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A NEW CLASS OF FUNCTIONS OF BOUNDED INDEX

BY

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ABSTRACT. Entire functions of strongly bounded index have been defined and it is shown that functions of genus zero and having all negative zeros satisfying a one sided growth condition belong to this class.

1. **Introduction.** Let $f(z)$ be an entire function and let

$$\Omega(z) = \Omega_s(z) = \max_{0 \le j \le s} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\} \quad (f^{(0)}(z) = f(z)). \tag{1.1}$$

Definition 1. An entire function $f(z)$ is said to be of *bounded index* (we shall also say of class B), if for some fixed s, $\{|f^{(n)}(z)|/n!\} \le \Omega_s(z)$ for all n and all z (see [3], [6]).

It is known that given any transcendental entire function f, there exists a transcendental entire function g of unbounded index such that [7]

$$\log M(r, f) \sim \log M(r, g) \qquad (r \to \infty).$$

In particular, given two numbers λ and ρ such that $0 \le \lambda \le \rho \le \infty$, there exists a function g of unbounded index such that g is of order ρ and of lower order λ. A result of this type cannot hold with g of bounded index since a function of bounded index must necessarily be of exponential type [8]. Furthermore, known examples of functions of bounded index and order one are all of regular growth, that is, the order of a function is equal to its lower order ([5], [9]). In this paper we show that there exist functions of bounded index, and of given order ρ and lower order λ provided $0 \le \lambda \le \rho \le 1$ (see also [10]). Our attempts to construct such functions have led us to the remark that a very simple subclass SB, of the class B, displays a particularly useful property. If $f \in SB$ and P is a polynomial then $fP \in SB$.

Definition 2. An entire function $f(z)$ is of *strongly bounded index* (we shall also say of class SB) if there exist quantities χ, $0 < \chi < 1$, r_0, and an integer $s \ge 0$ such that

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$$|f^{(n)}(z)|/n! \leq \chi \Omega_s(z), \tag{1.2}$$

for all $n \geq s+1$ and all z with $|z| \geq r_0$. For instance, $f(z) = e^z \in SB$. Here $\chi = \frac{1}{2}$, $s = 1$, $r_0 = 0$. We now state

Theorem 1. *Let $f(z)$ be entire and $f(z) \in SB$. Then*

(i) $f(z) \in B$,

(ii) *if $P(z)$ is a polynomial then $f(z)P(z) \in SB$,*

(iii) *$\{f(z)/P(z)\} \in SB$ provided $\{f(z)/P(z)\}$ is entire,*

(iv) *if a is any complex number and*

$$0 < \chi < e^{-2|a|} \tag{1.3}$$

where χ is the constant in (1.2), *then $e^{az}f(z) \in SB$.*

Our main result is

Theorem 2. *Let $\{a_n\}_{n=1}^{\infty}$ be a positive, strictly increasing sequence such that*

$$a_{n+1} - a_n \geq b_n \qquad (n \geq 1), \tag{1.4}$$

where $\{b_n\}_{n=1}^{\infty}$ is positive nondecreasing and

$$\sum_{n=1}^{\infty} \frac{1}{nb_n} < \infty. \tag{1.5}$$

Then

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right) \in SB. \tag{1.6}$$

Theorem 2 has four useful corollaries. Consider the Lindelöf functions

$$f(z) = \prod_{1}^{\infty} \left(1 + \frac{z}{a_n}\right), \tag{1.7}$$

where $a_n = \{n(\log n)^{\alpha}\}^{1/\lambda}$, $0 < \lambda \leq 1$ and $\alpha > 1$ if $\lambda = 1$, α an arbitrary real number if $0 < \lambda < 1$.

Corollary 2.1. *All Lindelöf functions defined by* (1.7) *belong to class SB.*

Corollary 2.2. *If α is any nonzero complex number and $f(z)$ satisfies the assumptions of Theorem* 2, *then*

$$f(\alpha z) = F(z) \in SB. \tag{1.8}$$

Corollary 2.3. *Let $\{a_n\}_{n=1}^{\infty}$ satisfy the conditions of Theorem* 2, *and let $\{a_{n_j}\}_{j=1}^{\infty}$ be any one of its infinite subsequences.*

Then

$$\prod_{j=1}^{\infty}\left(1+\frac{z}{a_{n_j}}\right)=g(z)\in SB.$$

From Corollary 2.2, we deduce that $F(z)\in SB\subset B$ and consequently there exists an index s such that

$$\frac{|F^{(n)}(z)|}{n!}\leq \max_{0\leq j\leq s}\left\{\frac{|F^{(j)}(z)|}{j!}\right\},$$

for all n and all z.

Assume now α real and greater than one so that (1.8) and (1.1) imply

$$\frac{\alpha^n|f^{(n)}(\alpha z)|}{n!}\leq \alpha^s\Omega_s(\alpha z).$$

Replacing αz by ζ, we obtain

$$|f^{(n)}(\zeta)|/n!\leq \Omega_s(\zeta)/\alpha^{n-s}\qquad (n=s+1,\ s+2,\cdots)$$

for all ζ.

We thus see that the functions in Theorem 2 belong to SB in a very special sense: the constant χ in (1.2) may be chosen arbitrarily small (a diminution of χ will of course increase, in general, the value of the index s).

In particular, if a is given, we can choose χ so as to satisfy (1.3). Consequently assertion (iv) of Theorem 1 leads to

Corollary 2.4. *If $f(z)$ satisfies the conditions of Theorem 2 then $e^{az+b}f(z)\in SB$.*

In Corollary 2.3 we can choose a subsequence $\{a_{n_j}\}$ by omitting from the given sequence $\{a_n\}$ long sections of consecutive terms. The entire function $h(z)$ corresponding to this subsequence belongs the the class SB and it is obvious that we may, by suitable choices of the gaps, obtain irregularities in the growth of $h(z)$. We are thus led to the following result which we state without proof.

Theorem 3. *Let $f(z)$ be given by (1.6) and let a sequence $\{a_n\}_{n=1}^{\infty}$ satisfy the conditions of Theorem 2. Let*

$$R_1,\ R_2,\cdots\qquad (R_m<R_{m+1},\ m=1,2,\cdots,\ R_m\to\infty) \tag{1.9}$$

be a given sequence.

It is always possible to select a subsequence $\{c_j\}_{j=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ and two subsequences $\{R_k'\}_{k=1}^{\infty}$, $\{R_k''\}_{k=1}^{\infty}$ of (1.9) such that

$$h(z)=\prod_{j=1}^{\infty}\left(1+\frac{z}{c_j}\right)\in SB, \tag{1.10}$$

and such that for all $k = 1, 2, \cdots$

$$\log M(R'_k, h) > (1 - 1/(k+1)) \log M(R'_k, f),$$

and

$$\frac{\log \log M(R''_k, h)}{\log R''_k} < \frac{1}{k+1}.$$

By an appropriate choice of gaps we can also construct a function h belonging to B and of given order ρ and of given lower order λ where $0 \leq \lambda \leq \rho \leq 1$. We omit the details of construction.

In §2 we give the proof of Theorem 1. §3 contains necessary lemmas and §4 gives the proof of Theorem 2.

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2. **Proof of Theorem 1. Proof of assertion (i).** By Definition 2, there exist fixed quantities χ, $0 < \chi < 1$, r_0 and $s \geq 0$ such that (1.2) holds for all $n \geq s + 1$ and all z with $|z| \geq r_0$.

We examine $f(z)$ and its successive derivatives in the closed disk

$$|z| \leq r_0. \tag{2.1}$$

Since the number of zeros of f in (2.1) is $n(r_0, 1/f)$, it is obvious that one of the quantities $f(z), f'(z), \cdots, f^{(N)}(z)$ $(N = n(r_0, 1/f))$ is different from zero.

Let

$$\Omega_N(z) = \max_{0 \leq j \leq N} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\}. \tag{2.2}$$

It is clear that $\Omega_N(z)$ is continuous and does not vanish in (2.1). Hence for some $\alpha > 0$,

$$\Omega_N(z) \geq \alpha \qquad (|z| \leq r_0). \tag{2.3}$$

Assume $|z| \leq r_0$. By Cauchy's formula, for the nth derivative,

$$\frac{|f^{(n)}(z)|}{n!} \leq \frac{1}{2^n} M(r_0 + 2, f). \tag{2.4}$$

If n is sufficiently large, say $n \geq n_0 \geq s + 1$, (2.3) and (2.4) imply

$$\frac{|f^{(n)}(z)|}{n!} \leq \frac{M(r_0 + 2, f)}{2^n} \leq \chi\alpha \leq \chi \Omega_N(z) \tag{2.5}$$

for all z such that $|z| \leq r_0$ and for all $n \geq n_0$. Let $p = \max(n_0, N)$. Then (2.5) and (2.2) imply

$$\frac{|f^{(n)}(z)|}{n!} \leq \chi \max_{0 \leq j \leq p} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\} \qquad (n \geq p+1), \tag{2.6}$$

provided $|z| \leq r_0$. On the other hand, since $n_0 \geq s+1$, and $f \in SB$, (2.6) holds for $n \geq p+1$ and $|z| \geq r_0$. Hence we can drop the restriction on the size of $|z|$ and this completes the proof.

Proof of assertion (ii). By hypothesis (1.2) holds for all $n \geq s+1$ and all z such that $|z| \geq r_0$. Let

$$g(z) = (z - z_0)f(z), \tag{2.7}$$

$$\Omega(z) = \max_{0 \leq j \leq s} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\}, \tag{2.8}$$

$$\Omega^*(z) = \max_{0 \leq j \leq s+1} \left\{ \frac{|g^{(j)}(z)|}{j!} \right\}. \tag{2.9}$$

Since

$$\frac{g^{(n)}(z)}{n!} = (z - z_0)\frac{f^{(n)}(z)}{n!} + \frac{f^{(n-1)}(z)}{(n-1)!} \tag{2.10}$$

we have, for $n \geq s+2$ and $|z| \geq r_0$,

$$\frac{|g^{(n)}(z)|}{n!} \leq \chi \Omega(z)\{1 + |z - z_0|\}. \tag{2.11}$$

From (2.7) we obtain, for $z \neq z_0$,

$$\frac{f^{(n)}(z)}{n!} = \frac{g^{(n)}(z)}{n!}\frac{1}{(z-z_0)} + \frac{g^{(n-1)}(z)}{1!(n-1)!}\frac{(-1)1!}{(z-z_0)^2} + \cdots + \frac{g(z)}{0!}\frac{(-1)^n n!}{n!(z-z_0)^{n+1}}. \tag{2.12}$$

(2.9) and (2.12) yield, for $z \neq z_0$,

$$\frac{|f^{(n)}(z)|}{n!} \leq \Omega^*(z)\left\{ \frac{1}{|z-z_0|} + \frac{1}{|z-z_0|^2} + \cdots + \frac{1}{|z-z_0|^{n+1}} \right\} \qquad (0 \leq n \leq s+1).$$

Consequently we have from (2.11) and (2.8), for $n \geq s+2$ and $|z| \geq r_0$, $z \neq z_0$,

$$\frac{|g^{(n)}(z)|}{n!} \leq \chi \Omega^*(z)\{1 + |z - z_0|\}\left\{ \frac{1}{|z-z_0|} + \cdots + \frac{1}{|z-z_0|^{n+1}} \right\}.$$

If $|z|$ is sufficiently large, say $|z| \geq R_1$, then

$$\chi\{1 + |z - z_0|\}\left\{ \frac{1}{|z-z_0|} + \cdots + \frac{1}{|z-z_0|^{n+1}} \right\} < \chi'$$

where $0 < \chi' < 1$. This shows that $g(z) \in SB$. Now if $P(z) = (z - z_0)\cdots(z - z_p)$

and $Q(z) = AP(z)$ (A a constant) then the above argument applied $(p+1)$ times shows that $fP \in SB$, $fQ \in SB$. This completes the proof.

Proof of assertion (iii). Let

$$g(z) = f(z)/(z - z_0) \tag{2.13}$$

and let $\Omega(z)$ and $\Omega^*(z)$ have the same meaning as in (2.8) and (2.9). Then for $n \geq s+1$ and $|z| \geq r_0$, $z \neq z_0$,

$$\frac{|g^{(n)}(z)|}{n!} \leq \Omega(z)\left\{\frac{\chi}{|z - z_0|} + \frac{1}{|z - z_0|^2} + \cdots + \frac{1}{|z - z_0|^{n+1}}\right\}. \tag{2.14}$$

From (2.13) we have, for $1 \leq n \leq s$,

$$\frac{|f^{(n)}(z)|}{n!} \leq \frac{|g^{(n)}(z)|}{n!}|z - z_0| + \frac{|g^{(n-1)}(z)|}{(n-1)!} \leq \Omega^*(z)\{1 + |z - z_0|\} \tag{2.15}$$

and

$$|f(z)| = |g(z)|\,|z - z_0| \leq \Omega^*(z)\{1 + |z - z_0|\}.$$

Hence (2.15) holds for $0 \leq n \leq s$ and

$$\Omega(z) \leq \Omega^*(z)\{1 + |z - z_0|\}. \tag{2.16}$$

The inequalities (2.14) and (2.16) imply

$$\frac{|g^{(n)}(z)|}{n!} \leq \Omega^*(z)\{1 + |z - z_0|\}\left\{\frac{\chi}{|z - z_0|} + \cdots + \frac{1}{|z - z_0|^{n+1}}\right\} \tag{2.17}$$

for $n \geq s+1$ and $|z| \geq r_0$, $z \neq z_0$. Hence for $|z|$ sufficiently large, say $|z| \geq R_2$, we have $|g^{(n)}(z)|/n! \leq \chi''\,\Omega^*(z)$, where $0 < \chi'' < 1$, for $n \geq s+1$ and $|z| \geq R_2$. This means that $g(z) \in SB$.

If $P(z) = \prod_{j=0}^{p}(z - z_j)$ and $Q(z) = AP(z)$, then the above argument shows that $f/P \in SB$, $f/Q \in SB$. This completes the proof.

Proof of assertion (iv). Let

$$g(z) = e^{az}f(z). \tag{2.18}$$

Then

$$\frac{g^{(n)}(z)}{n!} = e^{az}\left\{\frac{f^{(n)}(z)}{n!} + a\frac{f^{(n-1)}(z)}{(n-1)!} + \cdots + \frac{a^n}{n!}f(z)\right\}. \tag{2.19}$$

There is a similar relation where f and g are exchanged and a is replaced by $-a$. From this latter formula we deduce

$$\frac{|f^{(n)}(z)|}{n!} \leq |e^{-az}| \max_{0 \leq j \leq n} \left\{ \frac{|g^{(j)}(z)|}{j!} \right\} e^{|a|}.$$

In particular if

$$\Omega^{**}(z) = \max_{0 \leq j \leq s} \left\{ \frac{|g^{(j)}(z)|}{j!} \right\}, \tag{2.20}$$

we have, in view of (1.1),

$$\Omega(z) \leq |e^{-az}| e^{|a|} \Omega^{**}(z). \tag{2.21}$$

By assumption, (1.2) is satisfied for some fixed $\chi < 1$ and consequently (2.19) yields for all $n \geq s+1$ and all z such that $|z| \geq r_0$

$$\frac{|g^{(n)}(z)|}{n!} \leq |e^{az}| \left\{ \left(1 + \frac{|a|}{1!} + \cdots + \frac{|a|^{n-s-1}}{(n-s-1)!}\right) \chi \Omega(z) + \left(\frac{|a|^{n-s}}{(n-s)!} + \cdots + \frac{|a|^n}{n!}\right) \Omega(z) \right\}$$

$$\leq |e^{az}| e^{|a|} \left\{ \chi + \frac{|a|^{n-s}}{(n-s)!} \right\} \Omega(z).$$

Using (2.21) we find, for $n \geq s+1$ and $|z| \geq r_0$

$$\frac{|g^{(n)}(z)|}{n!} \leq e^{2|a|} \left(\chi + \frac{|a|^{n-s}}{(n-s)!} \right) \Omega^{**}(z).$$

Since s is fixed $|a|^{n-s}/(n-s)! \to 0$ as $n \to \infty$, and so we may select, in view of (1.3), an integer $s_0 \geq s$, so that

$$e^{2|a|} \left(\chi + \frac{|a|^{n-s}}{(n-s)!} \right) < \chi' < 1$$

as soon as $n \geq s_0 + 1$. Hence for all $n \geq s_0 + 1$ and all z with $|z| \geq r_0$,

$$\frac{|g^{(n)}(z)|}{n!} \leq \chi' \Omega^{**}(z) \leq \chi' \max_{0 \leq j \leq s_0} \left\{ \frac{|g^{(j)}(z)|}{j!} \right\}.$$

The proof of Theorem 1 is now complete.

3. **Lemmas.** We require several lemmas. The first two lemmas contain known results.

Lemma A [2, Example B. 18]. *If $\{b_n\}_1^\infty$ is positive nondecreasing and $\Sigma(nb_n)^{-1} < \infty$, then*

$$\lim_{n \to \infty} \frac{\log n}{b_n} = 0. \tag{3.1}$$

Lemma B [4, p. 261]. *If $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$, then*

$$\alpha_1\beta_{j_1} + \alpha_2\beta_{j_2} + \cdots + \alpha_n\beta_{j_n} \le \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n \tag{3.2}$$

for every permutation $j_1, \cdots, j_n$ *of* $1, 2, \cdots, n$.

Lemma 1. *Let* $a_{n+1} - a_n \ge b_n$ $(n \ge 1)$ *where* $\{b_n\}_1^\infty$ *is positive nondecreasing and* $\Sigma(nb_n)^{-1} < \infty$. *Given* $K \ge 1$ *and* $\epsilon > 0$, *it is possible to find an integer* $N \ge 1$ *and a positive nondecreasing sequence* $\{c_n\}_{n=1}^\infty$ *such that the following conditions hold simultaneously*:

$$c_n \ge 8 \quad (n \ge N), \tag{3.3}$$

$$\sum_{j=N}^{\infty} \frac{1}{jc_j} < \frac{\epsilon}{4}, \tag{3.4}$$

$$a_{n+1} - a_n \ge Kc_n + 8 \quad (n \ge N), \tag{3.5}$$

$$\sum_{j=1}^{\infty} \frac{1}{a_j} < +\infty, \tag{3.6}$$

$$a_{N+2m+1} - a_N > a_{N+2m} - a_N \ge (K+1)\, m\, c_{N+m} \quad (m \ge 1), \tag{3.7}$$

$$\frac{1}{c_{n+1}} + \frac{1}{c_{n+1} + c_{n+2}} + \frac{1}{c_{n+1} + c_{n+2} + c_{n+3}} + \cdots \le \epsilon \quad (n \ge N), \tag{3.8}$$

$$\frac{1}{c_{n-1}} + \frac{1}{2c_{n-2}} + \frac{1}{3c_{n-3}} + \cdots + \frac{1}{(n-N)\, c_N} < \epsilon \quad (n > N). \tag{3.9}$$

Proof. (i) Let $c_n = b_n/(K+1)$. Then $\{c_n\}_{n=1}^\infty$ is the required sequence such that $\Sigma(nc_n)^{-1} < +\infty$. By Lemma A and the convergence of this series we can choose N so large that (3.3) and (3.4) are satisfied.

(ii) $$a_{n+1} - a_n \ge b_n = (K+1)\, c_n \ge Kc_n + 8 \quad (n \ge N).$$

This proves (3.5).

(iii) Since

$$a_{p+2m} - a_p = \sum_{j=0}^{2m-1} (a_{p+j+1} - a_{p+j}) \ge (K+1) \sum_{j=0}^{2m-1} c_{p+j},$$

we have on taking $p = 2$, $p + 2m = 2n$,

$$a_{2n} > (K+1)(c_n + \cdots + c_{2n-1}) \ge (K+1)\, nc_n.$$

Hence

$$\frac{1}{a_{2n}} + \frac{1}{a_{2n+1}} < \frac{2}{K+1} \frac{1}{nc_n},$$

and the convergence of the series in (3.6) follows from (3.4).

(iv) Taking $p = N$ we get

$$a_{N+2m} - a_N \geq (K+1)m\, c_{N+m}.$$

Since $\{a_n\} \uparrow$, (3.7) follows.

(v) We have

$$c_{n+1} + \cdots + c_{n+2m-1} \geq c_{n+m} + \cdots + c_{n+2m-1} \geq m\, c_{n+m} \qquad (m \geq 1)$$

and so

$$\begin{aligned}\Sigma_n &= \frac{1}{c_{n+1}} + \frac{1}{c_{n+1} + c_{n+2}} + \cdots \\ &\leq 2 \sum_{j=1}^{\infty} \frac{1}{j\, c_{n+j}} < \frac{2}{c_n} \sum_{j=1}^{n} \frac{1}{j} + 2 \sum_{j=n+1}^{\infty} \frac{1}{j\, c_{n+j}} \\ &< \frac{2}{c_n}(1 + \log n) + 2 \sum_{j=n+1}^{\infty} \frac{1}{j\, c_{n+j}}.\end{aligned}$$

By Lemma A

$$2(1 + \log n)/c_n < \epsilon/2 \quad \text{if } n \geq N_1$$

and

$$2 \sum_{j=n+1}^{\infty} \frac{1}{j\, c_{n+j}} < \frac{\epsilon}{2} \quad \text{if } n \geq N_2.$$

Let $N = \max(N_1, N_2)$. Then for $n \geq N$, $\Sigma_n < \epsilon$ and (3.8) is proved.

(vi) Let $n > N$ and $r = 1/c_{n-1} + 1/2c_{n-2} + \cdots + 1/(n-N)c_N$. By (i),

(a) $$\frac{1}{c_{n-1}} \leq \frac{1}{c_{n-2}} \leq \cdots \leq \frac{1}{c_N},$$

and

(b) $$\frac{1}{n-N} < \frac{1}{n-N-1} < \cdots < \frac{1}{2} < \frac{1}{1}.$$

By applying Lemma B to (a) and (b) we have

$$\begin{aligned} r &\leq \frac{1}{c_N} + \frac{1}{2c_{N+1}} + \cdots + \frac{1}{(n-N)c_{n-1}} \\ &\leq \frac{1}{c_N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) + \frac{1}{(N+1)c_{N+N}} + \cdots + \frac{1}{(n-N)c_{n-1}} \\ &\leq \frac{1 + \log N}{c_N} + \sum_{j=1}^{\infty} \frac{1}{(N+j)c_{N+j}} < \epsilon \end{aligned}$$

for N sufficiently large. This proves (3.9) and also completes the proof of Lemma 1.

In what follows in this section and in the next section we shall take $\epsilon = 1/100$ and $N \geq 1$ such that (3.3)–(3.9) hold and also

$$\frac{1}{c_N} + \sum_{j=1}^{\infty} \frac{1}{jc_{N+j}} < \frac{K}{100}.$$

Lemma 2. *Let* $\Gamma_n = \{z: |z + a_n| < 4\}$, $n \geq N$, *and suppose* $z \notin \bigcup_{n=N}^{\infty} \Gamma_n$. *Then*

$$\sum_{j=N}^{\infty} \frac{1}{|z + a_j|} < \chi < 1 \tag{3.10}$$

where χ *is some fixed number.*

Proof. Let $z = x + iy$. Then either

(i) $x \geq -a_N$, or

(ii) $-a_{n+1} \leq x < -a_n$ for some well-determined $n \geq N$.

Suppose first that (i) holds. By assumption

$$1/|z + a_N| \leq 1/4 \tag{iii}$$

and

$$|z + a_{N+1}| \geq |x + a_{N+1}| \geq a_{N+1} - a_N \geq (K + 1)\, c_N.$$

Hence

$$1/|z + a_{N+1}| \leq 1/(K + 1)\, c_N < 1/100. \tag{iv}$$

For $j \geq N + 2$,

$$|z + a_j| \geq |x + a_j| \geq a_j - a_N.$$

Hence by (3.7) we have for $j = N + 2m$ or $N + 2m + 1$, $m \geq 1$,

$$|z + a_j| \geq (K + 1)m\, c_{N+m}.$$

Consequently

$$\sum_{j=N+2}^{\infty} \frac{1}{|z + a_j|} \leq \frac{1}{K + 1} \sum_{m=1}^{\infty} \frac{1}{m\, c_{N+m}} < \frac{1}{100} \tag{v}$$

and (3.10) follows from (iii)–(v). Suppose now (ii) holds. Then

$$1/|z + a_n| + 1/|z + a_{n+1}| < 1/2. \tag{vi}$$

For $j \geq n + 2$,

$$|z + a_j| \geq |x + a_j| \geq a_j - a_{n+1} \geq (K + 1)(c_{j-1} + c_{j-2} + \cdots + c_{n+1})$$

and so, by (3.8),

$$\text{(vii)} \qquad \sum_{j=n+2}^{\infty} \frac{1}{|z+a_j|} < \frac{1}{K+1}\left\{\frac{1}{c_{n+1}} + \frac{1}{c_{n+1}+c_{n+2}} + \cdots\right\} < \frac{1}{100}.$$

If $n > N$ we have for $N \le j \le (n-1)$

$$|z+a_j| \ge |x+a_j| \ge |a_n - a_j| \ge (K+1)(n-j)\,c_j$$

and (3.9) yields

$$\text{(viii)} \qquad \sum_{j=N}^{n-1} \frac{1}{|z+a_j|} \le \frac{1}{K+1}\left\{\frac{1}{c_{n-1}} + \frac{1}{2c_{n-2}} + \cdots + \frac{1}{(n-N)\,c_N}\right\} < \frac{1}{100}.$$

From (vi)–(viii) we get (3.10) in this case also. The proof of Lemma 2 is complete.

Lemma 3. *Let*

$$f(z) = \prod_{j=N}^{\infty}\left(1 + \frac{z}{a_j}\right)$$

and let $\{-d_j\}_{j=N}^{\infty}$ *be the zeros of* $f'(z)$. *Then for all* j *and* k $(j \ge N,\ k \ge N)$,

$$\text{(3.11)} \qquad |d_j - a_k| > 8.$$

Proof. We need to show that if $|z+a_k| \le 8$ for some k, then

$$\frac{f'(z)}{f(z)} = \sum_{j=N}^{\infty} \frac{1}{z+a_j} \ne 0.$$

For $j \ge k+1$, we have by (3.5),

$$|z+a_j| \ge |a_j - a_k| - |z+a_k| \ge (a_j - a_k) - 8 \ge K(c_{j-1} + c_{j-2} + \cdots + c_k).$$

Hence by (3.8)

$$\sum_{j=k+1}^{\infty} \frac{1}{|z+a_j|} < \frac{1}{K}\left\{\frac{1}{c_k} + \frac{1}{c_k + c_{k+1}} + \cdots\right\} < \frac{1}{50}.$$

If $k > N$ we have for $N \le j \le k-1$ (see (3.5)),

$$|z+a_j| \ge (a_k - a_j) - |z+a_k| \ge a_k - a_j - 8 \ge K(k-j)c_j.$$

This gives, by (3.9),

$$\text{(ii)} \qquad \sum_{j=N}^{k-1} \frac{1}{|z+a_j|} \le \frac{1}{K}\left(\frac{1}{c_{k-1}} + \frac{1}{2c_{k-2}} + \cdots + \frac{1}{(k-N)c_N}\right) < \frac{1}{100}.$$

Since $|z+a_k| \le 8$, (i) and (ii) imply

$$\frac{f'(z)}{f(z)} \ge \frac{1}{8} - \frac{1}{100} - \frac{1}{50} > 0.$$

This completes the proof of Lemma 3.

Lemma 4. *If* $z \in \Gamma_n = \{z: |z + a_n| < 4\}$ *for some* $n \geq N$, *then*

$$\sum_{j=N}^{\infty} \frac{1}{|z + d_j|} < \chi < 1 \tag{3.12}$$

where χ *is some fixed number.*

Proof. (a) We have, by Laguerre's theorem [1, p. 23], $a_N < d_N < a_{N+1} < \cdots$. Suppose first

$$|z + a_N| < 4. \tag{i}$$

By (3.11) and (i)

$$|z + d_N| \geq (d_N - a_N) - |z + a_N| > 8 - 4 = 4. \tag{ii}$$

For $j \geq N + 1$ we have by (3.5)

$$|z + d_j| \geq (d_j - a_N) - |z + a_N| \geq a_j - a_N - 4 \geq K\{c_{j-1} + c_{j-2} + \cdots + c_N\}.$$

Hence by (3.8),

$$\sum_{j=N+1}^{\infty} \frac{1}{|z + d_j|} < \frac{1}{K}\left\{\frac{1}{c_N} + \frac{1}{c_N + c_{N+1}} + \cdots\right\} < \frac{1}{50}. \tag{iii}$$

These two inequalities (ii) and (iii) give (3.12).

(b) Suppose now $|z + a_n| < 4$ for some $n \geq N + 1$. Then

$$|z + d_n| \geq (d_n - a_n) - |z + a_n| \geq 8 - 4 = 4.$$

Similarly $|z + d_{n-1}| > 4$ and so

$$1/|z + d_{n-1}| + 1/|z + d_n| < \tfrac{1}{2}. \tag{iv}$$

For $j \geq n + 1$, we have by (3.5)

$$|z + d_j| \geq (d_j - a_n) - |z + a_n| > (a_j - a_n) - 4 \geq K(c_{j-1} + c_{j-2} + \cdots + c_n).$$

This gives (see (3.8))

$$\sum_{j=n+1}^{\infty} \frac{1}{|z + d_j|} < \frac{1}{K}\left\{\frac{1}{c_n} + \frac{1}{c_n + c_{n+1}} + \cdots\right\} < \frac{1}{100}. \tag{v}$$

If $N \leq n - 2$ we have for $N \leq j \leq n - 2$,

$$|z + d_j| \geq |d_j - a_n| - |z + a_n| \geq a_n - a_{j+1} - 4 \geq K(n - j - 1)\, c_{j+1}.$$

Hence (3.9) yields

(vi) $$\sum_{j=N}^{n-2} \frac{1}{|z+d_j|} < \frac{1}{K}\left\{\frac{1}{c_{n-1}} + \frac{1}{2c_n} + \cdots + \frac{1}{(n-N-1)c_{N+1}}\right\} < \frac{1}{100}.$$

The inequalities (iv)–(vi) imply (3.12). The proof of Lemma 4 is complete.

Lemma 5. *Let*

$$f(z) = \prod_{j=N}^{\infty}\left(1 + \frac{z}{a_j}\right).$$

Then $f(z)$ is an entire function of genus zero and

$$f'(z) = f'(0)\prod_{j=N}^{\infty}\left(1 + \frac{z}{d_j}\right)$$

where $f'(0) = \Sigma_{j=N}^{\infty} a_j^{-1}$. If for some z at least one of the two inequalities

(3.13)
(i) $$\sum_{j=N}^{\infty} \frac{1}{|z+a_j|} < \chi < 1,$$
(ii) $$\sum_{j=N}^{\infty} \frac{1}{|z+d_j|} < \chi < 1,$$

where χ is a constant, holds, then for this z

(3.14) $$\frac{|f^{(n+1)}(z)|}{(n+1)!} \le \max\left\{\chi^{n+1}|f(z)|, \frac{\chi^n}{(n+1)}|f'(z)|\right\}$$
$$< \chi^n \max\{|f(z)|, |f'(z)|\}, \qquad n = 1, 2, \cdots.$$

Proof. Let

$$p(z) = \sum_{j=N}^{\infty} \frac{1}{(z+a_j)}$$

and suppose (i) of (3.13) holds. Then

(iii) $$|p(z)| < \chi < 1,$$
$$p^{(n)}(z) = (-1)^n n! \sum_{j=N}^{\infty} \frac{1}{(z+a_j)^{n+1}}, \qquad n = 1, 2, \cdots.$$

Hence

(iv) $$\frac{|p^{(n)}(z)|}{n!} \le \sum_{j=N}^{\infty} \frac{1}{|z+a_j|^{n+1}} \le \left(\sum_{j=N}^{\infty} \frac{1}{|z+a_j|}\right)^{n+1} < \chi^{n+1}.$$

Since $f' = fp$ we have

(v) $$|f'(z)| \le \chi|f(z)|$$

and

$$\text{(vi)}\quad \begin{aligned} \frac{|f^{(n+1)}(z)|}{(n+1)!} &= \frac{1}{n+1}\left|\left\{\frac{f^{(n)}(z)}{n!}\frac{p(z)}{0!} + \frac{f^{(n-1)}(z)}{(n-1)!}\frac{p'(z)}{1!} + \cdots + \frac{f(z)}{0!}\frac{p^{(n)}(z)}{n!}\right\}\right| \\ &\le \frac{1}{n+1}\left\{\frac{|f^{(n)}(z)|}{n!}\chi + \frac{|f^{(n-1)}(z)|}{(n-1)!}\chi^2 + \cdots + |f(z)|\chi^{n+1}\right\}. \end{aligned}$$

We now use an induction argument to show that

$$\text{(vii)}\qquad \frac{|f^{(n)}(z)|}{n!} \le \chi^n |f(z)|, \qquad n = 1, 2, \cdots.$$

For the inequality holds by (v) for $n = 1$. Suppose it is true for $n = 1, 2, \cdots, m$. Then by (vi),

$$\text{(viii)}\quad \begin{aligned} \frac{|f^{(m+1)}(z)|}{(m+1)!} &\le \frac{1}{m+1}\left\{\frac{f^{(m)}(z)}{m!}\chi + \cdots + |f(z)|\chi^{m+1}\right\} \\ &< \frac{1}{m+1}\left\{\chi^m\chi + \chi^{m-1}\chi^2 + \cdots + \chi^{m+1}\right\}|f(z)| = \chi^{m+1}|f(z)|. \end{aligned}$$

This proves (vii). Suppose now (ii) of (3.13) holds. We have then

$$f''(z) = f'(z)\sum_{j=N}^{\infty}\frac{1}{(z+d_j)}$$

and the above reasoning yields

$$\text{(ix)}\qquad \frac{|f^{(n+2)}(z)|}{(n+1)!} < \chi^{n+1}|f'(z)| \qquad (n = 0, 1, 2, \cdots).$$

From (vii) and (ix) we have

$$\begin{aligned} \frac{|f^{(n+1)}(z)|}{(n+1)!} &\le \max\left\{\chi^{n+1}|f(z)|, \frac{\chi^n}{n+1}|f'(z)|\right\} \\ &< \chi^n \max\{|f(z)|, |f'(z)|\} \qquad (n = 1, 2, \cdots). \end{aligned}$$

This completes the proof of Lemma 5.

4. **Proof of Theorem 2.** We have

$$f(z) = \prod_{j=1}^{\infty}\left(1 + \frac{z}{a_j}\right) = \prod_{1}^{N-1}\left(1 + \frac{z}{a_j}\right)\prod_{j=N}^{\infty}\left(1 + \frac{z}{a_j}\right) = P(z)f_N(z),$$

where $P(z)$ is a polynomial of degree $(N-1)$ and $f_N(z) = \prod_{j=N}^{\infty}(1 + z/a_j)$ and N is determined as in the remark following Lemma 1.

Let z be given. Then either $z \in \Gamma_n$ for some $n \ge N$ or $z \notin \bigcup_{n=N}^{\infty}\Gamma_n$. If $z \in \Gamma_n$ for some $n \ge N$, then by Lemma 4, (ii) of (3.13) holds and hence, by Lemma 5, (3.14) holds with f replaced by f_N.

If $z \notin \bigcup_{n=N}^{\infty}\Gamma_n$, then by Lemma 2, (i) of (3.13) holds and we have, again by Lemma 5, (3.14) with f replaced by f_N. Hence $f_N \in SB$ and so by Theorem 1,

$f(z) = P(z)f_N(z)$ belongs to SB. This completes the proof of Theorem 2.

The corollaries are almost obvious and we leave the proofs to the reader. Note that if $f(\alpha z) \in SB$ then $f(|\alpha|z) \in SB$ and conversely.

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