AUTOMORPHISMS OF \( \omega_1 \)-TREES

BY

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ABSTRACT. The number of automorphisms of a normal \( \omega_1 \)-tree \( T \), denoted by \( \sigma(T) \), is either finite or \( 2^{\aleph_0} \leq \sigma(T) \leq 2^{\aleph_1} \). Moreover, if \( \sigma(T) \) is infinite then \( \sigma(T) = 2^{\aleph_0} \) or \( \sigma(T) = 2^{\aleph_1} \). It is consistent that there is a Suslin tree with arbitrary prescribed \( \sigma(T) \) between \( 2^{\aleph_0} \) and \( 2^{\aleph_1} \), subject to the restriction above; e.g. \( 2^{\aleph_0} = \aleph_1 \), \( 2^{\aleph_1} = \aleph_3 \), and \( \sigma(T) = \aleph_7 \). We prove related results for Kurepa trees and isomorphism types of trees. We use Cohen's method of forcing and Jensen's techniques in \( L \).

0. Introduction. A normal \( \omega_1 \)-tree, being a partially ordered set with \( \aleph_1 \) elements, admits at most \( 2^{\aleph_1} \) automorphisms. Also, the number of isomorphism types of \( \omega_1 \)-trees is not larger than \( 2^{\aleph_1} \). In \( \S 1 \), we shall show that the number of automorphisms a normal \( \omega_1 \)-tree admits is either finite, or at least \( 2^{\aleph_0} \); as a matter of fact, it is \( 2^{\aleph_0} \) or \( 2^{\aleph_1} \), with the exception of Suslin trees.

In \( \S 2 \), we investigate Suslin trees. Inspired by Jensen's result [6], which says that it is consistent to have rigid Suslin trees, and Suslin trees which admit \( \aleph_1 \), or \( \aleph_2 \) automorphisms, we prove a general consistency result saying that the number of automorphisms of a Suslin tree can be anything between \( 2^{\aleph_0} \) and \( 2^{\aleph_1} \). We shall also show that the number of isomorphism types of Suslin trees can be \( 2^{\aleph_1} \).

In \( \S 3 \), we follow Jensen's line by showing that results analogous to those of \( \S 2 \) can be obtained for Suslin continua.

In \( \S 4 \), Kurepa trees will be considered. We shall establish the consistency of rigid Kurepa trees, of Kurepa trees with \( 2^{\aleph_0} \) of \( 2^{\aleph_1} \) automorphisms, and of as many as \( 2^{\aleph_1} \) isomorphism types of Kurepa trees.

Finally, we shall observe that there are Aronszajn trees which admit \( 2^{\aleph_1} \) automorphisms, and that we do not know whether the existence of rigid normal \( \omega_1 \)-trees can be proved.

Throughout the paper, we utilize the axiom \( \Diamond \) of Jensen, cf. [6]. Also, we use the method of forcing and assume the reader to be familiar with it.

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1. **Normal $\omega_1$-trees.** A partially ordered set $(T, \leq)$ is called a tree if for each $x \in T$, the set $\hat{x} = \{y \in T : y < x\}$ is well ordered by $\leq$. The order $\sigma(x)$ of $x \in T$ is the order type of $\hat{x}$. If $\alpha$ is an ordinal then $\U_{\alpha}$, the $\alpha$ the level of the tree, is the set of all $x \in T$ whose order is $\alpha$. The length of the tree $T$ is $l(T) = \sup \{\sigma(x) + 1 : x \in T\}$; an $\alpha$-tree is a tree whose length is $\alpha$. Maximal well-ordered subsets of $T$ are called branches of the tree; the length of a branch $b$ is the order type of $b$. An $\alpha$-branch is a branch of length $\alpha$.

If $T$ is a tree (we shall often write $T$ instead of $(T, \leq)$) and if $\alpha$ is an ordinal, then $T|\alpha$ is the tree consisting of all the elements of $T$ whose order is less than $\alpha$. If $T_1$ and $T_2$ are trees then $T_2$ is an extension of $T_1$ (and $T_1$ is a restriction of $T_2$) if $T_1 = T_2|\alpha$ for some $\alpha$. For each $x \in T$, the set $\{y \in T : y \geq x\}$ is a tree; we denote it $T_x$.

**Definition.** A normal $\omega_1$-tree is a tree $T$ with the following properties:

1. the length of $T$ is $\omega_1$,
2. for each $\alpha < \omega_1$, $\U_\alpha$ is at most countable; $\U_0$ consists of exactly one element,
3. for each $x \in T$, the length of $T_x$ is $\omega_1$,
4. each $x \in T$ has exactly $\aleph_0$ immediate successors,
5. if $\sigma(x) = \sigma(y) = \alpha$ is a limit ordinal, and if $\hat{x} = \hat{y}$, then $x = y$.

A normal binary $\omega_1$-tree is defined similarly, except that in (4), each $x$ has exactly two immediate successors. A normal countable (binary) tree is a countable tree which is a restriction of a normal (binary) $\omega_1$-tree.

Since a normal $\omega_1$-tree has $\aleph_1$ elements, we may always assume that the elements of the tree are countable ordinals. On the other hand, we have another representation of normal $\omega_1$-trees. Let $\omega^\omega = \U_{\alpha < \omega_1} \alpha$ be the set of all functions $s$ whose domain is a countable ordinal and whose values are the natural numbers; similarly, $\omega^2 = \U_{\alpha < \omega_1} \alpha^2$ is the set of all such functions $s$ with values 0 and 1. The length of $s$, $l(s)$, is the domain of $s$. Any normal $\omega_1$-tree can be represented by a set $T \subseteq \omega^\omega$ (ordered by the natural ordering of $\omega^\omega$), which satisfies the following conditions:

1. for each $s \in T$ and each $n \in \omega$, $\widehat{sn} \in T$,
2. for each $s \in T$ and each $\alpha < \omega_1$, there is $t \in T$ such that $l(t) = \alpha$ and $t \subseteq s$ or $t \supseteq s$,
3. for each $\alpha < \omega_1$, $T \cap \alpha$ is at most countable.

If we replace $n \in \omega$ by $n = 0, 1$ in (1) then we get a representation of normal binary $\omega_1$-trees. The order of each $s$ is $l(s)$. If $l(s)$ is a limit ordinal then $s = \U_{t \in b} t = \lim b$ where $b = \{t : t \subseteq s\}$ is a branch in $T|l(s)$; thus we shall often make no distinction between $b$ and $\lim b$. Also, we shall frequently use expressions like $T = \lim_{a < \omega_1} T | a$. All results we shall prove for normal trees hold for normal binary trees as well.
An isomorphism of two trees is a one-to-one mapping of $T_1$ onto $T_2$ which preserves the partial ordering. An automorphism of $T$ is an isomorphism of $T$ onto itself. A tree $T$ is called rigid if the only automorphism of $T$ is the identity. By $\sigma(T)$ we denote the cardinality of the set of all automorphisms of $T$.

A set $A$ of elements of $T$ is an antichain of $T$ if any two elements of $A$ are incomparable. A normal $\omega_1$-tree is called a Suslin tree if it has no uncountable antichain.

Before proceeding to our first theorem we make two more remarks on the terminology. A subtree naturally means a subset of a tree with the same partial ordering. An element $x \in T$ is said to split in $T$ if there are at least two immediate successors of $x$ in $T$.

Theorem 1. If $T$ is a normal $\omega_1$-tree then either $\sigma(T)$ is finite or $2^{\aleph_0} \leq \sigma(T) \leq 2^{\aleph_1}$. Moreover, if no subtree of $T$ is a Suslin tree then either $\sigma(T)$ is finite or $\sigma(T) = 2^{\aleph_0}$ or $\sigma(T) = 2^{\aleph_1}$.

Proof. Let $U$ be the set of all $x \in T$ such that $T_x$ is not rigid. $U$ is a subtree of $T$ and if $x < y$ and $y \in U$ then $x \in U$. Note that every automorphism of $T$ is an automorphism of $U$. We shall consider two cases.

Case I. $U$ has an infinite antichain $A$. For each $x \in A$, $T_x$ has at least one nontrivial automorphism $\pi_x$. If $Y \subseteq A$, let $\pi_Y$ be the automorphism of $T$ defined as follows: for $z \in \bigcup \{T_x : x \in Y\}$, let $\pi z = \pi_X z$ where $x \in Y$ is the unique $x$ such that $z \in T_x$; otherwise, let $\pi z = z$. For different $Y \subseteq A$ we get different $\pi_Y$; it follows that $\sigma(T) \geq 2^{\aleph_0}$.

Case II. $U$ does not have an infinite antichain. Then only finitely many elements of $U$ split in $U$ and $U$ has only finitely many branches. Let $U_0$ be the set of all $x \in T$ with the property that there exist two different immediate successors $x_1$ and $x_2$ of $x$, such that both $T_{x_1}$ and $T_{x_2}$ are rigid, and $T_{x_1}$ is isomorphic to $T_{x_2}$. Clearly, $U_0 \subseteq U$, and since $U$ has only finitely many branches, $U_0$ is cofinal in $U$. Now we consider three subcases:

Subcase (a). $U_0$ is infinite. Since $U_0$ does not have infinite antichains $U_0$ has an infinite chain $C$ (a linearly ordered subset). For each $x \in C$, let $x_1$, $x_2$ be successors of $x$, as in the definition of $U_0$, and let $\pi_x$ be an isomorphism of $T_{x_1}$ and $T_{x_2}$. If $Y \subseteq C$, let $\pi_Y$ be the automorphism of $T$ defined as follows: for $z \in \bigcup \{T_{x_1} : x \in Y\}$ let $\pi z = \pi_X z$, if $z \in \bigcup \{T_{x_2} : x \in Y\}$ let $\pi z = \pi_X^{-1} z$ (in either case, $x$ is unique); otherwise, let $\pi z = z$. For different $Y \subseteq A$ we get different $\pi_Y$; it follows that $\sigma(T) \geq 2^{\aleph_0}$.

Subcase (b). For each $x \in U$, let $s(x)$ be the set of all immediate successors $y$ of $x$ such that $T_y$ is rigid and is isomorphic to some $T_y'$, where $y' \neq y$ is another immediate successor of $x$. Assume that for some $x \in U$, $s(x)$ is infinite. It is obvious that $\sigma(T_x) \geq 2^{\aleph_0}$.
Subcase (c).  $U_0$ is finite and for each $x \in U_0$, $\sigma(x)$ is finite.  In this case, $\sigma(T)$ is finite.

Now suppose that the additional assumption of the theorem holds.  Let us consider the following two cases:

Case I.  At most countably many elements of $U$ split in $U$.  Let $\alpha < \omega_1$ be such that no element of $U$ of order $\alpha$ or bigger splits in $U$.  Then $U$ has only countably many branches longer than $\alpha$.

Subcase (a).  $U$ has an $\omega_1$-branch $B$.  We let $U_0 \subseteq U$ be as in Case II above.  $U_0$ is cofinal in $B$ and hence $U_0$ contains an uncountable chain.  By the same argument as in Subcase II(a) above, we get $2^{\aleph_1}$ automorphisms of $T$; thus $\sigma(T) = 2^{\aleph_1}$.

Subcase (b).  All branches of $U$ are countable.  Hence all branches of $U$ are shorter than some $\beta < \omega_1$.  Consequently, $\sigma(T)$ is at most the number of all permutations of the $\beta$th level, i.e. $\sigma(T) \leq 2^{\aleph_0}$.

Case II.  Uncountably many elements of $U$ split in $U$.  Using the assumption about Suslin trees, we shall show that $U$ has an uncountable antichain.  By the same argument as earlier in this proof, we conclude that $\sigma(T) = 2^{\aleph_1}$.

Assume that $U$ does not have an uncountable antichain; we shall construct a subtree which is a Suslin tree.  Let $V$ be the set of all $x \in U$ such that there are uncountably many $y \in U$ such that $y \geq x$ and $y$ splits in $U$.  $V$ has elements of arbitrary high order, hence $V$ is uncountable; also, if $x \in V$ and $y \leq x$ then $y \in V$.  We shall show that there is some $z \in V$ such that $T_z \cap U \subseteq V$.  If not, then $U - V$ is cofinal in $U$.  However, let $A$ be a maximal antichain in $U - V$, i.e. $A \subseteq U - V$ is an antichain and every $y \in U - V$ is comparable to some $a \in A$.  Since $A$ is at most countable, let $z \in V$ be of order bigger than the orders of all $a \in A$.  Let $y \geq z$, $y \in U - V$.  Now $y$ is comparable to some $a \in A$, i.e. $y \geq a$, and hence $z \geq a$; we reached the contradiction, since $z \in V$ and $a \notin V$.  Having $z \in V$ such that $T_z \cap U = T_z \cap V$, let $W = T_z \cap U$.  For each $x \in W$, there are uncountably many $y \in W$, $y \geq x$, such that $y$ splits in $W$.  Let $X$ be the set of all $y \in W$ which split in $W$.  $X$ is an uncountable tree, every $y \in W$ splits, every $T_x \cap X$ ($x \in X$) is uncountable and $X$ has no uncountable antichain.  As a matter of fact, $X$ is almost a Suslin tree, except that its elements may not have $\aleph_0$ immediate successors; all we know is that they split.  Thus we take $Y \subseteq X$ consisting of all elements of $X$ whose order (in the tree $X$) is a limit ordinal.  Now, $Y$ is a Suslin tree.

2. Suslin trees.  If $T$ is a Suslin tree then by Theorem 1, either $\sigma(T)$ is finite or $2^{\aleph_0} \leq \sigma(T) \leq 2^{\aleph_1}$.  If we are interested in what values $\sigma(T)$ takes, the best we can hope for is a relative consistency result.  This is because the existence of Suslin trees is independent of the axioms of Zermelo-Fraenkel set theory ZFC (see [15], [2], [13], [5], [7]).
The question of automorphism properties of Suslin trees was first investigated by Jensen in [6]. He proved that his axiom (\(\dagger\)) implies the existence of rigid Suslin trees and Suslin trees with \(\mathcal{K}_1\) automorphisms; also that \(V = L\) implies the existence of Suslin trees with \(\mathcal{K}_2\) automorphisms. Here we generalize this result to show, by the way of consistency, that \(\sigma(T)\) can be anything it is allowed to be.

First we prove a refinement of Theorem 1 to show what \(\sigma(T)\) is allowed to be.

**Theorem 2.** If \(T\) is a normal \(\omega_1\)-tree and if \(\sigma(T)\) is infinite, then \(\sigma(T)^{\mathcal{K}_0} = \sigma(T)\).

**Proof.** Let \(\sigma(T) = \kappa\) and let \(U = \{x \in T: \sigma(T_x) = \kappa\}\). We consider two cases.

*Case I.** \(U\) has an infinite antichain. Then we can construct \(\kappa\) automorphisms of \(T\).

*Case II.** \(U\) does not have an infinite antichain.

  *Subcase (a).* \(U\) has an \(\omega_1\)-branch \(B\). We let \(U_0\) be the set of all \(x \in U\) with the property that there exist different immediate successors \(x_1, x_2\) of \(x\) such that neither \(x_1\) nor \(x_2\) are in \(U\) and \(T_{x_1}\) is isomorphic to \(T_{x_2}\). \(U_0\) is cofinal in \(U\), and hence we may use the argument from Subcase II(a) Theorem 1, to get \(\sigma(T) = 2^{\mathcal{K}_1}\).

  *Subcase (b).* Each branch of \(U\) is at most countable. Thus there exists \(\alpha < \omega_1\) such that \(U \subseteq T|\alpha\). Let \(x_n, n \in \omega\), be all the elements of the \(\alpha\)th level. Obviously,

  \[
  \prod_{n=0}^{\infty} \sigma(T_{x_n}) \leq \sigma(T).
  \]

On the other hand,

\[
\sigma(T) \leq 2^{\mathcal{K}_0} \prod_{n=0}^{\infty} \sigma(T_{x_n}).
\]

This is proved as follows: \(2^{\mathcal{K}_0}\) is the number of all permutations of \(x_n, n \in \omega\); for any \(n, m \in \omega\), the number of isomorphisms between \(T_{x_n}\) and \(T_{x_m}\) is at most \(\sigma(T_{x_n})\). Thus

\[
\prod_{n=0}^{\infty} \sigma(T_{x_n}) \leq \kappa \leq 2^{\mathcal{K}_0} \prod_{n=0}^{\infty} \sigma(T_{x_n}).
\]

Now, using the fact that \(\kappa \geq 2^{\mathcal{K}_0}\), we see that either \(\kappa = 2^{\mathcal{K}_0}\) or \(\kappa = \prod_{n=0}^{\infty} \lambda_n\) where \(\lambda_n\) are cardinals less than \(\kappa\). A simple cardinal arithmetic shows that \(\kappa^{\mathcal{K}_0} = \kappa\).

Before we present the consistency proof, we shall discuss briefly the method we use. We assume that the reader is familiar with the method of forcing. Given a transitive model \(\mathcal{M}\) of ZFC, the *ground model*, we consider a partially ordered set \((P, \leq)\) of *forcing conditions* in \(\mathcal{M}\) and assume that there exists a set \(G \subseteq P\) which is *generic* (over \(\mathcal{M}\)). Then \(\mathcal{M}[G]\) is a model of ZFC, the *generic extension*
of \( \mathbb{M} \). By saying that there is a generic extension of \( \mathbb{M} \) which satisfies a certain theorem we mean that there is a \((P, \leq)\) such that \( \mathbb{M}[G] \) satisfies that theorem whenever \( G \) is generic.

A generic extension has the same ordinals as the ground model. Under some circumstances, \( \mathbb{M}[G] \) also has the same cardinals and the cofinality function of \( \mathbb{M} \) as \( \mathbb{M} \). We recall the following well-known fact: Call two conditions \( p, q \) compatible if there is some \( r \) such that \( p \geq r \) and \( q \geq r \). If \( \kappa \) is a cardinal then \((P, \leq)\) is said to satisfy the \( \kappa \)-chain condition (\( \kappa \)-c.c.) if every set of pairwise incompatible conditions has power less than \( \kappa \). \((P, \leq)\) is countably closed if every countable set of pairwise compatible conditions has a lower bound.

**Lemma 2.1.** (a) If, in \( \mathbb{M} \), \( \kappa \) is a regular cardinal and \((P, \leq)\) satisfies the \( \kappa \)-c.c., then \( \kappa \) is a regular cardinal in \( \mathbb{M}[G] \).

(b) If \((P, \leq)\) is countably closed in \( \mathbb{M} \), then every countable set of ordinals in \( \mathbb{M}[G] \) is an element of \( \mathbb{M} \). Consequently, \( \omega_1^{\mathbb{M}[G]} = \omega_1^{\mathbb{M}} \).

Now we present the theorem:

**Theorem 3.** Let \( \mathbb{M} \) be a transitive model of ZFC which satisfies \( 2^{\aleph_0} = \aleph_1 \); let \( \kappa \) be a cardinal in \( \mathbb{M} \) and assume \( \kappa^{\aleph_0} = \kappa \) in \( \mathbb{M} \). Then there is a generic extension \( \mathbb{M}[G] \) which has the same cardinals and the same cofinality and satisfies the following:

There exists a Suslin tree \( T \) which admits exactly \( \kappa \) automorphisms.

Thus, if we choose \( \mathbb{M} \) such that, e.g., \( 2^{\aleph_1} = \aleph_{17} \) and \( \kappa = \aleph_{51} \), then, in \( \mathbb{M}[G] \), we have a Suslin tree \( T \) with \( \sigma(T) = \aleph_{51} \), although \( 2^{\aleph_0} = \aleph_1 \) and \( 2^{\aleph_1} = \aleph_{17} \).

**Proof.** We use the following set \((P, \leq)\) of forcing conditions: \( P \) is the set of all pairs \((T, f)\) where

(i) \( T \) is a countable normal binary tree, \( T \subseteq \omega_1 \),

(ii) \( \text{dom}(f) \subseteq \kappa \), \( |\text{dom}(f)| \leq \kappa^{\aleph_0} \), \( f \) is one-to-one and \( \text{rng}(f) \) is a group of automorphisms of \( T \),

(iii) \( (T_2, f_2) \leq (T_1, f_1) \) iff \( T_2 \) is an extension of \( T_1 \) and, for all \( i \in \kappa \), \( f_2(i) \geq f_1(i) \).

Let \( G \) be a fixed generic set of conditions. To begin with, it is easy to see that \((P, \leq)\) is countably closed, since if \( \{(T_n, f_n) : n \in \omega\} \) is pairwise compatible, then \( (T, f) = (\lim_{n \to \omega} T_n, \lim_{n \to \omega} f_n) \) is a condition stronger than each \((T_n, f_n)\). Thus every countable set of ordinals in \( \mathbb{M}[G] \) is in \( \mathbb{M} \). We shall show that \((P, \leq)\) satisfies the \( \kappa_2 \)-c.c., and consequently, by Lemma 2.1, \( \mathbb{M}[G] \) has the same cardinals and cofinality as \( \mathbb{M} \). First, a combinatorial lemma:

**Lemma 2.2.** Assume \( 2^{\aleph_0} = \aleph_1 \). Let \( W \) be a family of countable subsets of \( \kappa \), \( |W| > \aleph_1 \). Then there exists \( Z \subseteq W \), \( |Z| > \aleph_1 \), and a countable \( S \subseteq \kappa \) such that \( X \cap Y = S \) for any two distinct elements \( X, Y \) of \( Z \).
Proof. Obviously, it is sufficient to prove the lemma for \( \kappa = \omega_2 \). The argument is essentially due to Marczewski [10]. We can assume \( \kappa = \aleph_2 \). For each \( x \in W \), let \( X_\alpha (\alpha < \omega_1) \) be the \( \alpha \)-th element of \( X \) (in the natural order). There exists \( \alpha \) such that \( |\{X_\alpha : x \in W\}| > \aleph_1 \); let \( \alpha_0 \) be the least one. We can construct \( W_1 \subseteq W \), \( |W_1| > \aleph_1 \) such that, for any distinct \( x, y \in W_1 \), the \( \alpha_0 \)-th element of \( x \) is greater than all elements of \( y \) (or conversely). Since \( |\{X_\alpha : \alpha < \alpha_0\} : x \in W_1| \leq 2^{\aleph_0} = \aleph_1 \), there is \( Z \subseteq W_1 \), \( |Z| > \aleph_1 \) such that \( \{X_\alpha : \alpha < \alpha_0\} \) is the same for each \( x \in Z \).

With this lemma at hand, it is not difficult to see that if \( W \) is a set of conditions and \( |W| > \aleph_1 \), then there exist \( Z \subseteq W_1 \), \( |Z| > \aleph_1 \), and a countable \( S \subseteq \kappa \) such that

if \( (T, f) \neq (T', g) \in Z \), then \( T = T', \text{dom}(f) \cap \text{dom}(g) = S \) and \( f \upharpoonright S = g \upharpoonright S \).

Thus \( (T, f) \) and \( (T', g) \) are compatible for any pair of conditions in \( Z \); hence the \( \aleph_2 \)-c.c. is satisfied.

Let us define \( (\mathcal{F}, \mathcal{P}) \) by

\[
(\mathcal{F}, \mathcal{P}) = \lim_{(T, f) \in G} (T, f).
\]

Obviously, \( \mathcal{F} \) is a normal binary \( \omega_1 \)-tree and \( \mathcal{P} \) is a one-to-one map of \( \kappa \) onto a group of automorphisms of \( \mathcal{F} \). To show that \( \mathcal{F} \) is a Suslin tree, we prove the following lemma:

Lemma 2.3. If \( (T, f) \) is a condition of limit length \( \alpha \) and if \( A \) is a maximal antichain in \( T \), then there is a condition \( (T', f') \) stronger than \( (T, f) \), of length \( \alpha + 1 \), such that \( A \) is maximal in \( T' \).

The argument which we used in [2] (see also [3]) and which we will not repeat here, shows that this lemma is enough to show that \( \mathcal{F} \) is a Suslin tree.

Proof. There is a certain pattern in the arguments used in this and subsequent proofs. In all cases we construct a branch of a countable normal tree of limit length, satisfying certain conditions. Rather than describing the construction, which is in each case based on diagonalization, we shall refer to the following (extramathematical)

Diagonal Principle. If \( T \) is a countable normal tree of limit length, then there exists a branch through \( T \) which satisfies a countable number of prescribed conditions.

(E.g., given a countable set \( B \) of branches of \( T \), there exists a branch which is not in \( B \).)

As for Lemma 2.3, we can use the Diagonal Principle to show that for each \( x \in T \) there exists an \( \alpha \)-branch \( b_x \) through \( x \) such that, for all \( \pi \in \text{rng}(f) \), \( \pi(b_x) \) intersects \( A \) (this is because \( A \) is maximal). We let
Further we set $T' = T \cup B$ and $f'$ the obvious extension of $f$ onto $T'$. Clearly, $T'$ is a normal tree of length $\alpha + 1$, $\mathrm{rng}(f')$ is a group of automorphisms of $T'$, and $A$ is a maximal antichain in $T'$.

It remains to be proved that $\mathcal{T}$ admits at most $\kappa$ automorphisms.

Lemma 2.4. (In $\mathbb{M}(\mathcal{G})$). For each automorphism $\rho$ of $\mathcal{T}$ and each $x \in \mathcal{T}$ there exist $y > x$ and $\pi \in \mathrm{rng}(f')$ such that $\rho|_{\mathcal{T}^y} = \pi|_{\mathcal{T}^y}$.

Proof. Assume that the lemma is false, that there exist $\rho$ and $x \in \mathcal{T}$ such that, for each $\pi \in \mathrm{rng}(f')$ and each $y \in \mathcal{T}$, $y > x$, there is some $z \in \mathcal{T}$, $z > x$, such that $\rho(z) \neq \pi(z)$. In other words, there exists a name $\bar{\rho}$ for $\rho$, a condition $(T_0, f_0) \in G$ and $x_0 \in T_0$ such that

$$(T_0, f_0) \models \bar{\rho}$$

is an automorphism of $\mathcal{T}$

and, for each $(T, f) \leq (T_0, f_0)$, each $\pi \in \mathrm{rng}(f)$ and each $y \in T$, $y > x_0$, there exist a condition $(T', f') \leq (T, f)$ and some $z \in T'$, $z > y$, such that

$$(T', f') \models \bar{\rho}(z) \neq \pi(z).$$

We shall construct a sequence $(T_0, f_0) \geq (T_1, f_1) \geq \cdots \geq (T_n, f_n) \geq \cdots$ of conditions and sequences $x_0 < x_1 < \cdots < x_n < \cdots$, $x'_0 < x'_1 < \cdots < x'_n < \cdots$ of elements of $T_0$, $T_1$, $\cdots$, $T_n$, $\cdots$. Let $n \mapsto (u(n), v(n))$ be some reasonable one-to-one correspondence between $\omega$ and $\omega \times \omega$. Having constructed $(T_n, f_n)$, $x_n$ and $x'_n$, we let $(T_{n+1}, f_{n+1})$, $x_{n+1}$ and $x'_{n+1}$ be such that

(a) $x_{n+1} > x_n$, $x_{n+1} \in T_{n+1} - T_n$,

(b) $(T_{n+1}, f_{n+1}) \models \bar{\rho}(x_{n+1}) = x'_{n+1} \neq \pi(x_{n+1})$,

where $\pi$ is the $v(n)$th element of $\mathrm{rng}(f_{v(n)})$ (each $\mathrm{rng}(f)$ is supposed to have a fixed enumeration by integers). We let

$$(T_\omega, f_\omega) = \lim_{n \to \omega} (T_n, f_n), \quad b = \lim_{n \to \omega} x_n, \quad \tilde{b} = \lim_{n \to \omega} x'_n.$$

Clearly, $b$ and $\tilde{b}$ are $\alpha$-branches of $T_\omega$ (where $\alpha$ is the length of $T_\omega$), and $(T_\omega, f_\omega) \models \rho(b) = \tilde{b}$, whereas $\tilde{b} \neq \pi(b)$ for each $\pi \in \mathrm{rng}(f_\omega)$.

As easily seen, there is a tree $T \supseteq T_\omega$ of length $\alpha + 1$ such that $b \in T$, $\tilde{b} \notin T$ and $T$ is closed under all $\pi \in \mathrm{rng}(f_\omega)$. Namely, consider

$$U_\alpha = \{ nb_z : \pi \in \mathrm{rng}(f_\omega), \ z \in T_\omega \} \cup \{ nb : \pi \in \mathrm{rng}(f_\omega) \},$$

where, for each $z \in T_\omega$, $b_z$ is an $\alpha$-branch such that $\pi b_z \neq \tilde{b}$ for each $\pi \in \mathrm{rng}(f_\omega)$. Also, $f_\omega$ naturally extends to an $f$ for $T$. Since $b \in T$ and $\tilde{b} \notin T$, we have $(T, f) \models b \in \mathcal{T} \land \tilde{b} \notin T$; thus $(T, f) \not\models \rho$ is not an automorphism of $\mathcal{T}$, a contradiction.

Now, if $\rho$ is any automorphism of $\mathcal{T}$, then by Lemma 2.4 we can find a
maximal antichain $A$ in $\mathcal{F}$ such that, for each $y \in A$, there is $\pi \in \text{rng}(\mathcal{F})$ such that $\pi|\mathcal{F}_y = \pi_0|\mathcal{F}_y$. Since $|A| \leq \kappa$, we can conclude that $\mathcal{F}$ admits at most $\kappa$ automorphisms. As mentioned above, every countable subset of $\kappa$ in $\mathcal{M}[G]$ is in $\mathcal{M}$, so that $((\kappa^{\aleph_0})[G]) = (\kappa^{\aleph_0}) = \kappa$. This completes the proof of Theorem 3.

Next we prove the consistency of $2^{\aleph_1}$ isomorphism types of Suslin trees. We use Jensen’s axiom (A) which was invented to show that there are Suslin trees in $L$ and to get the results on automorphism properties of Suslin trees in $L$, cf. [5], [6].

The formulation of (A) we use here is as it appears in [8].

A set $C$ of ordinals less than a limit ordinal $\alpha$ is closed unbounded in $\alpha$ if it is closed and unbounded in the order-topology of the ordinals less than $\alpha$. A set $A \subseteq \omega_1$ is stationary if it intersects every closed unbounded subset of $\omega_1$.

Axiom (A). There is a sequence $\{b_\alpha: \alpha \in \omega_1\}$ of functions with the property that, for every function $b \subseteq \omega_1 \times \omega_1$, the set $\{\alpha \in \omega_1: b|\alpha = b_\alpha\}$ is stationary.

We recall that (A) holds in $L$.

**Theorem 4.** If (A) holds, then there are $2^{\aleph_1}$ different isomorphism types of Suslin trees.

**Proof.** (2) We shall construct binary Suslin trees $T_F$ for every $F \in \omega_1$ such that, for any $F \neq G$, $T_F$ and $T_G$ are not isomorphic. To do that we construct countable normal trees $T_f$ for each $f \in \omega_1$ such that $\pi(T_f) = \alpha + 1$ where $\alpha = \text{dom}(f)$, and that $T_f \subseteq T_g$ whenever $f \subseteq g$. Then we let $T_F = \lim_{f \subseteq F} T_f$ for each $F \in \omega_1$. The elements of $T_F$ will be countable ordinals.

Having constructed $T_f$, $f \in \omega_1$, we construct $T_0$ and $T_1$ simply by adjoining $\aleph_0$ immediate successors to each element of the last level of $T_f$; moreover, we can easily make $T_0 \neq T_1$. If $\alpha$ is a limit ordinal and $f \in \omega_1$, let $\hat{T}_f$ be the tree $\hat{T}_f = \lim_{g \leq f} T_g$ of length $\alpha$; clearly, if $f \neq g \in \omega_1$ then $\hat{T}_f \neq \hat{T}_g$. To construct $T_f$ we consider three cases.

***Case I.*** The set $A = \text{rng}(b_\alpha)$ is a maximal antichain in $\hat{T}_f$. Then we construct $T_f$ in such a way that $A$ is maximal in $T_f$ (this is to make sure that the resulting $\omega_1$-trees are Suslin trees, as in [5]); to get $T_f$ we use the Diagonal Principle.

***Case II.*** The function $b_\alpha$ is an isomorphism of $\hat{T}_f$ and some $\hat{T}_g$, $g \neq f$. Then we construct $T_f$ and $T_g$ in such a way that $b_\alpha$ cannot be extended to an isomorphism of $T_f$ and $T_g$ : For each $x \in \hat{T}_f$, let $b_x$ be an $\alpha$-branch of $\hat{T}_f$ through $x$ and let $b$ be an additional $\alpha$-branch of $\hat{T}_f$. We construct $T_f$ by adjoining one

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(2) Originally (cf. [4]) we proved this theorem with $\kappa_2$ instead of $2^{\aleph_1}$ and got the consistency of $2^{\aleph_1}$ by a different method. The present proof was suggested by K. Hrbáček.
element on the top of each branch $b_x$, and also of $b$, and $T_b$ by adjoining elements to the branches $b_x(b_x)$ only.

Case III. If neither I or II hold, we let $T_f$ be an arbitrary normal $(\alpha + 1)$-tree which extends $\hat{T}_f$.

Now, each $T_F$ is a Suslin tree, as mentioned in Case I. To show that $T_F$ and $T_G$ are not isomorphic if $F \neq G$, assume that $\pi$ is an isomorphism of $T_F$ and $T_G$. Let $\alpha_0$ be an ordinal such that $F|\alpha_0 \neq G|\alpha_0$. By (\ref{6}), there is a limit ordinal $\alpha > \alpha_0$ such that $\pi|\alpha$ is an isomorphism of $T_F|\alpha$ and $T_G|\alpha$ and that $\pi|\alpha = b_\alpha$. By Case II, however, $b_\alpha$ cannot be extended to an isomorphism of $T_F$ and $T_G$; a contradiction.

3. Remarks on Suslin continua. A Suslin continuum is a complete dense linear ordering whose intervals satisfy the countable chain condition but it is not imbeddable in the real continuum. The notions of Suslin trees and Suslin continua are closely related; one can easily construct a Suslin continuum from a Suslin tree and vice versa (cf. [3]).

Jensen [6] investigated the automorphism properties of Suslin continua in $L$ (or, assuming \ref{6}). The results are the same as mentioned for Suslin trees in \S 2, because there is a relationship between automorphisms of Suslin trees and of Suslin continua.

Suppose that $T$ is a normal binary Suslin tree, we may assume that $T \subseteq \omega_1^2$. Consider the set of all branches of $T$, ordered by the lexicographical ordering of $\omega_1^2$. This is a linearly ordered set; moreover, the ordering is complete. If we remove the endpoints and the left sides of all gaps, we get a Suslin continuum $S$.

To get the results for Suslin continua from those for Suslin trees, Jensen uses the following relationship between automorphisms of Suslin trees and of Suslin continua:

If $C$ is a closed unbounded subset of $\omega_1$, let $T_C = \{x \in T: o(x) \in C\}$.

**Lemma 3.1.** (a) If $C$ is a closed unbounded subset of $\omega_1$, if $\pi$ is an automorphism of $T_C$ and if $\pi$ preserves the lexicographical ordering of $\omega_1^2$ then $\pi$ determines a unique automorphism $\mathcal{H}$ of $S$.

(b) If $\phi$ is an automorphism of $S$ then there exist a closed unbounded set $C \subseteq \omega_1$ and an automorphism $\pi$ of $T_C$ such that $\phi = \mathcal{H}$.

The automorphism $\mathcal{H}$ is defined in the natural way, since the branches of $T_C$ are dense in $S$. It follows that if $C_1 \subseteq C_2$ and $\pi_1 = \pi_2|T_C^1$ then $\mathcal{H}_1 = \mathcal{H}_2$.

Using Lemma 3.1 we can prove the analogs of Theorems 3 and 4.

**Theorem 3a.** Under the assumptions of Theorem 3 there is a generic extension which preserves the cardinals and cofinality and satisfies the following:

There exists a Suslin continuum $S$ which admits exactly $\kappa$ automorphisms.
The proof is analogous to that of Theorem 3. We modify the forcing conditions as follows: Let $A$ be the set of all limit countable ordinals. We let $\text{rng}(\mathcal{f})$ to a group of automorphisms of $T^A$, preserving the lexicographical ordering of $\omega_1^2$. Otherwise, $(T, \mathcal{f})$ is as before. The proof proceeds similarly. The same argument shows that $\mathcal{I}$ is a Suslin tree; $\mathcal{f}$ provides us with $\kappa$ automorphisms of the corresponding Suslin continuum $S$. To show that $S$ admits at most $\kappa$ automorphisms, consider an automorphism $\phi$ of $S$. By Lemma 3.1 there is a closed unbounded subset $C$ of $\omega_1$ and an automorphism $\rho$ of $\mathcal{I}^C$ such that $\phi = \rho$. Now, we can prove an analog of Lemma 2.4. For each $x \in \mathcal{I}$ there exist $y > x$ and $\pi \in \text{rng}(\mathcal{f})$ such that

$$\rho |_{\mathcal{I}^C \cap A} = \pi |_{\mathcal{I}^C \cap A}.$$  

(The proof goes exactly as before, we only have to carry a name $C$ for $C$ throughout.) By the remark following Lemma 3.1 it is obvious that $\mathcal{I}$ and $\mathcal{I}$ agree in the interval of $S$ determined by $y$. Thus $\phi$ is obtained as a combination of countably many elements of $\text{rng}(\mathcal{f})$ and hence $S$ admits at most $\kappa$ automorphisms.

In a similar way, we can modify the construction of the trees $T_F$ in the proof of Theorem 4 to obtain

**Theorem 4a.** If (i) holds then there are $2^{\aleph_1}$ different isomorphism types of Suslin continua.

To construct $T_F$ when $\text{dom}(\mathcal{f})$ is a limit ordinal $\alpha$, we slightly change Case II: There exist $g \notin \mathcal{f}$ and a set $C$ closed unbounded in $\alpha$ and $b_\alpha$ is an isomorphism of $\mathcal{I}_F^C$ and $\mathcal{I}_G^C$. Otherwise, we proceed as before. We establish that, for no closed unbounded subset $C$ of $\omega_1$, $T_F^C$ and $T_G^C$ are isomorphic if $F \neq G$. This implies that the Suslin continua corresponding to $T_F$ and $T_G$ are not isomorphic.

4. Kurepa trees. We may ask whether results similar to those in §2 can be proved for other types of $\omega_1$-trees, particularly for Kurepa trees.

A Kurepa tree is a normal $\omega_1$-tree which has at least $\aleph_2$ $\omega_1$-branches. By Silver [11], the existence of Kurepa trees is not provable in ZFC. However, by Stewart [14] and Solovay [12], the existence of Kurepa trees is consistent relative to ZFC. We prove the following theorem.

**Theorem 5.** The following is consistent relative to ZFC:

(a) there is a rigid Kurepa tree,

(b) there is a Kurepa tree $T$ with $\sigma(T) = 2^{\aleph_0}$,

(c) there is a Kurepa tree $T$ with $\sigma(T) = 2^{\aleph_1}$,

(d) there are $2^{\aleph_1}$ different isomorphism types of Kurepa trees.

Using hybrids of Kurepa and Suslin trees we could possibly get Kurepa trees with $\sigma(T)$ between $2^{\aleph_0}$ and $2^{\aleph_1}$, but if we are interested purely in Kurepa trees
then the expected infinite values of $\sigma(T)$ are $2^{\aleph_0}$ and $2^{\aleph_1}$.

**Proof.** We could prove (a) and (c) by refinement of Stewart's and Solovay's arguments. However, we use a different approach. First, the following lemma yields (c).

**Lemma 4.1.** If $T$ is a Kurepa tree then there is a Kurepa tree $T'$ with $\sigma(T') = 2^{\aleph_1}$.

**Proof.** Let $T_0$ be the following binary tree:

$$T_0 = \{ s \in \omega_1 \mid s(\alpha) = 1 \text{ for only finitely many } \alpha \}.$$

$T_0$ is a normal binary $\omega_1$-tree. Moreover, $\sigma(T_0) = 2^{\omega_1}$: for $f \in \aleph_1^1$, let $\pi_f$ be the following automorphism of $T_0$:

$$\pi_f(s) = \begin{cases} 
1 - s(\xi) & \text{if } f(\xi) = 0, \\
1 - s(\xi) & \text{if } f(\xi) = 1.
\end{cases}$$

Clearly, $\pi_f \neq \pi_g$ if $f \neq g$.

Now we define $T' = T \otimes T_0$ as follows:

$$T' = \{ (u, v) \mid u \in T, v \in T_0 \text{ and } \sigma(u) = \sigma(v) \},$$

$$\langle u, v \rangle \leq \langle u', v' \rangle \text{ iff } u \leq u' \text{ and } v \leq v'.$$

$T'$ is a normal $\omega_1$-tree. For each automorphism $\pi$ of $T_0$ we have an automorphism $\pi'$ of $T'$ defined as follows:

$$\pi'(u, v) = (u, \pi(v)).$$

Thus $\sigma(T') = 2^{\aleph_1}$. And finally, $b$ is a branch in $T'$ iff there are branches $b_1$ and $b_2$ in $T$ and $T_0$ such that $b = \{(u, v) \in T' \mid u \in b_1 \text{ and } v \in b_2\}$. Hence $T'$ is a Kurepa tree.

Next, call an $\omega_1$-tree $T$ **totally rigid** if for no $x \neq y \in T$, $T_x$ and $T_y$ are isomorphic. We show

**Lemma 4.2.** If $T$ is a Kurepa tree and if ($\xi$) holds, then there is a totally rigid Kurepa tree.

The assumptions of Lemma 4.2 are compatible since both are true in $L$.

**Proof.** We may assume that the elements of $T$ are countable ordinals and that there are uncountably many countable ordinals which are not elements of $T$. By recursion on the levels of $T$ we shall construct a tree $T'$ on countable ordinals which is "fatter" than $T$ and is totally rigid.
Let \( \{b_\alpha : \alpha < \omega_1\} \) be a sequence of functions given by the axiom (1). Suppose that we have already constructed \( T' | \alpha \ (\alpha < \omega_1) \); by the induction hypothesis, \( T' | \alpha \rightarrow T | \alpha \). We consider three cases.

**Case I.** For some \( \beta, \alpha = \beta + 1 \). Then we construct the \( \alpha \)-th level of \( T' \) in such a way that it includes the \( \alpha \)-th level of \( T \), and that every element of the \( \beta \)-th level of \( T' \) splits into \( \kappa \)-successors.

**Case II.** \( \alpha \) is a limit number, and for no \( x \neq y \in T' | \alpha \), \( b_\alpha \) is an isomorphism of \((T' | \alpha)_x \) and \((T' | \alpha)_y \). Then we construct the \( \alpha \)-th level of \( T \) such that for each \( x \in T' | \alpha \) there is some \( y > x \) in the \( \alpha \)-th level of \( T' \).

**Case III.** \( \alpha \) is a limit number, and there exist \( x \neq y \in T' | \alpha \) such that \( b_\alpha \) is an isomorphism of \((T' | \alpha)_x \) and \((T' | \alpha)_y \). First we do the same as in Case II; now, when we have extended countably many branches of \( T' | \alpha \), we may use the Diagonal Principle to find a branch \( b \) such that \( b \) goes through \( x \), \( b_\alpha(b) \neq b \) and \( b_\alpha(b) \) is not among those branches extended so far. In addition to the branches extended so far, we extend \( b \), to complete the construction of the \( \alpha \)-th level.

Clearly, \( b_\alpha \) does not extend to an isomorphism of \((T' | \alpha + 1)_x \) and \((T' | \alpha + 1)_y \).

We let \( T' = \lim_{\alpha \rightarrow \omega_1} T' | \alpha \). \( T' \) is totally rigid; for, if \( b \) is an isomorphism of \( T_x \) and \( T_y \) \((x \neq y)\), then by (1), there is a limit ordinal \( \alpha \) such that \( b | \alpha = b_\alpha \) and \( b_\alpha \) is an isomorphism of \((T' | \alpha)_x \) and \((T' | \alpha)_y \); this has been taken care of in Case III. Also, \( T' \) is a normal \( \omega_1 \)-tree, and obviously, \( T' \) has at least those \( \omega_1 \)-branches which \( T \) has; hence \( T' \) is a totally rigid Kurepa tree.

Now, we can easily prove (b) and (d) of Theorem 5.

(b) Consider the following tree \( T' \):

\[
\begin{array}{|c|c|c|c|c|}
\hline
\alpha & \beta & \gamma & \delta & \cdots \\
\hline
\end{array}
\]

where \( T \) is some rigid Kurepa tree. Obviously, \( \sigma(T') = 2^{\aleph_0} \).

(d) Let \( T \) be a totally rigid binary Kurepa tree. For every \( A \subseteq \omega_1 \), we construct a binary Kurepa tree \( T_A \) as follows. We start with a single \( \omega_1 \)-branch \( b = \{x_\alpha : \alpha < \omega_1\} \) and, to each \( x_\alpha \in b \), we graft a tree as follows, depending on whether \( \alpha \in A \) or not.

Clearly, if \( A \neq A' \) then \( T_A \) and \( T_A' \) are not isomorphic.
5. A remark on Aronszajn trees. An Aronszajn tree is a normal $\omega_1$-tree which has no $\omega_1$-branch. The first construction of such a tree was given by Aronszajn (cf. [9]); Gaifman and Specker [1] proved that there are $2^{\aleph_1}$ different isomorphism types of Aronszajn trees.

Given an Aronszajn tree $T$, we can use the argument in Lemma 4.1 to construct an Aronszajn tree $T = T \otimes T_0$ such that $\sigma(T') = 2^{\aleph_1}$.

If $T$ is a normal $\omega_1$-tree with $\sigma(T) < 2^{\aleph_1}$ and $T$ is not a Suslin tree then, for some $x \in T$, $T_x$ is rigid. By [13] the existence of Suslin trees is not provable in ZFC, so that we cannot prove the existence of a tree with $\sigma(T) < 2^{\aleph_1}$ without proving the existence of a rigid tree.

We conclude with the following problem; we are inclined rather to believe in the negative solution:

Problem. Is it provable in ZFC that there exists a rigid normal $\omega_1$-tree?

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