ITERATED FINE LIMITS AND ITERATED NONTANGENTIAL LIMITS

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ABSTRACT. Let $\Omega_k$, $k = 1$ to $n$, be harmonic spaces of Brelot and $u_k > 0$ harmonic functions on $\Omega_k$. For each $w$ in a class of multiply superharmonic functions it is shown that the iterated fine limits of $\left[\frac{w}{u_1 \cdots u_n}\right]$ exist up to a set of measure zero for the product of the canonical measures corresponding to $u_k$ and are independent of the order of iteration. This class contains all positive multiply harmonic functions on the product of $\Omega_k$'s. For a holomorphic function $f$ in the Nevanlinna class of the polydisc $U^n$, it is shown that the $n$th iterated fine limits exist and equal almost everywhere on $T^n$ the $n$th iterated nontangential limits of $f$, for any fixed order of iteration. It is then deduced that, with the exception of a set of measure zero on $T^n$, the absolute values of the different iterated limits of $f$ are equal. It is also shown that the $n$th iterated nontangential limits are equal almost everywhere on $T^n$ for any $f$ in $N_1(U^n)$.

Let $f$ be a holomorphic function belonging to the Nevanlinna class $N(U^n)$ of the polydisc $U^n$, i.e.

$$\int \log^+|f(r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n})| \, d\theta_1 \cdots d\theta_n$$

is bounded for $0 \leq r_j < 1$. A well-known result states that, for such a function $f$, the iterated nontangential limits exist except for a set $E$ contained in $T^n$ ($T = \{z: |z| = 1\}$) such that the repeated integral of the characteristic function of $E$ relative to $d\theta_1 \cdots d\theta_n$ (in the order reverse to that of taking limits) is zero. Further, if $f \in N_{n-1}(U^n)$, i.e.

$$\int \log^+|f| \log^+|f| \cdots \log^+|f|^{n-1} \, d\theta_1 \cdots d\theta_n$$

is bounded for $0 \leq r_j < 1$, then the iterated nontangential limits of $f$ are equal almost everywhere on $T^n$. It has been an open problem to decide whether the iterated limits of any $f \in N(U^n)$, for different orders of iteration, are equal up to a set of measure zero on $T^n$ [4]. In this paper, we shall show that the answer is in the affirmative to one part of the problem; viz., for any $f \in N(U^n)$, the absolute values of the iterated nontangential limits are equal on $T^n$ except for a set of measure zero (Theorem 11). We shall also show that the iterated nontangential limits are

Presented to the Society, October 30, 1971; received by the editors March 8, 1971.

AMS (MOS) subject classifications (1969). Primary 3115, 3120, 3150, 3225; Secondary 6062, 2820.

Key words and phrases. Polydisc, Nevanlinna class, holomorphic function, nontangential limit, fine limit, minimal boundary, multiply superharmonic functions, Radon measures.

(1) Part of this work was done when the author was a fellow at the Kingston Branch of the Summer Research Institute 1970, of the Canadian Mathematical Congress.
identical (almost everywhere) for all functions in a class wider than $N_{n-1}(U^n)$ (Theorem 12).

In §5, we prove that the iterated fine limits of $f \in N(U^n)$ exist as a measurable function on $T^n$ and equal almost everywhere the iterated nontangential limits of $f$, for each order of iteration. We then deduce the equality of the absolute values of the various iterated fine limits as a particular case of the results of §§4 and 3, which are of independent interest in the theory of harmonic functions.

We consider harmonic spaces $\Omega_1, \ldots, \Omega_n$ of Brelot and functions $w$ that are $n$-superharmonic on $\Omega_1 \times \cdots \times \Omega_n$. Let $u_k > 0$ be a harmonic function on $\Omega_k$ with 'canonical measure' $\mu_k$ charging some 'minimal boundary' $\Delta^k_1$, for $k = 1$ to $n$. In §3, we show that for a $n$-harmonic function $w > 0$ on the product space, $w/u_1u_2\cdots u_n$ has iterated fine limits for $(\mu_1 \times \cdots \times \mu_n)$-almost every element of $\Delta^1_1 \times \Delta^2_1 \times \cdots \times \Delta^n_1$ and these limit functions are equal almost everywhere to the Radon-Nikodym derivative of the 'canonical measure' of $w$ relative to $\mu_1 \times \cdots \times \mu_n$. This proves the uniqueness of the iterated limits. In §4, we consider the case of $n$-superharmonic functions. If a $n$-superharmonic function $v > 0$ has the property that the iterated fine limits exist in such a way that the $k$th iterated limit ($1 \leq k \leq n - 1$) of $(v/u_1u_2\cdots u_k)$ is $(n-k)$-superharmonic on the product $\Omega_{k+1} \times \cdots \times \Omega_n$, then the iterated fine limits are equal $(\mu_1 \times \mu_2 \times \cdots \times \mu_n)$-almost everywhere (once again by showing that these are identical to the Radon-Nikodym derivative of a fixed measure relative to $\mu_1 \times \cdots \times \mu_n$). In §5, we apply these results to $(-\log |f|)$ to derive the results corresponding to $f$.

In §2, we have a number of results on measurability of certain functions. The fine filters in general do not have countable bases; we have developed methods to prove the measurability of functions of the form $g(b, y)$, where, for every $b$ in a minimal boundary, $g$ is the fine limit (or lim sup, etc.) of $f(x, y)$ as $x$ tends to $b$, even when $f$ is with values in certain function spaces. These results are fundamental in our proofs.

1. Preliminaries. Let $\Omega$ be a locally compact (noncompact) locally connected, Hausdorff topological space with a countable base for its open sets. We shall say that $\Omega$ is a (Brelot) harmonic space if there is a system of harmonic functions defined on the open subsets of $\Omega$ satisfying the Axioms 1, 2 and 3 of Brelot [1, Part II]. All the harmonic spaces are assumed to have potentials $> 0$ existing on them (to avoid the trivial case). In the case of $U$, the unit disc, we shall consider the classical harmonic functions (viz., satisfying the Laplace equation). An open connected set $\delta$ of $\Omega$ is called regular if, for every continuous real valued function $f$ on $\partial\delta$, there is a continuous extension of $f$ into $\delta$ as a harmonic function $H_f$ such that, for $f \geq 0$, $H_f \geq 0$. The harmonic measure at $x \in \delta$ for such a domain is the Radon measure on $\partial\delta$ defined by $f \mapsto H_f(x)$.
Let $\Omega_1, \ldots, \Omega_n$ be $n$ harmonic spaces and $\delta$ an open subset of the product space. A real valued function (resp. extended real valued and $\neq +\infty$) is said to be $n$-harmonic (resp. $n$-superharmonic) on $\delta$ if, for every fixed value of any $(n-1)$-variables, $f$ is harmonic (resp. superharmonic or $+\infty$) and $f$ is continuous (resp. lower semicontinuous) on $\delta$.

Let $H^+(\Omega)$ (resp. $nH^+(\Omega_1 \times \cdots \times \Omega_n)$) be the cone of positive ($>0$) harmonic (resp. $n$-harmonic) functions on $\Omega$ (resp. the product). These cones are locally compact, separable and metrisable for the topology of local uniform convergence and consequently have compact bases. Let $\Lambda_1$ be the extreme (or minimal) elements belonging to such a compact base. Then, corresponding to each $u \in H^+(\Omega)$, there is a unique Radon measure $\mu$ on this compact base, charging $\Lambda_1$, such that, for every $x \in \Omega$, $u(x) = \int h(x) \mu(db)$ [2]. This measure, corresponding to $u$, is referred to as the canonical measure corresponding to $u$ on $\Lambda_1$. It is possible to choose a corresponding compact base for $nH^+(\Omega_1 \times \cdots \times \Omega_n)$ such that the extreme elements of this base are precisely of the form $b^1 \cdots b^n$, $b^i \in \Lambda^1_i$. And to any $w \in nH^+$, corresponds a unique 'canonical' Radon measure $\nu_w$, carried by $\Lambda^1_1 \times \cdots \times \Lambda^1_n$ such that $w = \int b^1 \cdots b^n \nu_w(db^1 \cdots db^n)$ [8].

For any nonnegative valued function $f$ defined on a set $E \subseteq \Omega$, the reduced function $R[f,E]$ is by definition $\inf \{v: v \geq f$ on $E$ and $v$ superharmonic or $+\infty$ on $\Omega\}$. (We use this rather than the standard $R_f$ for the sake of notational simplicity.)

Let $b \in \Lambda_1$. Then, $\mathcal{F}_b = \{E: R[f,E], b] \neq b\}$ is the fine filter on $\Omega$ corresponding to $b$. For any function $f$, the limit (lim sup, etc.) of the image filter $/(\mathcal{F}_b)$ is called the fine limit (fine lim sup, etc.) of $f(x)$ as $x$ tends to $b$. For any $u \in H^+(\Omega)$, with the canonical measure $\mu$ on $\Lambda_1$, and any superharmonic function $v \geq 0$, the fine limit of $v(x)/u(x)$ as $x$ tends to $b$ exists for $\mu$-almost every $b$ in $\Lambda_1$ ([6], [7]).

Let $X$ be a Hausdorff topological space. A Radon measure on $X$ is by definition a measure $\mu \geq 0$ defined on the Borel $\sigma$-algebra of $X$ such that (1) $\mu$ is locally finite and (2) for any Borel set $B \subseteq X$, $\mu(B)$ is the supremum of $\mu(K)$ for compact sets $K \subseteq B$ [11].

A function $f: X \rightarrow Y$, $Y$ any Hausdorff space, is said to be $\mu$-Borel measurable, $\mu$-Borel measurable or $\mu$-Lusin measurable according as $f^{-1}(B)$ is Borel in $X$ for every Borel set $B \subseteq Y$, $f^{-1}(B)$ is $\mu$-measurable for every Borel set $B \subseteq Y$ or, for any compact set $K \subseteq X$ and any $\epsilon > 0$, $\exists$ a compact subset $C$ such that $\mu(C) > \mu(K) - \epsilon$ and $f$ restricted to $C$ is continuous. In general, $\mu$-Lusin measurability of $f$ implies the $\mu$-Borel measurability, and the converse is true if $Y$ is a separable metrisable space.

In the sequel every topological space is assumed to be Hausdorff and with a countable basis for neighborhoods at each point.
2. Some measurability theorems.

Lemma 1. Let $\Omega$ be a harmonic space and $Y$ a topological space. Let $\psi$ be an extended real valued function $> -\infty$ on $\Omega \times Y$ such that it is lower semicontinuous in each variable, for every fixed value of the other. Then, for every $x$ in $\Omega$, and every $a \in \mathbb{R}$, the function $(b, y) \mapsto R[V_a(y), b](x)$ is lower semicontinuous on $\Delta_1 \times Y$, where $V_a(y) = \{\xi \in \Omega: \psi(\xi, y) > a\}$.

Proof. Let $(b_n, y_n) \in \Delta_1 \times Y$ converge to $(b_0, y_0) \in \Delta_1 \times Y$. The set $V_a(y_n)$ is open in $\Omega$ for every $n$ and $v_n = R[V_a(y_n), b_n]$ is superharmonic $> 0$ on $\Omega$ and equals $b_n$ on $V_a(y_n)$. Let $v = \liminf v_n$ as $n$ tends to $\infty$. Then, $v$, the lower semicontinuous regularisation of $v$, is $> 0$ and superharmonic on $\Omega$.

Let $x \in V_a(y_0)$. Since $y \mapsto \psi(x, y)$ is lower semicontinuous on $Y$ and $\psi(x, y_0) > a$, there is an integer $N(x)$ such that, for all $n \geq N(x)$, we have $\psi(x, y_n) > a$. Hence, $x \in V_a(y_n)$ for all $n \geq N(x)$. It follows that $V_a(y_0) \subset \bigcup_n \bigcap_n V_a(y_n)$.

Now, since $b_n$ converges to $b_0$ (in particular pointwise), for all $x \in V_a(y_0)$,

$$v(x) = \liminf v_n(x) = \liminf b_n(x) = b_0(x).$$

Now, $V_a(y_0)$ is open and $b_0$ is continuous and we deduce that $w = b_0$ on $V_a(y_0)$. Hence, $x \in V_a(y_0)$ for all $n \geq N(x)$. It follows that

$$\lim\inf R[V_a(y_n), b_n](x) \geq v(x) \geq w(x) \geq R[V_a(y_0), b_0](x),$$

for every $x$ in $\Omega$, proving the lemma.

Theorem 1. Let $\Omega$, $Y$ and $\psi$ be as in the previous lemma. Then, the function $\psi_1$, defined by $\psi_1(b, y) = \limsup \psi(x, y)$ as $x \to b$, is Borel measurable on $\Delta_1 \times Y$.

Proof. It is enough to show that, for every real number $b$, the set $A(b) = \{(b, y): \psi_1(b, y) \leq b\}$ is Borel in $\Delta_1 \times Y$. Let, for every $n$,

$$E_n = \{(b, y): R[V_{b+1/n}(y), b] \neq b\}.$$

Then, we assert that $A(b) = \bigcap_{n=1}^{\infty} E_n$. For, suppose $(b, y) \in A(b)$. Then since $\psi_1(b, y) \leq b < b + 1/n$, there is a set $F$ belonging to $\mathcal{F}_b$ such that, for every $x \in F$, $\psi(x, y) \leq b + 1/n$. Hence, $F \cap V_{b+1/n} = \emptyset$ and $b \neq R[V_{b+1/n}(y), b]$, i.e. $(b, y) \in E_n$. Conversely, if $(b, y)$ belongs to $E_n$, for every $n$, then fine lim sup $\psi(x, y) \leq b + 1/n$ as $x$ tends to $b$. We deduce that $(b, y)$ belongs to $A(b)$.

To complete the proof of the theorem we shall show that, for every real number $a$, the set $E_a = \{(b, y): R[V_a(y), b] \neq b\}$ is a Borel set in $\Delta_1 \times Y$. For this, let $(\delta_m)$ be a sequence of regular domains of $\Omega$ forming a covering of the space and, for every $m$, $x_m$ an arbitrary element in $\delta_m$. From Lemma 1 and Fatou's lemma, we deduce that, for every $m$,
(b, y) \mapsto \int R[V_a(y), b](\xi)\rho_{x_m}^m(d\xi)

is a lower semicontinuous function, where $\rho_{x_m}^m$ is the harmonic measure on $\partial \Omega_m$ corresponding to $x_m$. Hence, it follows that

$$E_{a, m} = \{(b, y) : \int R[V_a(y), b](\xi)\rho_{x_m}^m(d\xi) < b(x_m)\}$$

is a Borel subset of $\Delta_1 \times Y$, in fact, a countable union of closed sets. But $E_a = \bigcup_{n=1}^{\infty} E_{a, m}$ [6, Theorem II.1]. It follows that $E_a$ is a Borel subset of $\Delta_1 \times Y$.

The proof is complete.

The following two corollaries are immediate consequences.

Corollary 1. Let $v$ be a $n$-superharmonic function on $\Omega_1 \times \cdots \times \Omega_n$. Then, the function $(b, x^2, x^3, \ldots, x^n) \mapsto \text{fine lim sup } v(x, x^2, \ldots, x^n)$ as $x$ tends to $b$ is a Borel measurable function on $\Delta_1 \times \Omega_2 \times \cdots \times \Omega_n$.

Corollary 2. Let $u$ be a $n$-harmonic function on $\Omega_1 \times \cdots \times \Omega_n$. Then the set of points $(b, x^2, \ldots, x^n)$ of $\Delta_1 \times \Omega_2 \times \cdots \times \Omega_n$ for which the fine limit of $u(x, x^2, \ldots, x^n)$ exists as $x$ tends to $b$ is a Borel subset of this space.

Theorem 2. Let $\Omega$ be a harmonic space, $Y$ any topological space and $X$ a polish space (separable, complete metrisable). Let $f : \Omega \times Y \mapsto X$ be a separately continuous mapping. Then the set $E$ defined by

$$E = \{(b, y) : \text{fine lim } f(x, y) \text{ exists in } X \text{ as } x \to b\}$$

is a Borel subset of $\Delta_1 \times Y$.

Proof. Let $d$ be a complete metric compatible with the topology of $X$. Let $X' = (x_n)_1^{\infty}$ be a countable dense subset of $X$. Let $(\delta_m)_1^{\infty}$ be a countable family of regular domains of $\Omega$ forming a base for the open sets and $\xi_m \in \delta_m$ an arbitrary element, for every $m$. Let, for any integer $k > 0$,

$$V_{n, k}(y) = \{\xi \in \Omega : d[x_n, f(\xi, y)] > 1/k\}.$$

We assert that

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{l=1}^{\infty} \left\{(b, y) : \int R[V_{n, k}(y), b](\xi)\rho_{\xi_m}^m(d\xi) < b(x_m)\right\}.$$

For, suppose $(b, y)$ belongs to $E$. Let $\xi$ be the fine limit in $X$ of $f(\xi, y)$ as $\xi$ tends to $b$. Given $k \in N$, let $l$ be a positive integer such that $2k < l$. Let $x_N \in X'$ such that $d[\xi, x_N] < 1/l$. Let $F \subset \Omega$ be such that $F \in \mathcal{F}_k$ and $d[\xi, f(\xi, y)] < 1/l$, for every $\xi \in F$. Consider $V_{N, k}(y)$. Since, for every $\xi \in F$,

$$d[x_N, f(\xi, y)] \leq d[x_N, \xi] + d[\xi, f(\xi, y)] < 1/k$$

we conclude that $F \subset \Omega - V_{N, k}(y)$, i.e. $\mathcal{C} F \supset V_{N, k}(y)$. Hence, $R[V_{N, k}(y), b] \neq b$ on $\Omega$ and we conclude that there exists an $M \in N$ such that
Thus \( E \) is contained in the set on the right. Conversely, suppose \((b, y)\) belongs to the right side. To prove that \((b, y) \in E\), it is enough to show that the image of \( \mathcal{F}_b \) under the mapping \((\xi, y) \mapsto f(\xi, y)\) (\( y \) is fixed) is the base of a \( d\)-Cauchy filter. Given \( \varepsilon > 0 \), choose \( k \in \mathbb{N} \) such that \( 2/k < \varepsilon \). For this \( k \), we can find integers \( N \) and \( M \) such that

\[
\int R[V_N, k(y), b] \rho^M_{\xi, M}(d\xi) < \varepsilon \rho^M_{\xi, M}(b(\xi_M)).
\]

This, in particular, implies that \( F = \mathcal{C}(V_N, k(y)) \) belongs to \( \mathcal{F}_b \). Further, for every \( \xi, \eta \in \Omega \), since \( d[x_N, /((\xi, y)) < 1/k \) and \( d[x_N, f((\eta, y)) < 1/k \) we get by triangle inequality that \( d[f(\xi, y), f((\eta, y)) < 2/k < \varepsilon \). This proves the assertion.

Now, to complete the proof of the theorem, it is enough to show that, for every \( \xi \in \Omega \), \( k \) and \( n \), the mapping \((b, y) \mapsto R[V_N, k(y), b](\xi) \) is lower semicontinuous on \( \Delta_1 \times Y \). But this follows from Lemma 1, since \((\xi, y) \mapsto d[x_N, f((\xi, y))] \) is also separately continuous for every \( n \). The theorem is proved.

**Theorem 3.** Let \( \lambda \) be a Radon measure on a topological space \( Y \) and \( \Omega \) a harmonic space. Let \( f \) be a nonnegative real valued (resp. complex valued) function defined on \( \Omega \times Y \) (resp. \( \Omega \times Y \)) such that (1) for every \( x \in \Omega \) (resp. \( U^n \)), \( y \mapsto f(x, y) \) is \( \lambda \)-measurable and (2) for every \( y \in Y \), \( x \mapsto f(x, y) \) is harmonic on \( \Omega \) (resp. holomorphic on \( U^n \)). Then, the mapping \( y \mapsto f(\cdot, y) \) of \( Y \) into \( H^+(\Omega) \cup \{0\} \) (resp. \( \mathcal{H}(U^n) \)) is \( \lambda \)-Lusin measurable, where the function spaces are provided with the topology \( r \) of uniform convergence on compact subsets of the respective sets.

**Proof.** The spaces \( H^+(\Omega) \cup \{0\} \) and \( \mathcal{H}(U^n) \) provided with the \( r \)-topology are polish. Hence, the class of Borel sets for any weaker Hausdorff topology on these sets is the same as the class of \( r \)-Borel sets. Consider one such topology \( \tau_1 \) defined as follows. Let \( Z \subset \Omega \) (resp. \( U^n \)) be a countable dense subset. The topology \( \tau_1 \) is the topology of simple convergence on \( Z \). We observe that there is a countable base for the \( \tau_1 \)-open sets: the finite intersections of sets of the form \( \{u: u(z) \in V\} \) where \( z \in Z \) and \( V \) belongs to a countable base for open sets of \( \mathbb{R} \) (resp. \( \mathbb{C} \)).

Consider the mapping \( Y \mapsto H^+(\Omega) \cup \{0\} \) (resp. \( \mathcal{H}(U^n) \)). For every \( \tau_1 \)-open set of the form \( \{u: u(z) \in V\} \) (as above), we have

\[
\{y \in Y: f(z, y) \in V\} = \{y \in Y: f(\cdot, y) \in \{u(z) \in V\}\}.
\]

Hence, the inverse image of such \( \tau_1 \)-open sets and hence every \( \tau_1 \)-open set is \( \mu \)-measurable in \( Y \). Hence this mapping is \( \mu \)-Borel measurable when \( H^+(\Omega) \cup \{0\} \) (resp. \( \mathcal{H}(U^n) \)) is provided with \( \tau_1 \). Now, by the earlier remark, we see that the same is true with \( r \)-topology. However, the spaces involved on the right are polish; hence we get the required \( \mu \)-Lusin measurability. The proof is complete.

**Remark.** In particular, if, in the above theorem, \( \lambda \) is a finite Radon measure, then, given \( \varepsilon > 0 \), we can find a compact set \( K \subset Y \) satisfying (1) \( \lambda(Y - K) < \varepsilon \) and (2) \( y \mapsto f(\cdot, y) \) is continuous on \( K \) into \( H^+(\Omega) \cup \{0\} \) (resp. \( \mathcal{H}(U^n) \)).
Theorem 4. Let $\Omega$ be a harmonic space, $u > 0$ a harmonic function on $\Omega$ with the canonical measure $\mu$ on $\Delta_1$; $\lambda$ a finite Radon measure on a topological space $X$. Let $w \geq 0$ be a function defined on $\Omega \times X$ such that, for every $x \in X$, $w(\cdot, x)$ is harmonic and, for every $y \in \Omega$, $x \mapsto w(y, x)$ is $\lambda$-measurable. Then the following two functions $f$ and $g$ on $\Delta_1 \times X$ are $(\mu \times \lambda)$-measurable, where $f(b, x) =$ fine lim inf $[w(y, x)/u(y)]$ and $g(b, x) =$ fine lim sup $[w(y, x)/u(y)]$ as $y$ tends to $b$.

Proof. Let $\epsilon > 0$. From the above theorem, we have a compact set $K$ satisfying $\lambda(X - K) < \epsilon$ and $x \mapsto w(y, x)$ of $K \mapsto H^d(\Omega)$ is continuous. This certainly implies the separate continuity of $(y, x) \mapsto [w(y, x)/u(y)]$. We deduce from Theorem 1 that the functions $f$ and $g$ restricted to $\Delta_1 \times K$ are Borel measurable. Let, for $\epsilon = 1/n$, $n = 1, 2, \ldots$, $K_n$ be the corresponding compact subset of $X$. We conclude that $f$ and $g$ are Borel measurable on $\Delta_1 \times F$ where $F = \bigcup_{n=1}^{\infty} K_n$. Now $\Delta_1 \times (X - F)$ is of $\mu \times \lambda$ measure zero since it is a measurable (in fact, Borel) rectangle, $\mu$ is a totally finite measure and $\lambda(X - F) = 0$. This completes the proof.

Corollary. Let $w$, $u$ and $\lambda$ be as in the theorem. Then, except for a set of $\mu \times \lambda$ measure zero on $\Delta_1 \times X$, for every $(b, x)$, the fine limit of $[w(y, x)/u(y)]$ as $y$ tends to $b$ exists and is finite. Further, this fine limit is $H^d$-measurable.

Proof. Let $E = \{(b, x): f(b, x) < g(b, x)\} \cup \{(b, x): g(b, x) = \infty\}$. Clearly, $E$ is $(\mu \times \lambda)$-measurable. Further, for every $x \in X$, for almost $\mu$-every $b \in \Delta_1$, $f(b, x) = g(b, x) < +\infty [\text{7, Theorem 8}]$. It follows, by Fubini's theorem, that $(\mu \times \lambda)(E) = 0$. This proves the Corollary.

3. Limits of $n$-harmonic functions. We shall consider, for the sake of simplicity of notation, 3-harmonic functions. In the general case, the proofs are absolutely similar. We shall fix harmonic functions $u_k > 0$ defined on harmonic spaces $\Omega_k$ for $k = 1, 2, 3$. Let $\mu_k$ be the canonical measure corresponding to $u_k$, on some convenient compact base of positive harmonic functions on $\Omega_k$, charging the extreme elements $\Delta_k^1$ in that base.

Lemma 2. Let $w > 0$ be a 3-superharmonic function on $\Omega_1 \times \Omega_2 \times \Omega_3$ such that, for every fixed $x^1 \in \Omega_1$, $w$ is 2-harmonic on $\Omega_2 \times \Omega_3$. Then, except for a set $E$ of $\mu_2$ measure zero on $\Delta_1^1$, for every $(x^2, x^3)$ in $\Omega_2 \times \Omega_3$, the fine limit of $[w(x^1, x^2, x^3)/u_1(x)]$ exists (and is finite) as $x^1$ tends to $b$ in $(\Delta_1^1 - E)$. Further, this fine limit is 2-harmonic on $\Omega_2 \times \Omega_3$ and Borel measurable in all the three variables together (on $\Delta_1^1 \times \Omega_2 \times \Omega_3$).

Proof. Let $X^2$ and $X^3$ be respectively countable dense subsets of $\Omega_2$ and $\Omega_3$. Consider the positive superharmonic function $w(\cdot, x^2, x^3)$ on $\Omega_1$, for every $x^2 \in X^2$ and $x^3 \in X^3$. We can find a set $E_{n, m}$ of $\mu_1$ measure zero on $\Delta_1^1$ such that, for every $b \notin E_{n, m}$, the fine limit of $[w(x^1, x^2, x^3)/u_1(x)]$ exists (and is finite) as $x^1$ tends to $b$ [7, Theorem 8].
Let 
\[ E = \bigcup \{ E_{n,m} : x^2_n \in X^2, x^3_m \in X^3 \}. \]

Let \( b \in \Delta^1_1 - E \). Let \( \mathcal{G}_b \) be the filter, on the cone of positive 2-harmonic functions on \( \Omega_2 \times \Omega_3 \), generated by the image of \( \mathcal{F}_b \) under the mapping \( x \mapsto [w(x, x^2_1, x^3_1)/u_1(x)] \). Since \( [w(x, x^2_1, x^3_1)/u_1(x)] \) converges to a finite limit following \( \mathcal{F}_b \), we deduce that there is a set \( A \in \mathcal{F}_b \) such that, for \( x \in A \), 0 < \( [w(x, x^2_1, x^3_1)/u_1(x)] \) for some positive number \( M \). But, the subset of positive 2-harmonic functions bounded above at some point of \( \Omega_2 \times \Omega_3 \) is relatively compact and hence there is a filter finer that \( \mathcal{G}_b \) converging to some \( v \in (2 - H)(\Omega_2 \times \Omega_3) \), in the topology of uniform convergence on the compact subsets. However, any two adherent points \( v_1 \) and \( v_2 \) of \( \mathcal{G}_b \) coincide on \( X^2 \times X^3 \) which is a dense subset of \( \Omega_2 \times \Omega_3 \); hence \( v_1 = v_2 \). We deduce, therefore, that \( \mathcal{G}_b \) convergent to an element \( w^b \), i.e. we get that \( w^b(x^2, x^3) = \text{fine lim}[w(x, x^2, x^3)/u_1(x)] \) as \( x \) tends to \( b \), for every \( (x^2, x^3) \), and the convergence is uniform for compact subsets of \( \Omega_2 \times \Omega_3 \). The Borel measurability of the limit function is an immediate consequence of Theorem 1. The lemma is proved.

**Theorem 5.** Let \( w > 0 \) be a 3-harmonic function on \( \Omega_1 \times \Omega_2 \times \Omega_3 \). Then, except for a set of \( \mu_1 \times \mu_2 \times \mu_3 \) measure zero, for every \( (b^1, b^2, b^3) \) in \( \Delta^1_1 \times \Delta^2_1 \times \Delta^3_1 \), the iterated fine limit of \( (w/u_1 u_2 u_3) \) exists and is finite; and any iterated limit function, defined \( (\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere, is \( (\mu_1 \times \mu_2 \times \mu_3) \)-measurable.

**Proof.** It is enough to prove the results for some order of iteration. Without loss of generality, let us consider the natural order. From Lemma 2, we deduce the existence of a set \( E \subset \Delta^1_1 \) of \( \mu_1 \) measure zero such that, for every \( b \notin E \), \( w^b(x_1^2, x_3^3) = \text{fine lim}[w(x, x_1^2, x_3^3)/u_1(x)] \) as \( x \to b \) and 2-harmonic and Borel measurable in all the three variables. Now, from the corollary to Theorem 4, we deduce that, for every \( x_3^3 \in X^3 \), \( \Delta^1_1 - E \times \Delta^2_1 \) of \( \mu_1 \times \mu_2 \) measure zero such that, for \( (b^1, b^2) \) not in \( F'_k \), the fine limit of \( [w^{b_1}(x, x_3^3)/u_1(x)] \) as \( x \to b^2 \) exists and is finite (may be zero). Further, this limit function is \( (\mu_1 \times \mu_2) \)-measurable. We observe that \( E \times \Delta^2_1 \) is of \( \mu_1 \times \mu_2 \) measure zero since \( \mu_2 \) is a totally finite measure. Let \( F_k = F'_k \cup E \times \Delta^2_1 \) for every \( x_3^3 \in X^3 \) and \( F = \bigcup F_k \). Then, \( F \) is a \( (\mu_1 \times \mu_2) \)-measurable set of measure zero; also, for every \( x_3^3 \in X^3 \), finite limit of \( [w^{b_1}(x, x_3^3)/u_2(x)] \) as \( x \) tends to \( b^2 \) exists and is finite for every \( (b^1, b^2) \in F \). Now, using the fact that \( X^3 \) is dense in \( \Omega_3 \), we deduce (exactly as in the proof of Lemma 2) that, for every \( (b^1, b^2) \in F \), the fine limit of \( [w^{b_1}(x, x_3^3)/u_2(x)] \) as \( x \) tends to \( b^2 \) exists locally uniformly for \( x_3 \in \Omega_3 \). (We use the fact that the set of positive harmonic functions on \( \Omega_2 \) bounded at one point is relatively compact for the local uniform convergence topology on \( \Omega_2 \) [2].) This limit \( w^{b_1, b^2} \) is harmonic.
on \( \Omega_3 \) for every \((b^1, b^2) \in F\) and \((\mu_1 \times \mu_2)\)-measurable for every \(x_3 \in \Omega_3\). Hence, by the corollary to Theorem 4, we get that the fine limit of \([w^{b_1,b_2}(x)/u_3(x)]\) exists and is finite except for \((b^1, b^2, b^3) \in G\), where \(G \subset (\Delta_1^1 \times \Delta_1^2 - F) \times \Delta_3^2\) and is of \(\mu_1 \times \mu_2 \times \mu_3\) measure zero. Further the above limit is a \((\mu_1 \times \mu_2 \times \mu_3)\)-measurable function on \([(\Delta_1^1 \times \Delta_1^2 - F) \times \Delta_3^2] - G\). Once again, \(F \times \Delta_3^2\) is of \(\mu_1 \times \mu_2 \times \mu_3\) measure zero, since \(\mu_3\) is totally finite. Hence the iterated fine limit of \((w/u_1u_2u_3)\) exists as a \((\mu_1 \times \mu_2 \times \mu_3)\)-measurable function, except for all \((b^1, b^2, b^3)\) in \([G \cup (F \times \Delta_3^2)]\). The proof is complete.

**Lemma 3.** Let \(w > 0\) be a 3-harmonic function on \(\Omega_1 \times \Omega_2 \times \Omega_3\) such that \(w \leq M u_1u_2u_3\) for some positive \(M\). Then, except for a set of \(\mu_1 \times \mu_2 \times \mu_3\) measure zero, the various (3!) iterated fine limits of \((w/u_1u_2u_3)\) are identical and the common value is a Radon-Nikodym derivative of the canonical measure of \(w\) relative to \(\mu_1 \times \mu_2 \times \mu_3\).

**Proof.** Since the canonical measure \(\nu_w\) of \(w\) is \(\leq M(\mu_1 \times \mu_2 \times \mu_3)\), \(\nu_w\) is absolutely continuous relative to \(\mu_1 \times \mu_2 \times \mu_3\). Now, using the notation of Theorem 5, \((w^{b_1,b_2}/u_3)\) is bounded on \(\Omega_3\) and, for \((\mu_1 \times \mu_2)\)-almost every element of \(\Delta_1^1 \times \Delta_1^2\), as \(x\) tends to \(b^3\), fine \(\lim [w^{b_1,b_2}(x)/u_3(x)] = \hat{w}(b_1, b_2, b_3)\) \(\mu_3\)-almost everywhere. Hence, we get

\[
\int_{\Delta_1^3} \hat{w}(b_1, b_2, b_3)b_3(x)\mu_3(db_3) = w^{b_1,b_2}(x) \quad \text{on} \quad \Omega_3 \quad [7, \text{Theorem 7}].
\]

Again, \((b_1, b_2)\rightarrow w^{b_1,b_2}(x_3)\) is measurable and, for any \(x_3 \in \Omega_3\) and \(b_1 \in \Delta_1^1 - E\) (\(E\) as in Theorem 5), \([w^{b_1}(x, x_3)/u_2(x)]\) is bounded on \(\Omega_2\) and its limit following \(\mathbb{F}_{b_2}\) equals \(w^{b_1,b_2}(x^3)\) for \(\mu_2\)-almost every \(b_2 \in \Delta_2^1\). Hence, for \(x \in \Omega_2\),

\[
w^{b_1}(x, x_3) = \int_{\Delta_2^1} w^{b_1,b_2}(x_3)b_2(x)\mu_2(db_2).
\]

Finally, going one more step backward, we deduce that, for all \((x^1, x^2, x^3)\),

\[
w(x^1, x^2, x_3) = \int_{\Delta_1^1} \mu_1(db_1)\int_{\Delta_2^1} \mu_2(db_2)\int_{\Delta_3^1} \hat{w}(b_1, b_2, b_3)b_1(x^1)b_2(x^2)b_3(x^3)\mu_3(db_3),
\]

\[
= \iiint \hat{w}(b_1, b_2, b_3)b_1(x_1)b_2(x_2)b_3(x_3)(\mu_1 \times \mu_2 \times \mu_3)(db_1db_2db_3)
\]

[Fubini's theorem].

It follows from the uniqueness of integral representation that \(d\nu_w = \hat{w}(b_1, b_2, b_3)d(\mu_1 \times \mu_2 \times \mu_3)\) [8, Theorem 7]. We conclude that \(\hat{w}\) is a Radon-Nikodym derivative of \(\nu_w\) relative to \(\mu_1 \times \mu_2 \times \mu_3\). The Radon-Nikodym derivative of \(\nu_w\) relative to \(\mu_1 \times \mu_2 \times \mu_3\) is unique up to a set of \(\mu_1 \times \mu_2 \times \mu_3\) measure zero; the proof is complete.

**Theorem 6.** Let \(w\) and \(u_1, u_2, u_3, \text{etc.}\) be as in Theorem 5. Then, the iterated fine limits of \((w/u_1u_2u_3)\) are equal \((\mu_1 \times \mu_2 \times \mu_3)\)-almost everywhere.
Proof. Let \( \nu \) be the canonical measure on \( \Delta_1^1 \times \Delta_2^2 \times \Delta_3^3 \) corresponding to \( w \).
Let \( \nu = \nu_1 + \nu_2 \) where \( \nu_1 \ll \mu_1 \times \mu_2 \times \mu_3 \) and \( \nu_2 \) is singular relative to \( \mu_1 \times \mu_2 \times \mu_3 \). Let \( f \) be a Radon-Nikodym derivative of \( \nu_1 \) relative to \( \mu_1 \times \mu_2 \times \mu_3 \). We note that \( f \geq 0 \) and is finite \((\mu_1 \times \mu_2 \times \mu_3)\)-almost everywhere. Let \( \Sigma_f \) be the 3-harmonic function on \( \Omega_1 \times \Omega_2 \times \Omega_3 \) with the canonical measure \( \nu_1 \) on \( \Delta_1^1 \times \Delta_2^2 \times \Delta_3^3 \).
Consider the iterated limit \( f_1 \) (resp. \( w_1 \)) of \( \Sigma_f \) (resp. \( w \)) for the same order of iteration say, for the natural order. Since \( w \geq \Sigma_f \), we get that \( w_1 \geq f_1 \) everywhere. However, if \( f_n = \inf (f, n) \), for any positive integer \( n \geq 2 \), then \( f_n \geq 0 \) and is bounded, and we get, from Lemma 3, that the iterated fine limits of \( \Sigma_{f_n} \) equal \( f \)
\((\mu_1 \times \mu_2 \times \mu_3)\)-almost everywhere. (\( \Sigma_{f_n} \) is the 3-harmonic function with the canonical measure \( \nu_1(\mu_1 \times \mu_2 \times \mu_3) \).) It follows that \( f_1 \geq f_n \) almost everywhere. This is true for every \( n \) and we deduce that \( f_1 \geq f(\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere. On the other hand, (using the notation of Theorem 5) \( w^{b_1, b_2} \geq 0 \) harmonic and the fine limit \( w_{b_1, b_2}(x)=w_1(b_1, b_2, b^3) \mu_3\)-almost everywhere. Hence [7, Theorem 8], \( w^{b_1, b_2}(x)=\int w_1(b_1, b_2, b^3) \mu_3(x) \mu_3 \) for all \( x \in \Omega_3 \). All this is true for almost every \((b_1, b_2)\) in \( \Delta_1^1 \times \Delta_2^2 \).
Proceeding backwards and repeating the argument, we deduce that \( w \geq \Sigma w_1 \). In view of the uniqueness of integral representation, we conclude that
\((\nu_1, \mu_2 \times \mu_3) \geq w_1(\mu_1 \times \mu_2 \times \mu_3) \) (the canonical measure of \( \Sigma w \)),
i.e., \( f(\mu_1 \times \mu_2 \times \mu_3) + \nu \geq w_1(\mu_1 \times \mu_2 \times \mu_3) \). But \( w_1 \geq f_1 \geq f(\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere and, hence, the positive measure \((w_1 - f)(\mu_1 \times \mu_2 \times \mu_3) \leq \nu_2 \), a measure singular relative to \( \mu_1 \times \mu_2 \times \mu_3 \). It follows that \( w_1 = f(\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere. This completes the proof of the theorem.

We deduce immediately

**Corollary 1.** If \( \phi \) is the difference of two positive 3-harmonic functions on \( \Omega_1 \times \Omega_2 \times \Omega_3 \), then the iterated fine limits of \( (\phi/u_1 u_2 u_3) \) exist and are equal (and finite) \((\mu_1 \times \mu_2 \times \mu_3)\)-almost everywhere.

In the course of the proof of the theorem it is shown that the iterated fine limit of \( w \) is a Radon-Nikodym derivative of the absolutely continuous part of \( \nu \) relative to \( \mu_1 \times \mu_2 \times \mu_3 \). In particular, we have

**Corollary 2.** Let \( w \geq 0 \), a 3-harmonic function, having canonical measure \( \nu \) absolutely continuous relative to \( \mu_1 \times \mu_2 \times \mu_3 \). Let \( f_w \) be the iterated fine limit of \( w/u_1 u_2 u_3 \) (for some order of iteration). Then, \( dv = f_w d(\mu_1 \times \mu_2 \times \mu_3) \); equivalently, for every \((x, y, z)\),
\[ w(x, y, z) = \prod f_w(b_1, b_2, b^3) \mu_3(\mu_1 \times \mu_2 \times \mu_3) (db_1, db^2, db^3). \]

Another consequence is the following result.

**Theorem 7.** Let \( w, u_1, u_2 \) and \( u_3 \) be as in Theorem 5. Then the following are
equivalent.

(a) The greatest 3-harmonic minorant of \( \inf(w, u_1 u_2 u_3) \) is 0.
(b) The canonical measure \( \nu \) of \( w \) is singular relative to \( \mu_1 \times \mu_2 \times \mu_3 \).
(c) The iterated fine limit of \( (w/u_1 u_2 u_3) \) equals zero \( (\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere.

Proof. We shall show that (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) \( \Rightarrow \) (a).

Suppose (a) is true. Let \( \lambda \) be the Radon-Nikodym derivative of the absolutely continuous part of \( \nu \) relative to \( \mu_1 \times \mu_2 \times \mu_3 \). Note that \( \lambda \geq 0 \). The 3-harmonic functions with canonical measures \( f_n/n \) \( (\mu_1 \times \mu_2 \times \mu_3) \), where \( n \in \mathbb{N} \), \( f_n = \inf(f, n) \) minorise both \( w \) and \( u_1 u_2 u_3 \). Hence \( f_n/n = 0 \), i.e. \( f_n = 0 \) \( (\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere. This implies (b). The fact that (b) \( \Rightarrow \) (c) is an immediate consequence of Theorem 6.

Now, assume that (c) holds. Let \( \lambda \) be the canonical measure on \( \Delta_1 \times \Delta_2 \times \Delta_3 \) corresponding to the greatest 3-harmonic minorant \( w_1 \) of \( \inf(w, u_1 u_2 u_3) \) and \( \lambda \) a Radon-Nikodym derivative of \( \lambda \) relative to \( \mu_1 \times \mu_2 \times \mu_3 \).

From Lemma 3 and (c) we deduce that 0 = iterated fine limits of \( w/u_1 u_2 u_3 \) \( \Rightarrow \) those of \( w_1/u_1 u_2 u_3 \) \( \geq \) 0 \( (\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere. Hence \( \lambda = 0 \) almost everywhere and \( w_1 \equiv 0 \). The proof is complete.

4. The case of \( n \)-superharmonic functions. We consider 3-superharmonic functions like in the earlier section. Let \( \Omega_k \), \( u_k \), \( \Delta_k \) and \( \mu_k \) be as in \$3$, for \( k = 1, 2, 3 \). First we need the following result (Lemma 4). We state and prove it for functions of two variables, but obviously the proof is valid for functions of several variables.

Definition 1. Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be respectively a base of regular domains of \( \Omega_1 \) and \( \Omega_2 \). An extended real valued function \( f \) on \( \Omega_1 \times \Omega_2 \) is said to be an MS-\((\mathcal{B}_1, \mathcal{B}_2)\) function if (1) \( f \) is locally lower bounded at every point and (2) for every \( \delta \in \mathcal{B}_1 \), \( \omega \in \mathcal{B}_2 \) and all \( (x, y) \in \delta \times \omega \),

\[
f(x, y) \geq \int_{\delta}^{\star} f(\xi, \eta) \left( \rho_\delta^x \times \rho_\omega^y \right) (d\xi \, d\eta).
\]

Lemma 4. Let \( \nu \neq +\infty \) be an MS-\((\mathcal{B}_1, \mathcal{B}_2)\) function on \( \Omega_1 \times \Omega_2 \). Then, \( w \) the lower semicontinuous regularisation of \( \nu \) is a 2-superharmonic function on \( \Omega_1 \times \Omega_2 \).

Further, for every \( (x, y) \in \Omega_1 \times \Omega_2 \),

\[
w(x, y) = \sup \left\{ \int_{\delta}^{\star} \nu(\xi, \eta) \left( \rho_\delta^x \times \rho_\omega^y \right) (d\xi \, d\eta) : x \in \delta, \, y \in \omega; \, \delta \in \mathcal{B}_1, \, \omega \in \mathcal{B}_2 \right\}.
\]

In particular, a lower semicontinuous MS-\((\mathcal{B}_1, \mathcal{B}_2)\) function is 2-superharmonic.

Proof. Let us first consider a lower semicontinuous MS-\((\mathcal{B}_1, \mathcal{B}_2)\) function \( g \). Fix \( x \in \Omega_1 \) and let \( \omega \in \mathcal{B}_2 \) contain \( y \in \Omega_2 \). Let \( (\delta_n)_{n=1}^{\infty} \) be a sequence of regular domains in \( \mathcal{B}_1 \) such that \( \delta_{n+1} \subset \delta_n \) and \( \bigcap_{n=1}^{\infty} \delta_n = \{ x \} \). It is easily seen that the harmonic measures \( \rho_\delta^x \) (on \( \partial_\delta \subset \partial_\mathcal{B}_1 \)) converge weakly to the Dirac measure \( \delta_x \) at
for every $n$. In deducing the above inequality the use of Fubini's theorem is justified since $w$ is locally lower bounded, measurable and $\rho_x^n$ and $\rho_y^\omega$ are totally finite measures. But, $\xi \mapsto \int w(\xi, \eta) \rho_y^\omega(d\eta)$ is lower semicontinuous; hence,

$$w(x, y) \geq \liminf \int \rho_x^n(d\xi) \left[ \int w(\xi, \eta) \rho_y^\omega(d\eta) \right]$$

$$\geq \int \epsilon_x(d\xi) \left[ \int w(\xi, \eta) \rho_y^\omega(d\eta) \right]$$

$$= \int w(x, \eta) \rho_y^\omega(d\eta).$$

Now, from the local criterion for the superharmonicity of lower semicontinuous functions of one variable, we deduce that $y \mapsto w(x, y)$ is superharmonic (or identically $+\infty$) on $\Omega_2$. The symmetry of the argument in the variables involved shows that $w$ is a 2-superharmonic function.

Let $v$ be as in the hypothesis of the lemma. Now, for any $(x, y) \in \delta \times \omega$ with $\delta \in B_1$, $\omega \in B_2$, $(a, b) \mapsto \int \int u(\xi, \eta)(\rho_a^\delta \times \rho_b^\omega)(d\xi d\eta)$ is a 2-harmonic function on $\delta \times \omega$, hence, in particular, continuous [8, Theorem 2]. Hence $w$, the lower semicontinuous regularisation of $v$, satisfies

$$w(x, y) \geq \int \int u(\xi, \eta)(\rho_a^\delta \times \rho_b^\omega)(d\xi d\eta).$$

But, given $\alpha < w(x, y)$, there is a neighbourhood $V$ of $(x, y)$ such that $v > \alpha$ on $V$. We deduce that

$$\sup \{ \int \int u(\xi, \eta)(\rho_a^\delta \times \rho_b^\omega)(d\xi d\eta) : \delta \in B_1, \omega \in B_2, \delta \times \omega \subseteq V \}$$

$$\geq \lim \alpha \int \int \rho_a^\delta(d\xi) \rho_b^\omega(d\eta) = \alpha.$$

Thus, $w$ is the stated supremum. Further, for any $\delta \in B_1$, $\omega \in B_2$, $(x, y) \in \delta \times \omega$,$$
 w(x, y) \geq \liminf v(a, b) \quad \text{[as } (a, b) \text{ tends to } (x, y)\text{]}$$

$$\geq \liminf \int \int u(\xi, \eta)(\rho_a^\delta \times \rho_b^\omega)(d\xi d\eta) \quad \text{[}(a, b) \in \delta \times \omega \rightarrow (x, y)\text{]}$$

$$\geq \liminf \int \int w(\xi, \eta)(\rho_a^\delta(d\xi) \rho_b^\omega(d\eta))$$

$$= \lim \int \int w(\xi, \eta) \rho_a^\delta(d\xi) \rho_b^\omega(d\eta) = \int \int w(\xi, \eta) \rho_a^\delta(d\xi) \rho_b^\omega(d\eta).$$

Thus $w$ is also an $MS<$-$(B_1, B_2)$ function and we deduce from the first part that $w$ is 2-superharmonic. The proof is complete.

Remark. It is immediate from the above lemma that the lower semicontinuous regularisation of the lower envelope of an arbitrary family of locally lower bounded $n$-superharmonic functions is again $n$-superharmonic.

We recall that $\{0\} \cup S^+(\Omega)$ (the set of positive superharmonic functions on $\Omega$)
is locally compact, separable and metrisable in the $T$-topology of Mme. Hervé [9, Theorems 21.1, 21.2]. Further, for any regular domain $\omega$, $x \in \omega$ and $\alpha > 0$,

$$\Lambda_\alpha = \left\{ w \in S^+ \cup \{0\}; \int w(\xi) \rho^\omega_x(d\xi) \leq \alpha \right\}$$

is compact [9, Theorem 21.2]. The Cartan-Brelot topology (we shall write C-B-topology) on $S^+ \cup \{0\}$ is the coarsest topology which makes the functions $w \mapsto \int w(\xi) \rho^\delta_x(d\xi)$ continuous, where $\delta$ belongs to a countable base $B$ of regular domains of $\Omega$ and $x$ belongs to a countable dense subset $A$ of $\Omega$ (naturally $x \in \delta$ for every $\delta$). We now have

**Proposition 1.** The $\sigma$-algebras of $T$-Borel sets and $\sigma$-algebras of C-B-Borel sets are identical.

**Proof.** The $T$-topology is coarser than the C-B-topology [9, Proposition 24.6]. Hence, every $T$-Borel set is necessarily a C-B-Borel set. However, the mapping $w \mapsto \int w(\xi) \rho^\delta_x(d\xi)$ is $T$-lower semicontinuous, for every $\delta$ and all $x \in \delta$ [9, Proposition 24.1]. Hence, every C-B-open set belonging to the following subbase is a $T$-Borel set: $\forall x \in A, \delta \in B, V$ any open interval with rational endpoints, $\{w: \int w \rho^\delta_x \leq V\}$. This subbase is clearly countable and we deduce, by standard measure theoretic arguments, that every C-B-Borel set is also a $T$-Borel set. The proof is complete.

**Theorem 8.** Let $\Omega$ be a harmonic space, $u > 0$ a harmonic function on $\Omega$ and $\mu$ the corresponding canonical measure carried by $\Lambda_1$. Let $\lambda$ be a finite positive Radon measure on a topological space $Y$. Let $v \geq 0$ be an extended real valued function on $\Omega \times Y$, measurable with respect to the product $\sigma$-algebra of the Borel sets of $\Omega$ and $\lambda$-measurable sets of $Y$. Further, suppose that, for every $y \in Y$, $x \mapsto v(x, y)$ is superharmonic on $\Omega$. Then, the function

$$(b, y) \mapsto 1 \text{fine lim sup} \left\{ v(x, y)/u(x) \right\},$$

as $x$ tends to $b$, is $(\mu \times \lambda)$-measurable on $\Lambda_1 \times Y$.

**Proof.** Let $\mathcal{B}$ be a countable base for the open sets of $\Omega$ consisting of regular domains and $A$ a countable dense subset of $\Omega$. Let us assume to start with that $3 a z \in \Omega$ and an $\omega \in \mathcal{B}$, $z \in \omega$, such that $\int v(x, y) \rho^\omega_z(dx) \leq 1$ for every $y \in Y$.

Let $\Lambda = \{w: \int w \rho^\omega_z \leq 1\}$.

Now, consider the mapping $\phi: Y \mapsto \Lambda$ defined by $(\phi(y))(x) = v(x, y)$. For every $\delta \in \mathcal{B}$, $x \in A$, $y \mapsto \int (\phi(y))(\xi) \rho^\delta_x(d\xi)$ is $\lambda$-measurable (Fubini). From this, it is easy to deduce by standard measure theoretic arguments, that the mapping $\phi: Y \mapsto \Lambda$ is $\lambda$-Borel measurable, when $\Lambda$ is provided with the C-B-topology. This certainly implies that $\phi: Y \mapsto \Lambda$ is $\lambda$-Borel measurable when $\Lambda$ is provided with the $T$-topology. But $\Lambda$ with the $T$-topology is compact and metrisable, hence polish. Hence, $\phi$ is $\lambda$-Lusin measurable. Given $\epsilon > 0$, let $C$ be compact $\subset Y$ such that

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λ(Y) < λ(C) + ε/2. By the Lusin measurability we can find a compact subset K of C satisfying λ(K) > λ(C) − ε/2 and ϕ restricted to K is continuous. Now, the mapping w → w(x) is T-lower semicontinuous from S^+ → R, for every x ∈ Ω. For every x ∈ Ω, therefore, y → ϕ(y)(x) = v(x, y) is lower semicontinuous on Y. Now we conclude, from Theorem 1, that f is Borel measurable on Δ_1 × C.

Choose a compact set K_n ⊂ Y as above, for the choice of ε = 1/n, for n = 1, 2, 3, · · ·. Let B = ∪ K_n. Then B is Borel on Y and λ(Y − B) = 0. The function f restricted to each Δ_1 × K_n is Borel measurable and, hence, f restricted to Δ_1 × B is Borel measurable. Further, (μ × λ)(Δ_1 × Y − B) = 0 since μ is a finite measure. We conclude that f^−1(E) is (μ × λ)-measurable, for every Borel set E ⊂ R. This completes the proof for the v's satisfying the condition stated at the beginning.

Now, for any general w satisfying the hypothesis of the theorem, let v(x, y) = [w(x, y)/α(y)] where α(y) = sup [1, ∫ w(x, y)ρ_z^ω(dx)]. Then v satisfies the additional condition that ∫ v(x, y)ρ_z^ω(dx) ≤ 1. Now, for (b, y) ∈ Δ_1 × Y, the fine lim sup [w(x, y)/u(x)] as x tends to b equals the fine lim sup [w(x, y)/u(x)] divided by α(y). And, y → α(y) is clearly λ-measurable. The required measurability is now easily seen to be true, completing the proof.

Now, we proceed to consider the iterated fine limit operations on a 3-superharmonic function v > 0 defined on Ω_1 × Ω_2 × Ω_3. As an immediate consequence of Theorem 1, we deduce that (b, y, z), going to fine lim sup of [v(x, y, z)/u_(1)(x)] as x tends to b, is a Borel measurable function. The next lemma shows that except for a set of μ_1 measure zero this is an MS-(B_2, B_3) function.

Lemma 5. Let B_2 and B_3 be countable bases of regular domains of the spaces Ω_2 and Ω_3 respectively. Then, a set E ⊂ Δ_1 of μ_1 measure zero can be chosen such that, for every b ∈ E, v(b, ·, ·) is an MS-(B_2, B_3) function.

Proof. Consider the function w(x, y, z) = ∫ ∫ v(x, η, ζ)ρ_y^ω(dη)ρ_z^ω(dζ), corresponding to δ ∈ B_2 and ω ∈ B_3. Clearly, for every x ∈ Ω, w(x, ·, ·) is a 2-harmonic function on δ × ω. Any easy application of Fubini's theorem and Fatou's lemma lets us conclude that, for every fixed (y, z) ∈ δ × ω, w(·, y, z) is superharmonic. Hence w is a 3-superharmonic function on Ω × δ × ω [8, Theorem 2]; also v ≥ w. Let us consider the three harmonic functions on Ω_1 × δ × ω. w_1 is defined by

\[ w_1(x, y, z) = \int \mu_1(db) \int b(x) v^b(\eta, \zeta) \rho_y^ω(d\eta) \rho_z^ω(d\zeta) \]

where v^b(η, ζ) = fine lim sup [v(x, η, ζ)/u_1(x)] as x tends to b. For every fixed η and ζ in δ and ω, v(x, η, ζ) ≥ ∫ b(x)v^b(η, ζ)μ_1(db) [7, Theorems 6, 7]. Hence, by Fubini's theorem, we get that w ≥ w_1 on Ω_1 × δ × ω. Now, from Lemma 2, we deduce the existence of a set G (depending on δ and ω) of μ_1 measure zero such that, for every b ∈ Δ_1 − G, as x tends to b the fine limit of [w_1(x, y, z)/u_1(x)]
\(w^b(y, z)\) exists for all \((y, z) \in \delta \times \omega\) and is 2-harmonic on \(\delta \times \omega\); also \((b, y, z) \mapsto w^b(y, z)\) is a Borel measurable function. In view of the regularity of the solution of Dirichlet problems by the Perron method [7, Theorem 7] we may assume that this set \(G\) was chosen such that, for all \(b \in \Delta_1^1 - G\) and every \((y, z)\) belonging to a countable dense subset of \(\delta \times \omega\), the equality \(w^b(y, z) = \int \int v^b(\eta, \zeta) \rho_y^5(d\eta) \rho_z^\omega(d\zeta)\) holds to be good. Now, \(w^b(y, z)\) and \(\int \int v^b(\eta, \zeta) \rho_y^5(d\eta) \rho_z^\omega(d\zeta)\) are two 2-harmonic functions on \(\delta \times \omega\) and they coincide on a dense set, hence the two functions are identical. Hence, for all \(b \in \Delta_1^1 - G, \ y \in \delta, \ z \in \omega, \)
\[
v^b(y, z) = \text{fine lim sup } \left[ \frac{v(x, y, z)}{u_1(x)} \right]
\geq \text{fine lim } \left[ \frac{w_1(x, y, z)}{u_1(x)} \right]
= \int \int v^b(\eta, \zeta) \rho_y^5(d\eta) \rho_z^\omega(d\zeta).
\]
Let \(F_{n,m}\) be the exceptional set corresponding to \(\delta_n \in \mathcal{B}_2, \omega_m \in \mathcal{B}_3\) and \(F\), the union of these countably many sets. Then \(F\) is of \(\mu_1\) measure zero and, for all \(b \in \Delta_1^1 - F\) and any \(\delta_n \in \mathcal{B}_2, \omega_m \in \mathcal{B}_3, \)
\[
v^b(y, z) \geq \int \int v^b(\eta, \zeta) \rho_y^5(d\eta) \rho_z^\omega(d\zeta), \quad \forall y \in \delta_n, \ z \in \omega_m.
\]
Since \(v'\) is, in addition, \(\geq 0\), we conclude that \(v'\) is an MS-(\(\mathcal{B}_2, \mathcal{B}_3\)) function, completing the proof.

**Lemma 6.** The lower semicontinuous regularisation (in \(y, z\)) \(w^b(y, z)\) of \(v^b(y, z)\), for every \(b \in \Delta_1^1 - F\), is 2-superharmonic; further \((b, y, z) \mapsto w^b(y, z)\) is a measurable function on \((\Delta_1^1 - F) \times \Omega_2 \times \Omega_3\).

**Proof.** The first part is a consequence of Lemma 4 and Lemma 5. To prove the measurability, consider \(\delta \in \mathcal{B}_2\) and \(\omega \in \mathcal{B}_3\). The function \((b, y, z) \mapsto \int \int v^b(\eta, \zeta) \rho_y^5(d\eta) \rho_z^\omega(d\zeta)\) is measurable in the three variables together (since it is \(\mu_1\)-measurable on \(\Delta_1^1 - F\), for every fixed \(y, z\) and 2-harmonic (\(\geq 0\)) on \(\delta \times \omega\) for every fixed \(b\)) (Theorem 3). Again, from Lemma 4, we get, for every \(a \in \mathbb{R}, \)
\[
\{(b, y, z) : w^b(y, z) > a\} = \bigcup \left\{ (b, y, z) \in (\Delta_1^1 - F) \times \delta \times \omega : \int \int v^b(\eta, \zeta) \rho_y^5(d\eta) \rho_z^\omega(d\zeta) > a \right\}
\]
where the union on the right is taken over all \(\delta \in \mathcal{B}_2\) and \(\omega \in \mathcal{B}_3\). This clearly proves the measurability of \(w^b(y, z)\). The lemma is proved.

**Lemma 7.** There exists a set \(G \subset \Delta_1^1 \times \Delta_1^2\) of \(\mu_1 \times \mu_2\) measure zero such that, for every \((b^1, b^2) \notin G, \ \theta(b^1, b^2, z) = \text{fine lim sup } [v^b(\eta, z)/u_2(\eta)]\) as \(\eta\) tends to \(b^2\) is a \(\mathcal{B}_3\)-nearly superharmonic function on \(\Omega_3\). Let, for every \((b^1, b^2) \in (\Delta_1^1 \times \Delta_1^2 - G), \ \phi(b^1, b^2, z)\) be the lower semicontinuous regularisation of \(\theta(b^1, b^2, z)\) in the \(z\) variable. Then, \(\phi\) is a measurable function.
Proof. The proof of measurability is exactly similar to the proof of Lemma 6, and we omit it. To prove the first part, we observe that $\theta$ is a $(\mu_1 \times \mu_2 \times \lambda)$-measurable function on $(\Delta_1 \times F) \times \Delta_2 \times \Omega_3$ (Theorem 8), for any finite Radon measure $\lambda$ on $\Omega_3$. Consider $\delta \in B_3$. The function $\psi$ defined by

$$\psi(x, y, z) = \int \int b_1(x) b_2(y) \theta(b_1, b_2, z) \mu_1(db_1) \mu_2(db_2) \rho_\delta(dz)$$

is a 3-harmonic function on $\Omega_1 \times \Omega_2 \times \delta$, and $\psi \leq \nu$ (Theorem 6). Imitating the first part of the proof of Theorem 5, we can find a set $G^\delta$ of $\mu_1 \times \mu_2$ measure zero on $\Delta_1 \times \Delta_2$ such that the second iterated fine limit (as $x \to b_1$, $y \to b_2$) of $[\psi(x, y, z)/u_1(x)u_2(y)]$ is harmonic on $\delta$ and further this function coincides with $\int \theta(b_1, b_2, z) \rho_\delta(dz)$, for every $(b_1, b_2) \in G^\delta$. (The latter assertion is deduced using Corollary 2, Theorem 6 and a countable dense subset of $\delta$, as in the proof of Lemma 5.) Now, let $G = F \times \Delta_1 \cup \delta G^\delta$ where $\delta \in B_3$ and $F$ is as in Lemma 5. Then $G$ is of $\mu_1 \times \mu_2$ measure zero and, for every $(b_1, b_2) \in (\Delta_1 \times \Delta_2 - G)$,

$$\theta(b_1, b_2, z) \geq \text{fine lim}_{y \to b_1} \left[ \text{fine lim}_{x \to b_1} (\psi(x, y, z)/u_1(x)u_2(y)) \right]$$

This together with the fact that $\theta$ is lower bounded, in fact $\geq 0$, we conclude that $\theta$ is an $S(B_3)$ function. The lemma is proved.

Theorem 9. Let $f_v(b_1, b_2, b_3) = \text{fine lim sup}_{z \to \delta_3} [\phi(b_1, b_2, b_3)/u_3(z)]$ as $z$ tends to $b_3$, for $(b_1, b_2, b_3) \in (\Delta_1 \times \Delta_2 - G) \times \Delta_3$ ($G$ and $\phi$ as in the previous lemma). Then, $f_v$ is $(\mu_1 \times \mu_2 \times \mu_3)$-measurable and is the Radon-Nikodym derivative of (the absolutely continuous part of the) canonical measure $\nu$ on $\Delta_1 \times \Delta_2 \times \Delta_3$ of the greatest 3-harmonic minorant of $\nu$ relative to $\mu_1 \times \mu_2 \times \mu_3$.

Proof. The $(\mu_1 \times \mu_2 \times \mu_3)$-measurability of $f_v$ is an immediate consequence of Theorem 8. Now, $\nu(x, y, z) \geq \int \phi(b_1, b_2, z)b_1(x)b_2(y) \mu_1(db_1)\mu_2(db_2)$ and, since $\phi \geq 0$ is superharmonic in the $z$-variable, $\phi(b_1, b_2, z) \geq \int f_v(b_1, b_2, b_3)\mu_3(db_3)$ [7, Theorems 6, 7]. Hence, if $u$ is the greatest 3-harmonic minorant of $\nu$, then $u \geq \Sigma(f_v, \nu)$ where

$$\Sigma(f_v, \nu) = \int f_v(b_1, b_2, b_3) b_1b_2b_3(\mu_1 \times \mu_2 \times \mu_3)(db_1 db_2 db_3).$$

By the uniqueness of the integral representation of 3-harmonic functions $\geq 0$, we get that the canonical measure $\nu$ of $u$ (on $\Delta_1 \times \Delta_2 \times \Delta_3$) majorises the canonical measure of $\Sigma(f_v, \nu)$. But, since $f_v$ is measurable $\geq 0$ and the measure $f_v d(\mu_1 \times \mu_2 \times \mu_3)$ is on $\Delta_1 \times \Delta_2 \times \Delta_3$, it is clear that this is indeed the canonical measure of $\Sigma(f_v, \nu)$. Hence, $\nu \geq f_v d(\mu_1 \times \mu_2 \times \mu_3)$. It follows that the Radon-Nikodym derivative $g$ of (the absolutely continuous part of) $\nu$ relative to $\mu_1 \times \mu_2 \times \mu_3$ is $\geq f_v$ at $(\mu_1 \times \mu_2 \times \mu_3)$-almost every element of $\Delta_1 \times \Delta_2 \times \Delta_3$. 

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On the other hand, if we do the same operation on \( u \) (as on \( v \)) and arrive at the function \( f_u \), then it is clear that \( f_u \) is nothing but the iterated fine limit of \( u \); and by Theorem 6, Corollary 2, \( f_u = g \) almost everywhere. However, \( v \geq u \) and hence \( f_v \geq f_u \) almost everywhere, i.e. \( g = f_v (\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere. The proof is complete.

We deduce immediately

**Corollary 1.** If \( v \) is \( 3 \)-superharmonic > 0 and has 0 as the greatest \( 3 \)-harmonic majorant, then \( f_v = 0 (\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere.

**Corollary 2.** If \( v \) is \( 3 \)-superharmonic > 0 and moreover if the fine limit of \( v/u \), etc. exists as a lower semicontinuous function in the rest of the variable(s) (as usual a set of measure zero excepted), then the iterated fine limit of \( v/u_1 u_2 u_3 \) exists almost everywhere and is the Radon-Nikodym derivative as in the above theorem. If, further, the same holds for different orders of iteration then the various iterated limits are equal \( (\mu_1 \times \mu_2 \times \mu_3) \)-almost everywhere.

**Remark.** If \( v \) is a \( 3 \)-superharmonic function with a \( 3 \)-harmonic minorant \( w \), \( w \leq 0 \) everywhere, then similar results are true for \( v \). We note that \( f_v = f_{(v - w)^+}/f \); here we have equality (and not '\( \leq \)') since at every stage the limits exist for \( w/u_1 \), etc.

5. Functions on the polydisc. Let us now consider a polydisc \( U^n, n \geq 2 \), and functions belonging to the Nevanlinna class \( N(U^n) \). For any \( f \in N(U^n) \), the function \( \log^+|f| \) is a positive \( n \)-subharmonic function and has a positive \( n \)-harmonic majorant; see, for instance, [10]. We shall apply our earlier results on the existence and equality of iterated fine limits to the function \( -\log|f| \) and deduce the corresponding conclusions for \( f \). To do this, first of all we observe that \( T \), the unit circle, is precisely the Martin boundary of \( U \) (for the system of harmonic functions satisfying the Laplace equation); and the Martin boundary consists entirely of minimal functions; viz., \( T \) is also the minimal boundary (belonging to the compact base of positive harmonic functions taking the value 1 at the origin). Let \( l \) denote the linear normalized (i.e., \( l(T) = 1 \)) Lebesgue measure on \( T \). Then, \( l \) is the canonical measure corresponding to the harmonic function 1. For the sequel, \( l_k \) will stand for the completion of the product measure \( l \times l \times \cdots \times l \) \( k \)-times on \( T \).

Before proceeding to give our proof, we recall below three results which we use repeatedly.

**Theorem Z** [12, Theorem 5.16, Chapter II]. Let \( f \in N(U^n), n \geq 2 \). Then, there exists a set \( E \subset T \) such that \( l(T - E) = 0 \) and, for every \( t \in E \), as \( z \) tends to \( t \) nontangentially, \( |z| < 1 \), the limit of \( f(z, z^2, \ldots, z^n) \) exists for every \( (z^2, \ldots, z^n) \in U^{n-1} \); and the convergence is uniform on every compact subset of \( U^{n-1} \). Further the function \( f(t, \cdot) \) belongs to \( N(U^{n-1}) \) for every \( t \in E \).
Theorem B and D [3, Theorem 9]. Let $X$ be a metric space and $f: U \mapsto X$.
Let, for every $t$ belonging to an $l$-measurable set $E$, the limit of $f(z)$ exists in $X$ as $z, |z| < 1$, tends to $t$ nontangentially. Then, for $l$-almost every $t \in E$, the fine limit of $f(z)$ as $z$ tends to $t$ exists and equals the nontangential limit at this point.

Theorem C and C [5, p. 62]. Let $f$ be a holomorphic function on $U$. Let $t \in T$ be such that the fine limit of $f(z)$ as $z$ tends to $t$ equals $a$. Then, as $z$ tends to $t$ nontangentially, $f(z)$ tends to the limit $a$.

We shall now consider the iterated limits of $f$. As before, we consider the natural order of iteration.

Theorem 10. Let $f \in N(U^n)$, $n \geq 2$. Then, for every integer $k$, $1 \leq k \leq n - 1$, the $k$th iterated fine limit of $f$ exists for every $t \in E_k \subset T_k$ and every $(z_k, z_{k+2}, \ldots, z_n) \in U^{n-k}$ such that

1. the complement of $E_k$ is of $l_k$ measure zero,
2. for every fixed $(z_{k+1}, \ldots, z_n) \in U^{n-k}$, the iterated limit is a $l_{n-k}$ measurable function,
3. for every $t \in E_k$, the limit is holomorphic belonging to the Nevanlinna class of $U^{n-k}$ and
4. the (above) iterated fine limit equals the $k$th iterated nontangential limits (independent of $(z_{k+1}, \ldots, z_n)$ in $U^{n-k}$).

Proof. Let us first prove the result for the case $k = 1$. We consider the mapping $\phi: U \mapsto \mathcal{H}(U^{n-1})$, the space of holomorphic functions on $U^{n-1}$, defined by $z \mapsto f(z, \cdot)$. The space $\mathcal{H}(U^{n-1})$ is separable and complete metrisable. The Theorem Z states precisely that $\phi(z)$ tends to a limit in this space (in fact in $N(U^{n-1})$), as $z$ approaches nontangentially almost every point of $T$. Hence, from Theorem B and D, we deduce the existence of the fine limit for almost every $t \in T$, satisfying (3) and (4). Now, (2) is an immediate consequence of Theorem 1.

Let us now assume that the result is true for all integers $1, 2, \ldots, k, k < n - 1$; we shall show the validity for $k + 1$. From Theorem 3, for $\epsilon = 1/m$, $m = 1, 2, 3, \ldots$, we can find a compact set $C_m \subset E_k$ such that (i) $l_k(E_k - C_m) < 1/m$ and, restricted to $C_m$, the mapping $t \mapsto f_t$ of $C_m \mapsto \mathcal{H}(U^{n-k})$ is continuous, where $f_t(z_{k+1}, \ldots, z_n)$ is the $k$th iterated fine limit (or nontangential limit) of $f$ at $t$.

This, in particular, implies the separate continuity of the mapping $(r, z_{k+1}) \mapsto f_{r}(z_{k+1}, \ldots, z_n)$ of $C_m \times U \mapsto \mathcal{H}(U^{n-k-1})$. (The spaces of holomorphic functions are provided with the topology of uniform convergence on the compact subsets of the corresponding sets.) If $E_{k,m}$ is the set of all $(r, t)$ belonging to $C_m \times T$ such that fine limit of $f_t(z_{k+1}, \cdot)$ exists in $\mathcal{H}(U^{n-k-1})$ as $z_{k+1}$ tends to $t$, then, from Theorem 2, we deduce that $E_{k,m}$ is $l_{k}$-measurable. However, for every $r \in C_m$, since $f_r \in N(U^{n-k})$, we deduce from Theorem Z and Theorem B and D, as in the case
k = 1, that the section through \( \tau \) of \( E_{k,m} \) is of full \( l \)-measure. Now, from Fubini's theorem, we get that
\[
\int_{E_{k+1}} l_k(E_{k,m}) = \int_{E_{k+1}} l_k(C_m) > \int_{E_{k+1}} l_k(E_k) - 1/m = 1 - 1/m.
\]
Now, let
\[
E_{k+1} = \bigcup E_{k,m}.
\]
Then, \( l_{k+1}(E_{k+1}) = 1 = l_{k+1}(T^{k+1}) \) and for every \( (\tau, t) \in E_{k+1} \), the fine limit of \( f(z^{k+1}, \cdot) \) exists in \( N(U^{n-k-1}) \) as \( z^{k+1} \) tends to \( t \). Once again (2) is an immediate consequence of Theorem 1. Also, property (4) of the above limit is a consequence of Theorem C and C.

The proof is complete.

**Theorem 11.** Let \( f \in N(U^n) \). The \( n \)th iterated fine limit of \( f \) exists for \( l_n \)-almost every element of \( T^n \), and this iterated limit is an \( l_n \)-measurable function. Further, the \( n \)th iterated nontangential limit of \( f \) equals the iterated fine limit \( l_n \)-almost everywhere.

**Proof.** Let \( E_{n-1} \) be the set furnished by the above theorem and \( f_T(z^n) \) the \((n-1)\)th iterated fine (= nontangential) limit of \( f \) at \( r \in E_{n-1} \). As before, by Theorem 3, we can find a compact set \( K_m \subset E_{n-1} \), for every \( m = 2, 3, \ldots \), such that
\[
l_{m-1}(K_m) > l_{m-1}(E_{n-1}) - 1/m = 1 - 1/m,
\]
and \( r \mapsto f_T(\cdot) \) is a continuous map from \( K_m \mapsto \overline{\mathbb{H}}(U) \). Once again, by Theorem 2, the sets \( F_m \),
\[
F_m = \{(r, \tau) \in E_{n-1} \times T : \text{fine lim } f_T(z^n) \text{ exists as } z^n \to t\}
\]
are \( l_n \)-measurable. But, for every \( r \in E_{n-1} \), since \( f_T(\cdot) \in N(U) \), for almost every \( t \in T \), both the fine and the nontangential limits of \( f_T(\cdot) \) exist and are equal as \( z \) tends to \( t \). In particular
\[
l_n(F_m) = l_{n-1}(K_m) \cdot l(T) > 1 - 1/m \quad \text{(Fubini's theorem)}.
\]
Let \( F \) be the union of \( F_m \), for \( m = 2 \) to \( \infty \). Then \( l_n(F) = 1 \). Further, for every \( r \in F \) the \( n \)th iterated fine and nontangential limits of \( f \) exist and are equal. Also we deduce from Theorem 4 that the iterated limit is \( l_n \)-measurable. The theorem is proved.

**Theorem 12.** Let \( f \in N(U^n) \). Then, there is a set \( E \subset T^n \) of \( l_n \)-measure zero such that, if \( g_1 \) and \( g_2 \) are two iterated fine (or nontangential) limit functions, for two different orders of iteration, then \( |g_1| = |g_2| \) outside \( E \).

**Proof.** The function \( -\log|f| \) is \( n \)-superharmonic on \( U^n \) and let \( w \) be the greatest \( n \)-harmonic minorant of \( -\log|f| \). Since \( w \) majorises the negative (or zero) \( n \)-harmonic function which is the greatest minorant of \( -\log^+|f| \), we conclude that \( w \) is the difference of two positive \( n \)-harmonic functions. By Corollary 1 to Theorem 6, we know that the \( n \)th iterated fine limits of \( w \) exist and are all identically \( l_n \)-almost everywhere. Further, the \( k \)th iterated limit of \( w \) is \( (n-k) \)-harmonic, in particular, continuous on \( U^{n-k} \). We know that the \( k \)th iterated fine limit of \( -\log|f| \) exists for all \( k = 1 \) to \( n \) and it is easy to see that, for every \( k \) between \( 1 \) and \( n-1 \), the \( k \)th iterated limit is a \( (n-k) \)-superharmonic function on \( U^{n-k} \). Hence the same is true for \( (-\log|f| - w) \). However, \( (-\log|f| - w) \) is a positive
n-superharmonic function with greatest n-harmonic minorant zero and hence by Corollary 1 to Theorem 9, we deduce that the iterated fine limit of \((-\log |f| - w)\) is zero \(l_n\)-almost everywhere, whatever be the iteration order. We deduce that, with the exception of a set of \(l_n\) measure zero, the different iterated fine limits of \(\log |f|\) are identical. The proof is complete.

**Corollary.** For a \(f \in N(U^n)\), if the \(n\)th iterated fine or nontangential limit is zero on a set of positive \(l_n\) measure, then \(f = 0\).

**Proof.** From the above theorem we deduce that if \(f \neq 0\), then the iterated fine or nontangential limit of \(\log |f|\) is finite \(l_n\)-almost everywhere. The Corollary follows.

**Remark.** The equality of the iterated limits could be proved for functions \(f \in N(U^n)\) such that \(f\) has no zeroes in \(U^n\).

**Theorem 13.** Let \(f \in N_1(U^n)\); viz., \(\int \log^+ |f| (\log^+ \log^+ |f|) \, d\theta_1 \cdots d\theta_n\) is bounded independent of \(r_1, r_2, \ldots, r_n\) between 0 and 1. Then, the different iterated (fine or) nontangential limits of \(f\) are equal \(l_n\)-almost everywhere on \(T^n\).

**Proof.** For every function \(g \in N_1(U^2)\), the two iterated nontangential limits of \(g\) are equal \(l_2\)-almost everywhere [12, p. 328]. We shall prove the theorem by induction on \(n\).

Now, \(x \mapsto x \log^+ x\) is an increasing convex function on \([0, \infty)\) (in fact, it is a strongly convex function). Hence, \((\log^+ |f|) (\log^+ \log^+ |f|)\) is a \(n\)-subharmonic function on \(U^n\). Hence, \(\int \log^+ |f| (\log^+ \log^+ |f|) \, d\theta_1 \cdots d\theta_n\) is increasing in each \(r_j\). Since \(f\) is in \(N_1(U^n)\) there is an upper bound \(M > 0\) for these integrals. Suppose that except for a set \(E_k\) of \(l_k\) measure zero, some \(k\)th iterated nontangential limit of \(f\) exists for every \(r \in T^n - E_k\). Then, using the fact the radial limit exists at every stage and by repeated application of Fatou's lemma, we deduce that

\[
\int l_k(d\tau) \cdots \int \log^+ |f| \tau(r_{k+1} e^{i\theta_{k+1}}, \ldots, r_n e^{i\theta_n}) \log^+ \log^+ |f| \tau(r_{k+1} e^{i\theta_{k+1}}, \ldots, r_n e^{i\theta_n}) \, d\theta_{k+1} \cdots d\theta_n \leq M.
\]

Using the increasing nature of \(\int \cdots \int \log^+ |f| (\log^+ \log^+ |f|) \, d\theta_{k+1} \cdots d\theta_n\), we deduce easily that, for \(l_k\) almost every element \(\tau\) of \(T^n - E_k\), \(f \in N_1(U^n - k)\). (Observe that \(f\) may be assumed to belong to \(N(U^n - k)\) by Theorem 10.) This is true whatever be \(k < n\).

Now, assume that, for all \(k = 1, 2, \ldots, n\) \((n \geq 2)\), whatever be \(g \in N_1(U^k)\), the \(k\)th iterated fine limits of \(g\) are all equal \(l_k\)-almost everywhere. We shall show that, for any \(g \in N_1(U^{n+1})\), a similar result holds to be good. To prove this consider the iterated limits \(g_1\) of \(g\) for the natural order and \(g_2\) for an order \(\sigma\) which
1972] ITERATED FINE AND NONTANGENTIAL LIMITS 91

ends with say the \( k \)th variable \( k < n + 1 \). We can choose a countable dense sub-
set, say \( (z^k_m) \) contained in \( U \) (for the \( k \)th variable) such that, for every \( z^k_m \), the
\( n \)th iterated fine (and nontangential) limits of \( f \) exist except for a set of measure
zero on \( T^n \) and independent of the order of iteration. But then, by Theorem 10,
this limit could be so chosen that it is holomorphic in the \( k \)th variable on \( U \).

Hence, the limit could be chosen, independent of the order of iteration and \( z^k \in U \),
except for a set of \( l^n \) measure zero on \( T^n \). Hence we may assume that, in the
above \( \sigma \), the all but the last one limit is taken in the \( (n + 1) \)th variable, i.e. \( g_2 \)
is the iterated limit where all but the last two variables are in that order \( z^{n+1} \) and \( z^k \).

Similarly, we can suppose that \( g_1 \) is obtained by the iterated limit order \( \tau \), where
all but the last two variables (in that order) in which the limits are taken are pre-
cisely \( z^k \) and \( z^{n+1} \). Again, by an argument similar to the above and the induction
hypothesis, we get that with a set of measure zero excepted on \( T^{n-1} \), for every
t \( T^{n-1} \), the \( (n - 1) \)th iterated limit functions \( h^t(\tau, z^k, z^{n+1}) \) and \( h^t(\sigma, z^k, z^{n+1}) \)
following respectively \( \tau \) and \( \sigma \) satisfy:

\[
b^t(\tau, z^k, z^{n+1}) = h^t(\sigma, z^k, z^{n+1}) \quad \text{for every } (z^k, z^{n+1}) \in U^2
\]

and this common function belongs to \( N_1(U^2) \). Now, the two iterated limits of \( h^t \)
as \( z^{n+1} \) tends to the boundary nontangentially and then \( z^k \) tends to the bound-
ary nontangentially and vice versa) are equal \( l_2 \)-almost everywhere. Now, since
the \( (n + 1) \)th iterated nontangential limits are measurable, we deduce that \( g_1 =
g_2 l_{n+1} \)-almost everywhere. The proof is complete.

Remark. If, for every \( f \in N(U^2) \), the two iterated nontangential limits are equal
\( l_2 \)-almost everywhere, then the result is true in general for any function in \( N(U^n) \).
An exactly similar proof based on induction holds to be good. Probably the re-
sult is true for functions belonging to \( N(U^2) \).

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