iff $C \subseteq G_{\overline{\omega}}$, so by (4), $C^* = \bigcap \{G_{\overline{\omega}} \mid C \subseteq G_{\overline{\omega}} \text{ and } G_{\overline{\omega}} \text{ closed}\} = \bigcap \{G_{\overline{\omega}} \mid \overline{\omega} \in \overline{FxC} \cap \overline{\Omega}_c\} = G(\overline{FxC} \cap \overline{\Omega}_c)$, proving (3). Finally, if (3) holds, then since $\overline{Fx}\{1\} = \overline{\Omega}$, we have $\{1\} = \{1\}^* = G(\overline{\Omega}_c) = L(\mathcal{C})$, which is (2).

A representation of an l -group H on a chain Σ is an l -isomorphism θ of H into $A(\Sigma)$. θ is complete if it preserves arbitrary sups that exist in H , or equivalently, if $H\theta$ is a complete l -subgroup of $A(\Sigma)$. H has a complete representation iff H is completely distributive [2, Theorem 3.10].

By the normal kernel of a convex l -subgroup C of an l -group H we mean the intersection of all the conjugates of H , which is the largest l -ideal of G contained in C . A collection $\{K_\lambda \mid \lambda \in \Lambda\}$ of primes of H whose intersection has trivial normal kernel is called a Holland kernel of H . These kernels were used by Holland to obtain representations of H as l -permutation groups [3, Theorem 2].

Corollary 11. *Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be a Holland kernel of an abstract l -group H , so that the normal kernel of $\bigcap K_\lambda$ is $\{1\}$. Then the normal kernel of $\bigcap K_\lambda^*$ is $D(H)$. Hence if H is completely distributive, $\{K_\lambda^*\}$ is also a Holland kernel.*

Proof. Use the Holland representation to represent H as an l -permutation group on a chain Ω so that each K_λ is the stabilizer of some point ω_λ and the union of the orbits $\omega_\lambda H$ is Ω . Then by Theorem 9, the K_λ^* 's are stabilizers of σ -blocks in the smallest convex H -congruence \mathcal{C} . Hence the collection of all conjugates of K_λ^* 's is precisely the collection of stabilizers of all σ -blocks in \mathcal{C} , and the intersection of these is $L(\mathcal{C})$, which by Theorem 9 is $D(H)$.

4. **Transitive l -permutation groups.** Recall that in a transitive group (G, Ω) , all the o -blocks of a given convex G -congruence are translates of each other, and that the o -block systems form a tower, so that the concept of the smallest closed system \mathcal{C} is particularly nice. Further, for any $\omega \in \Omega$, the map $\Delta \rightarrow \bar{\Delta}$ is an o -isomorphism from the tower of o -blocks Δ containing ω onto the tower of o -block systems.

Theorem 12. *Let (G, Ω) be a transitive l -permutation group. Then*

- (1) \mathcal{C} is the trivial system iff G is a complete subgroup of $A(\Omega)$.
- (2) The closed l -ideals of G form a tower under inclusion.
- (3) If G has only a finite number of closed o -block systems, every closed prime $\neq G$ is $G_{\bar{\omega}}$ for some cut $\bar{\omega}$ in \mathcal{C} (i.e., (6) of Theorem 9 cannot occur).
- (4) If G is completely distributive, the closed l -ideals of G are precisely the $L(\mathcal{D})$'s.

Proof. \mathcal{C} is the trivial system iff stabilizers of points are closed, and by [8, Theorem 7] this holds iff G is complete in $A(\Omega)$. (2) follows from (8) of Theorem 9 since the o -block systems form a tower.

For (3), let q be the number of closed o -block systems, and suppose that P is as in (6) of Theorem 9, and thus has no prime cover. Then there exists above P a stabilizer $G_{\bar{\omega}}$ above which there lies a tower of $q + 1$ distinct closed stabilizers. Theorem 4 supplies a tower of $q + 1$ distinct o -blocks whose systems are closed, a contradiction.

By Theorem 9, every closed l -ideal is $L(\mathcal{D})$ for some \mathcal{D} ; and if $\mathcal{D} \geq \mathcal{C}$, then \mathcal{D} and hence $L(\mathcal{D})$ are closed. If $\mathcal{D} < \mathcal{C}$, then $L(\mathcal{D}) = L(\mathcal{C}) = \{1\}$ if G is completely distributive, completing the proof.

Question. Does the conclusion of (4) hold even without complete distributivity? Nontrivial $L(\mathcal{D})$'s can be closed even when \mathcal{D} is not, and perhaps they always are.

A representation θ of an l -group H on a chain Σ is *transitive* if $H\theta$ is transitive on Σ . A *representing subgroup* of H is a one element Holland kernel, i.e., a prime subgroup which contains no nontrivial l -ideal of H . If H has a representing subgroup K , then the Holland representation obtained from K is transitive and has K as the stabilizer of a point; and conversely, if (G, Ω) is transitive, the stabilizers of points are representing subgroups of G . Thus if H has a complete transitive representation, H is completely distributive and has a representing subgroup. Conversely, we have

Corollary 13. *Let H be a completely distributive abstract l -group. Then*

- (1) If θ is a transitive representation of H on a chain Σ , then $(H\theta, \Sigma)$ can be faithfully "reduced" to the complete transitive group $(H\theta, \mathcal{C})$, where \mathcal{C} is the smallest closed system of $(H\theta, \Sigma)$.

(2) If K is a representing subgroup of H , so is K^* . Hence maximal representing subgroups are closed.

Proof. Since $H\theta$ is completely distributive, its action on \mathcal{C} is faithful, and the stabilizers in this action are closed in G , so G is complete in $A(\mathcal{C})$, proving (1). Since a representing subgroup is precisely a one element Holland kernel, (2) follows from Corollary 11.

A transitive group (G, Ω) is said to be *weakly o-primitive* [4] if $L(\mathcal{D}) = \{1\}$ only when \mathcal{D} is the trivial system, so that (G, Ω) has no proper faithful reductions.

Corollary 14. *Conditions (1), (2), (3), and (4) of Theorem 7 are equivalent for weakly o-primitive groups.*

Proof. We need only show that (4) implies (3). But by Theorem 10, complete distributivity implies $L(\mathcal{C}) = \{1\}$, so that by weak o-primitivity, \mathcal{C} must be the trivial system, i.e., (3) holds.

Corollary 15. *Let H be an abstract l-group having a representing subgroup. Then the closed l-ideals of H form a tower under inclusion.*

Proof. Since H has a transitive representation, we may use part (8) of Theorem 9.

6. **The support property.** If an o-block Δ of a transitive group G supports some $1 \neq g \in G$, so does every $\Gamma \in \tilde{\mathcal{X}}$, and by Lemma 6, $\tilde{\mathcal{X}}$ must be closed. We shall say that G has the *support property* if each of its nontrivial o-blocks supports some $1 \neq g \in G$. Clearly groups having the support property are weakly o-primitive.

A transitive group (G, Ω) is called *locally o-primitive* if it has a smallest nontrivial o-block system \mathcal{D} . In this case the o-blocks of \mathcal{D} are called *primitive segments*. The $(G_\Delta | \Delta, \Delta)$'s for the various primitive segments Δ are all isomorphic as l-permutation groups, and are o-primitive; and a property enjoyed by them is said to be enjoyed *locally* by (G, Ω) . If G is not locally o-primitive, then $\{\alpha\} = \bigcap \{\Delta | \alpha \in \Delta, \Delta \text{ a nontrivial o-block}\}$, so that the support property implies that every nondegenerate interval of Ω supports some $1 \neq g \in G$. In view of the next lemma, this implication also holds if G is locally nonpathologically o-2-transitive. (An o-2-transitive group is *pathological* if it lacks elements $\neq 1$ of bounded support.)

Lemma 16. *Let (G, Ω) be a locally o-primitive group having the support property. Let Δ be a primitive segment, let $K = G_\Delta | \Delta$, and let $L = \{g | \Delta : \Delta \text{ supports } g\}$. Then if G is locally o-2-transitive, L is an o-2-transitive l-ideal of K containing all elements of bounded support; and otherwise $L = K$.*

Proof. It is easily checked that L is an l -ideal of K , and the support property states that $L \neq \{1\}$. In pathologically o -2-transitive groups, all nontrivial l -ideals are (pathologically) o -2-transitive [11, Theorem 6], and in nonpathologically o -2-transitive groups, the o -2-transitive l -ideal consisting of the elements of bounded support is contained in every nontrivial l -ideal (by the proof of [3, Theorem 6]). O -primitive groups which are not o -2-transitive have no proper l -ideals [10, Corollary 46]. The lemma follows.

The following theorem generalizes [12, Theorem 4], which states that an o -primitive group fails to be complete iff it is pathologically o -2-transitive.

Theorem 17. *Let (G, Ω) be a transitive l -permutation group having the support property. Then (G, Ω) fails to be complete if and only if it is locally pathologically o -2-transitive; and in this case, the smallest closed o -block system \mathcal{C} is the smallest nontrivial system.*

Proof. Since by Lemma 6 the support by an o -block Δ of some $1 \neq g \in G$ implies the closure of \tilde{X} , the support property guarantees that either \mathcal{C} is the trivial system (i.e., G is complete), or G is locally o -primitive and \mathcal{C} is the smallest nontrivial system. It remains to show that when G is locally o -primitive, the stabilizers G_α fail to be closed iff G is locally pathologically o -2-transitive.

Let $\alpha \in \Omega$, let Δ be the primitive segment containing α , and let K and L be as in the lemma. Suppose $s = \sup\{s_i\}$, with each $s_i \in G_\alpha$ and $s \in G \setminus G_\alpha$. Since G_Δ is closed, $\alpha s \in \Delta$, (K, Δ) is o -primitive, and if it is not pathologically o -2-transitive, K_α is closed in K [12, Theorem 4]. Thus, letting $\hat{s} = s|\Delta$ and $\hat{s}_i = s_i|\Delta$, there exists $1 < \hat{t} \in K$ such that, for each i , $\hat{s}_i \leq \hat{s}\hat{t}^{-1} < \hat{s}$. By the lemma, $\hat{t} \in L$. (If (K, Δ) is nonpathologically o -2-transitive, \hat{t} exceeds some $1 < k \in K$ having bounded support, so we may assume that \hat{t} has bounded support.) Hence Δ supports some $t \in G$ such that $t|\Delta = \hat{t}$. st^{-1} agrees with $\hat{s}\hat{t}^{-1}$ on Δ and with s outside Δ , so for each i , $s_i \leq st^{-1} < s$, a contradiction. Therefore G_α is closed in G .

Conversely, suppose that G is locally pathologically o -2-transitive. By the lemma, L is (pathologically) o -2-transitive, so there exists $\{\hat{s}_i\} \subseteq L_\alpha$ having in L a sup $\hat{s} \notin L_\alpha$. Δ supports s , $s_i \in G$ such that $s|\Delta = \hat{s}$ and $s_i|\Delta = \hat{s}_i$. Then $s = \sup\{s_i\}$ in G , for if $s_i \leq t < s$, then Δ supports t , so that $t|\Delta$ yields a contradiction. Therefore G_α is not closed in G .

If an l -permutation group (G, Ω) has the support property and is locally o -primitive, but not locally pathologically o -2-transitive, the other transitive representations of G can be very precisely described. (The description is due to Holland [4, Theorem 7]. In [11, Theorem 7], the hypotheses are relaxed to those just mentioned, and it is shown that pathologically o -2-transitive groups need not satisfy

the description.) Theorem 17 says that the groups (G, Ω) satisfying these hypotheses are precisely those locally \circ -primitive groups having the support property which are complete.

Corollary 18. *Let $\{(G_\gamma, \Omega_\gamma) \mid \gamma \in \Gamma\}$ be a collection of transitive l -permutation groups indexed by a chain Γ , and let (W, Ω) be its ordered wreath product [5, § 3]. The transitive l -permutation group (W, Ω) fails to be complete if and only if Γ has a least element 0 and (G_0, Ω_0) is pathologically \circ -2-transitive.*

7. Examples. Our first example shows that completely distributive transitive groups need not have closed stabilizers (cf. Theorem 7 and Corollary 14). Holland [4, p. 433] pointed out that if Ω is the chain of real numbers, the stabilizer subgroups of $A(\Omega)$ are not minimal representing subgroups. Pick a representing subgroup K properly contained in a stabilizer. By part (3) of Theorem 12, K is not closed. Thus in the Holland representation of $A(\Omega)$ on $R(K)$, K is the stabilizer of a point and is not closed, even though $A(\Omega)$ is completely distributive.

Next, we exhibit a transitive group (G, Ω) with G totally ordered, so that $\{1\}$ is a closed prime, but with no stabilizer $G_{\bar{\omega}} = \{1\}$ (cf. Theorem 9). Let I be the \circ -group of integers. Let J be the \circ -group

$$\overleftarrow{\dots \oplus I \oplus I \oplus \dots},$$

the small direct sum (indexed by I) of copies of I , ordered lexicographically from the right. Let f be the \circ -group automorphism of J which shifts each sequence one place to the right. Map $1 \in I$ onto f and extend the map to a group homomorphism of I into the group of \circ -group automorphisms of J . Take G to be the semi-direct sum $\overleftarrow{J \oplus I}$ (i.e., $\langle j_1, i_1 \rangle + \langle j_2, i_2 \rangle = \langle j_1/i_1^2 + j_2, i_1 + i_2 \rangle$), ordered lexicographically from the right to give an \circ -group. Let $K = \{\langle j, 0 \rangle \mid j(n) = 0 \text{ for all } n \geq 0\}$. K is a representing subgroup of G , for if $\langle 0, 0 \rangle \neq \langle j, 0 \rangle \in K$, and if $n < 0$ is the greatest integer such that $j(n) \neq 0$, then $\langle 0, n \rangle + \langle j, 0 \rangle + \langle 0, -n \rangle \notin K$. Let G act on $\Omega = R(K)$ via the Holland representation. Letting $\alpha = K$, we have $K + \langle j, i \rangle \in FxG_\alpha$ iff $i \geq 0$, so that FxG_α is a cofinal segment of Ω extending properly below α . For any $\bar{\omega} \in \bar{\Omega}$, we can pick $\beta < \bar{\omega}$, and then $\{1\} \neq G_\beta \subseteq G_{\bar{\omega}}$. Therefore $\{1\}$ is not a stabilizer $G_{\bar{\omega}}$.

The argument that K is a representing subgroup also shows that K is properly contained in certain of its conjugates. (In [10, § 3], groups exhibiting this pathology were called *imbalanced*, but the present example, which had been pointed out to the author by Charles Holland, was omitted for the sake of brevity.) Therefore, $K (= G_\alpha)$ is not a maximal representing subgroup, so that (G, Ω) is not weakly \circ -primitive. However, G_α is closed in G since G is an \circ -group, so that (G, Ω) is complete. Thus among completely distributive transitive groups, weak \circ -primitivity is properly stronger than completeness (cf. Corollary 14).

Closed primes which are not stabilizers can occur even in groups which are complete and weakly and locally \mathcal{O} -primitive: Form an ordered wreath product (W, Σ) of two factors, with the previous group (G, Ω) as top factor, and $(A(\Lambda), \Lambda)$, Λ the reals, as bottom factor. W is complete by Corollary 18. But $L(\mathcal{D})$, \mathcal{D} the smallest nontrivial \mathcal{O} -block system, is a closed prime (since G is an \mathcal{O} -group) which is not a stabilizer.

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