ABSTRACT. In the search for an easily-classified Baire set of diffeomorphisms, all the studied classes have had the property that all maps close enough to any diffeomorphism in the class have the same number of periodic points of each period. The author constructs an open subset $U$ of $\text{Diff}^r(T^3)$ with the property that if $f$ is in $U$ there is a $g$ arbitrarily close to $f$ and an integer $n$ such that $f^n$ and $g^n$ have a different number of fixed points. Then, using the open set $U$, he illustrates that having a rational zeta function is not a generic property for diffeomorphisms and that $\Omega$-conjugacy is an ineffective means for classifying any Baire set of diffeomorphisms.

A. Introduction and statement of theorems. Let $\text{Diff}^r(M^n)$ be the space of $C^r$ diffeomorphisms of a compact $C^\infty$ $n$-manifold $M$ with the $C^r$ topology, $1 \leq r \leq \infty$. Central problems in the study of differentiable dynamical systems, as formulated by Smale ([24], [26]) are

(a) Find a Baire subset $B$ of $\text{Diff}^r(M^n)$ with strong stability properties.

(b) Find a practical means of classifying the elements of $B$.

Let $f \in \text{Diff}^r(M)$. The nonwandering set of $f$, $\Omega(f)$, is the invariant set $\{x \in M: \text{for any neighborhood } U \text{ of } x \text{ there is a positive integer } n \text{ with } f^n U \cap U \neq \emptyset\}$. $f$ satisfies Axiom A if the periodic points of $f$ are dense in $\Omega(f)$ and if $\Omega(f)$ has a hyperbolic structure, i.e., there is an invariant splitting of the tangent bundle of $M$ restricted to $\Omega(f)$

$$TM|_{\Omega(f)} = E^s \oplus E^u$$

with $Tf: E^u \to E^u$ an expansion and $Tf: E^s \to E^s$ a contraction. Hirsch and Pugh [9] have shown that if $f$ satisfies Axiom A, then for each $x \in \Omega(f)$ the stable manifold of $x$, $W^s(x, f) = \{y \in M: d(f^m x, f^m y) \to 0 \text{ as } m \to \infty\}$, is a smooth, injectively immersed open cell through $x$ and depends smoothly on $x$ and $f$. The unstable manifolds of $f$, $W^u(x, f)$, are the stable manifolds of $f^{-1}$. $f$ is structurally stable ($\Omega$-stable) if for each $g$ in some neighborhood of $f$ in $\text{Diff}^r(M)$ there is a homeomorphism $h: M \to M$ ($h: \Omega(f) \to \Omega(g)$) with $gh = hf$ on $M$ (on $\Omega(f)$). A generic property is a property that holds for a Baire subset of $\text{Diff}^r(M)$. For a general
Finally, the reader is referred to [8] and [31] for the definition and properties of a $k$-foliation $F$ on $M$. $f: M \rightarrow M$ respects the foliation $F$ if the image of a leaf of $F$ by $f$ is another leaf of $F$. $f$ preserves the foliation $F$ if $f$ maps each leaf onto itself.

To put the results of this paper into perspective, we discuss briefly the recent history of problems (a) and (b). There have been a number of unsuccessful candidates for $B$, beginning with Morse-Smale maps, [20], i.e., diffeomorphisms whose nonwandering set is hyperbolic and consists of a finite number of points, whose stable and unstable manifolds intersect only transversally (strong transversality condition). Such maps were later shown to be structurally stable [15] but by no means dense in $\text{Diff}'(M)$ ([22], [24]). Smale showed that structurally stable maps are not dense in [23], where he conjectured that diffeomorphisms that satisfy Axiom A and the strong transversality condition might form a Baire subset of $\text{Diff}'(M)$. Later, he demonstrated [25] that maps satisfying Axiom A and the "no-cyle property" were $\Omega$-stable. However, in 1968 Abraham and Smale [2] showed that neither $\Omega$-stable maps nor ones satisfying Axiom A form a Baire subset of $\text{Diff}'(M^n)$ for $r \geq 1$, $n \geq 4$. Newhouse [13] has the corresponding result for $r \geq 2$, $n = 2$. However, both Abraham and Smale [26] have emphasized that many more such counterexamples must be constructed and analyzed for the theory to advance, especially since each new conjecture for $B$ has arisen from careful analysis of past counterexamples. The examples we construct in this paper are the first $C^1$ counterexamples to the genericity of Axiom A and $\Omega$-stability on 3-manifolds. More significantly, all the above classes of diffeomorphisms conjectured to solve problem (a) have had the following property: all maps close enough to any diffeomorphism in the class have the same number of periodic points of each period as the original map. Theorem 1 below illustrates that this is not a generic property, i.e. there is an open set in $\text{Diff}'(T^3)$ with the property that as close as you wish to any map in the set there is another map with a different number of periodic points of some period.

**Theorem 1.** Let $1 \leq r < \infty$. For $f \in \text{Diff}'(T^3)$ and positive integer $n$, let $N_n(f) =$ number of fixed points of $f^n = f \circ \cdots \cdots (n \text{ times} ) \cdots \circ f: T^3 \rightarrow T^3$. Then, there exists an open set $U$ in $\text{Diff}'(T^3)$ such that if $f_0 \in U$ and $U_0$ is any neighborhood of $f_0$ in $U$, there are $f_1 \in U_0$ and integer $n$ such that $N_n(f_0) \neq N_n(f_1)$ and all periodic points of $f_1$ of period $\leq n$ are hyperbolic.

The proof of Theorem 1 is contained in §B–K. First, let us see what effect it has on problem (b), the classification problem. In [24], Smale conjectured that an effective means of classifying the maps in $B$ might be the zeta function. The zeta function of a diffeomorphism $f$ is

$$\zeta(f) = \zeta_f(t) = \exp \left( \sum_{i=1}^{\infty} \frac{N_i(f)}{i} t^i \right)$$

where $N_i = N_i(f)$.
as in Theorem 1. Artin and Mazur [3] demonstrated that a dense (not Baire) set of
diffeomorphisms have zeta functions with a positive radius of convergence. Meyer
[12] and Shub [19] showed that if \( f \) satisfies Axiom A, \( \zeta_f(t) \) has a positive radius
of convergence. Williams [28] demonstrated that if \( \Lambda \) is a hyperbolic attractor of
\( f \), \( \zeta_f(\Lambda) \) is rational. Bowen and Lanford ([4], [5]) showed the same for \( \Lambda \) zero-
dimensional and hyperbolic. Recently, Guckenheimer [7] has shown that if \( f \) satis-
ﬁes Axiom A and the no-cycle property, \( \zeta_f(\epsilon) \) is rational. However, in order to be
at all effective and practical as a means of classiﬁcation, \( \zeta_f(\epsilon) \) must be rational for
a Baire set of diﬀeomorphisms. Whether or not \( \zeta_f(\epsilon) \) is generally rational was asked
in [24, Problem 4.5], [29], [27], and [28]. Theorem 2 uses Theorem 1 to answer this ques-
tion.

Theorem 2. Diﬀeomorphisms with rational zeta functions do not form a Baire
subset of \( \text{Diﬀ}^r(T^3) \), \( 1 \leq r \leq \infty \).

Proof of Theorem 2. Since there are only a countable number of rational zeta
functions [5], enumerate them as \( Z_1, Z_2, \ldots \). Say \( Z_j(t) = \exp(\sum_{i=1}^{\infty} N_i^j t^i/i) \).

Let \( U \) be the open set in \( \text{Diﬀ}^r(T^3) \) from Theorem 1. Let \( V_j = \{ f \in U \mid \text{for some } k \}
\in \mathbb{N}, (1) N_k(f) \neq N_k^j \) and (2) \( f^k \) has only hyperbolic ﬁxed points \( \ell \). So, if \( f \in V_j \),
\( \zeta(f) \neq Z_j \). By the hyperbolicity in the deﬁnition of \( V_j \), each \( V_j \) is open. We claim
each \( V_j \) is also dense. Then, we will have \( V = \bigcap V_j \), a Baire subset of \( U \); and no
diffeomorphism in \( V \) can have a rational zeta function.

Suppose the above claim is false, i.e. that there is an open set \( W \) in \( U \) with
\( W \cap V_j = \emptyset \). By the Kupka-Smale Theorem [21], there is \( g_1 \in W \) with all periodic
points hyperbolic. Since \( g_1 \notin V_j \), \( N_k^j(g_1) = N_k^j \) for all \( k \). By Theorem 1, there are
\( g_2 \in W \) and integer \( i \) with \( N_i^j(g_2) \neq N_i^j(g_1) = N_i^j \) and \( \text{Fix}(g_2^i) \) hyperbolic. Thus, \( g_2 \in V_j \), contradicting \( W \cap V_j = \emptyset \).

Finally, Theorem 3 below deals with another aspect of the classiﬁcation prob-
lem. It states that \( \Omega \)-conjugacy is not a reasonable equivalence relation to use in
classifying diﬀeomorphisms. The same result holds for any equivalence relation
which has all \( N_n(f) \) constant in each equivalence class. The proof of Theorem 3
is the same as that of Theorem 2 with \( V_j \) replaced by \( \{ f \in U \mid \text{for some } k \in \mathbb{N}, (1)
\in \mathbb{N}, \text{ and } (2) f^k \) has only hyperbolic ﬁxed points \( \ell \).

Theorem 3. There do not exist a countable set \( \{ h_j \} \) and a Baire subset \( B \) in
\( \text{Diﬀ}^r(T^3) \) such that each \( f \) in \( B \) is \( \Omega \)-conjugate to some \( h_j \).

Let us outline the construction used to prove Theorem 1. In §B, we construct
a hyperbolic “D-A” diﬀeomorphism \( g \) of \( T^2 \). \( \Omega(g) \) consists of a ﬁxed point source
\( \theta \) and a one-dimensional expanding attractor \( \Sigma \). The one-dimensional \( \{ W^r(x, g) \mid x \in \Sigma \} \)
ﬁll up \( T^2 \setminus \{ \theta \} \) and extend to a \( g \)-invariant foliation \( S \) of \( T^2 \). If \( b : S^1 \to S^1 \) has
\( \{ +1 \} \) as a ﬁxed point source, \( g \times b \) is a diﬀeo of \( T^3 \) respecting the foliation \( \mathcal{F} \)
whose leaves are a product of \( S^1 \) and the leaves of \( S \). In §D, we construct \( b : T^3 \)
which is the identity on $\Sigma_1 = \Sigma \times \{+1\}$, which preserves $\mathcal{F}$, and which forces
the two-dimensional local unstable manifolds of points of $\Sigma_1$ to intersect the one-
dimensional stable manifolds from $\Sigma_1$ transversally. $\mathcal{F}$ is a normally-hyperbolic
foliation ($\S F$) for $f = b \circ (g \times h)$. So, maps near $f$ will respect foliations $\mathcal{F}'$ near $\mathcal{F}$.

In $\S D$, we single out an open subset $B_1$ of $\Sigma_1$ and for each $x \in B_1$ a 2-disk
$F(x)$ in the leaf of $\mathcal{F}$ through $x$, so that $\bigcup F(x) : x \in B_1 \bigcup$ is a 3-disk. Each $f|_F(x)$
contains a Smale "horseshoe" as drawn in Figures 6, 9, and 10, yielding a one-param-
eter family of horseshoe maps. In $\S D$, I, and J, we show how an arbitrarily small
change in $f$ can radically change the topological type of one of these horseshoes so
that, for some $x$, $f|_F(x)$ will have a different number of periodic points than the
corresponding $f'|_F(x')$. In $\S J$, we achieve the hyperbolicity of Theorem 1 by using
the Kupka-Smale Theorem. See also [32].

Theorems 1, 2, 3 hold at least for all manifolds which are the product of $T^2$
with any manifold. The author has benefited from many valuable and encouraging
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spiration and counsel of R. F. Williams.

B. Anosov diffeomorphisms and derived-from-Anosov diffeomorphisms. Let $A_0$
be a $2 \times 2$ matrix with all integer entries, determinant 1, and no eigenvalues of
norm one. $A_0$ induces a hyperbolic automorphism $A$ of the 2-torus via the canonical
quotient map $\pi: R^2 \rightarrow T^2$. $A_0$ has eigenvalues $\lambda, \mu$ with $0 < |\lambda| < 1 < |\mu|$ and eigen-
spaces $L_0, M_0$ respectively. Let $\mathcal{L}$ and $\mathcal{M}$ be the families of all lines in $T^2$ parallel
to $\pi(L_0)$ and $\pi(M_0)$ respectively. $\mathcal{L}$ and $\mathcal{M}$ become the stable and unstable mani-
folds for $A$ giving us two transversal foliations of the torus. For example, $W^s(\theta, A) = \pi(L_0)$ where $\theta = \pi(0, 0)$.

We now construct a $C^0$ perturbation of $A$, using a surgery described by Smale
[24] and Williams [30].

Theorem (Smale-Williams). Let $A: T^2 \rightarrow T^2$ be a hyperbolic toral automorphism.
Then there exists $g: T^2 \rightarrow T^2$ such that
(a) $g$ is smoothly isotopic to $A$,
(b) nonwandering set $\Omega(g) = \{\theta\} \cup \Sigma$, where $\theta = \pi(0,0)$ is a point source and
$\Sigma$ is a one-dimensional attractor with hyperbolic structure,
(c) the stable manifolds of $g|_\Sigma$ are the lines of $\mathcal{L}$ except for $L_0$ which divided
by $\theta$ now forms two stable manifolds,
(d) $g$ respects the foliation $\{W^s(x, A): x \in T^2\}$.

$g$ is usually called a D-A map, since it is derived from the Anosov diffeomor-
phism $A$. In the construction of $g$, one chooses a small rectangle $Q$ (in the canoni-
cal coordinates of [24]) about $\theta$. Then, $g = \phi \circ A$ where $\phi$ is a $C^\infty$ diffeomorphism
of $T^2$ that is the identity outside $Q \cap A(Q)$ and on $D_0$, the path component of $M_0$.
\( \cap Q \) containing \( \theta \). One requires that \( \phi(C) = C \) for each path component \( C \) of members of \( \overline{Q} \) in \( Q \) and that, on each \( C \), \( \phi \) is expanding away from \( D_0 \cap C \). The expansive constant of \( \phi \) on the path component of \( L_0 \cap Q \) containing \( \theta \) need be greater than \( \mu \). In effect, one changes \( A \) on \( Q \) so that \( g \) has 2 saddle-like fixed points \( \{ x_0, \overline{x}_0 \} \) in \( Q \) and one point source \( \theta \), as in Figure 1; while \( A \) had only one fixed point in \( Q \), the saddle point \( \theta \). Williams ([27], [30]) has shown that \( \Sigma \), a "generalized solenoid," is locally the product of a Cantor set and an interval, periodic points of \( g \) are dense in \( \Sigma \), \( W^u(\Sigma, g) = \Sigma \), and \( \Sigma = \overline{W^u(x_0, g)} \).

![Figure 1](image)

The leaves of our foliation are now the generalized stable manifolds of points of \( \Sigma \) with the exception that \( W^s(x_0, g) \cup W^s(\overline{x}_0, g) \cup \{ \theta \} \) forms one leaf. Now \( \Sigma \) is a basic set for \( g \), i.e. a closed invariant subset of \( \Omega(g) \) with a hyperbolic structure, a dense orbit, and a dense subset of periodic points. So, \( T^\Sigma M \) has an invariant splitting \( E^+ \oplus E^- \) and there are constants \( 0 < \lambda_1 < 1 < \mu_1 \) such that \( |TgX| \leq \lambda_1 |X| \) for \( X \in E^- \) and \( |TgX| \geq \mu_1 |X| \) for \( X \in E^+ \). By choosing \( \phi \) so that the rate of expansion of \( g = \phi \circ A \) on all the above-mentioned intervals \( C \) is less than \( \mu_2 \) where \( 1 < \mu_2 < \mu_1 \), one makes the rate of expansion normal to the foliation larger than any rate of expansion on any leaf.

Consider now \( g^k \) for any integer \( k > 0 \). \( \Omega(g^k) = \Omega(g) \). \( x_0 \) is a fixed point of \( g^k \) and \( W^s(x, g) = W^s(x, g^k) \) for all \( x \in \Omega(g) \). \( g^k \) respects the above foliation. In addition, \( |T(g^k)X| < \lambda_1^k |X| \) for \( X \in E^- \) and \( |T(g^k)X| > \mu_1^k |X| \) for \( X \in E^+ \). If \( r \) in Theorem 1 is finite, choose \( k \) so that \( \mu_1^k > 4^r \) and \( \lambda_1^k < \frac{1}{4} \). If \( r = \infty \), make \( \mu_1^k > 16 \). \( g^k \) will be denoted as \( g \) in the remainder of this paper.

C. \( g \times h: T^3 \rightarrow T^3 \). Let \( h: S^1 \rightarrow S^1 \) be a \( C^\infty \) diffeomorphism of the circle with exactly two (hyperbolic) fixed points: \( \{ +1 \} \) a source and \( \{ -1 \} \) a sink. Choose \( b \) so that \( T_{+1} b(s) = as \) where \( 3 < a < 4 \) and \( b \) increases no arc of \( S^1 \) by a factor greater than 4.
$g \times h$ is a hyperbolic $C^\infty$ diffeomorphism of the 3-torus, $T^3 \cong T^2 \times S^1$. Since

$$\Omega(g \times h) = \Omega(g) \times \Omega(h) [24, \S 10], \Omega(g \times h) = \Sigma \times \{+1\} \cup \{\theta, +1\} \cup \Sigma \times \{-1\} \cup \{\theta, -1\}.$$

For convenience, we introduce the following notation: $T^2_+ = T^2 \times \{+1\}$, $\Sigma = \Sigma \times \{+1\} \cup \{\theta, 1\} \cup \Sigma \times \{-1\} \cup \{\theta, -1\}$.

Since $g$ respected the foliation $\{W^s(x, A)\}$ on $T^2$, $g \times h$ respects the foliation $\{W^s(x, A) \times S^1\}$ on $T^2 \times S^1$. We will denote this $C^\infty$ foliation with cylindrical leaves by $\mathcal{F}$ and the leaf of $\mathcal{F}$ containing $x \in T^3$ by $F(x)$.

Note also that, around the fixed point $x_0$, $W^s(x_0, g \times h)$ is a 1-disk lying in $T^2_+$ and equal to $W^s(x_0, g_+)$, $W^u(x_0, g \times h)$ is a 2-disk transversal to $W^s(x_0, g \times h)$, and equal to $W^u(x_0, g_+) \times [S^1 - \{-1\}]$.

D. The bump function $b$ with support near $x_0$. Choose 2-disk $B_1$ in $T^2_+$ such that

1. $x_0 \in$ interior (as 2-disk) of $B_1$.
2. $\theta \notin B_1$.
3. $B_1 \subset Q$, where $Q$ is as in $\S B$.
4. If $x \in J^g$, the path component of $W^s(x_0, g \times h) \cap B_1$ containing $x_0$, $d(gx, x_0) < 1/3d(x, x_0)$. If $y \in J^g$, the path component of $W^u(x_0, g_+) \cap B_1$ containing $x_0$, $d(gy, x_0) > 3d(y, x_0)$. This is possible because eigenvalues $\lambda$ and $\mu$ of $T^g g$ are such that $|\lambda| < \frac{1}{4}$ and $|\mu| > 4$.
5. $B_1 = J^g \times J^g$ in $T^2$.
6. Let $v_0$ be the point of $\partial J^g$ closest to $\theta$ as in Figure 2. $\{v_0\} \times J^g \subset W^u_{\text{loc}}(\theta, g_+)$, a fixed local unstable manifold of $\theta$ for $g_+$, while $g^n_+(\{v_0\} \times J^g) \cap W^u_{\text{loc}}(\theta, g_+) = \emptyset$ for all $n \geq 1$.
7. For each $x \in B_1 \cap \Sigma$, let $W^s_+(x, g)$ be the path component of $W^s(x, g_+) \cap B_1$ containing $x$. Choose $B_1$ so that, for $x \in B_1 \cap \Sigma$, $g(W^s_+(x, g)) \subset W^s_+(gx, g)$ or misses $B_1$.

Choose interval $B_2$ in $S^1$ so that $\theta + 1 \in \text{int} B_2 \subset S^1$ and $z \in B_2 \Rightarrow d(bz, +1) > 3d(z, 1)$. Then, $B = B_1 \times B_2$ is a 3-disk about $x_0$ in $T^3$.

Notation. The following notation will be helpful:

$\Sigma = \text{path component of } \Sigma \cap B$ containing $x_0$, i.e. $J^\Sigma$;
$\Sigma = \text{path component of } F(x) \cap B$ containing $x$ for $x \in B$;
$W^s_-(x, g \times h) \equiv \text{the local stable manifold of } x$, i.e. path component of $W^s(x, g \times h)$
$\cap B$ containing $x$, for $x \in \Sigma \cap B$;
$W^u_-(x, g \times h) \equiv \text{the local unstable manifold of } x$, i.e. path component of $W^u(x, g \times h)$
$\cap B$ containing $x$, for $x \in \Sigma \cap B$;
$W^s_{\Sigma}(x, g) \equiv B_1 = \bigcup W^s(x, g)$ for $x \in \Sigma$.

Note that $W^s_+(x, g \times h)$ is an interval and equals $\mathcal{F}$ \cap $T^2_+$, while $W^u_-(x, g \times h)$ is a 2-disk. Now, choose 2-disk $N_1$ in $B_1$ so that
(a) \( N_1 \cap \Sigma = \emptyset \),
(b) \( g_+^{-n}(N_1) \cap B_1 = \emptyset \) and \( g_+^n(N_1) \cap N_1 = \emptyset \) for all \( n > 0 \),
(c) \( W^s_L(x_0, g \times b) \) divides \( N_1 \) into two 2-disks (as in Figure 2),
(d) if \( W^s_L(x, g \times b) \cap N_1 = \emptyset \), then \( g_+^n W^s_L(x, g \times b) \cap B_1 = \emptyset \).

Figure 2

Figure 2A
In addition, as in Figure 2, about \( x_0 \) choose 2-disk \( N_0 \) in the interior (as 2-disk) of \( B_1 \) so that \( N_0 \cap N_1 = \emptyset \) but \( W^s_L(x_0, g \times b) \) meets \( N_0 \) iff it meets \( N_1 \) for \( x \in \Sigma \).

At this point, it will be helpful to name a collection of intervals in \( S^1 \). First, write \( S^1 \) as the union of two intervals, \( S^+ \) and \( S^- \), where \( S^+ \cap S^- = \{-1, +1\} \).

Then, choose open intervals \( N_2 \) and \( N_3 \) in \( B_2 \subset S^1 \) such that

(i) \( +1 \in N_2 \),
(ii) \( \overline{N}_3 \subset [B_2 - N_2] \cap S^+ \),
(iii) \( \partial N_3 \cap \partial N_5 = \emptyset \),
(iv) \( \partial N_2 \cap \partial N_3 \).

Also, let \( N_2 \) be an interval in \( B_2 \) such that

(v) \( N_2 \subset \text{interior (as 1-disk)} \) of \( N_5 \),
(vi) \( \overline{N}_3 \cap \overline{N}_5 = \emptyset \).

Let \( c \) be the point \( \partial N_5 \cap S^+ \). Finally, let \( N_4 \) be a subinterval of \( N_2 \) about \( +1 \), contained in \( \partial^{-1} N_5 \) with length at most \( 1/3 \) the length of \( N_2 \).

![Figure 3](https://example.com/figure3.png)

Let \( D_1 = (N_0 \times N_3) \cap W^u_L(x_0, g \times b) \), a 2-disk in \( T^3 \). Finally, choose open set \( N \) in \( T^3 \) such that

1. \( N \cap (T^2 \times N_2) = N_1 \times N_2 \),
2. \( N \cap (N_0 \times S^1) = N_0 \times N_3 \).
(3) \( \overline{N} \subset \text{interior } B \),
(4) \( B \cap (g \times b)^{-1} \{ N \} \subset B_1 \times b^{-1}(B_2 - N) \),
(5) \([g, B_1 \times \{ c \}] \cap N = \emptyset\).
So, \( N \cap W^u_L(x_0, g \times b) = D_1 \) and \( N \cap T^2_+ = N_1 \). Pictorially, we want \( N \cap \tilde{F}(x) \) to be empty or as in Figure 4 for \( x \in \Sigma \), where \( a_1 = N \cap \tilde{F}(x) \cap N_1 \subset T^+_2 \) and \( a_2 = N \cap \tilde{F}(x) \cap D_1 \subset W^u_L(x, g \times b) \).

One now can construct a \( C^\infty \) diffeomorphism \( b \) of \( T^3 \), a "bump function" whose main purpose is to force \( W^u_L(x_0, g \times b) \) to intersect \( T^2_+ \) transversally. \( b \) need have the following properties:
(a) \( b = \) identity outside \( N \),
(b) \( b(D_1) \) intersects \( T^2_+ \) transversally (in \( N_1 \), of course),
(c) \( b(\tilde{F}(x)) \subset \tilde{F}(x) \) for all \( x \in B \), i.e. \( b \) preserves the foliation \( \tilde{F} \),
(d) \( b([x_0] \times N_3) \) intersects \( W^u_L(x_0, g \times b) \) in two points,
(e) the largest increase of arc length under \( b \) occurs at \([x_0] \times N_3 \) where length \( b([x_0] \times N_3) / \text{length of } [x_0] \times N_3 = P \),
(f) for all \( x \in \Sigma \), \( \tilde{F}(x) \) intersects \( b(D_1) \) transversally in \( N_1 \times N_2 \).

Pictorially, \( b \) sends points from left to right in \( N \cap \tilde{F}(x) \) in Figure 4; and for \( x = x_0 \), \( b(a_2) \) intersects \( a_1 \) in two points. Finally, choose \( k \) at the end of \( \S B \) so that \( \mu_1^k > [4(1 + P)]^l \) and again denote \( g^k \) by \( g \). \( N_0 \), \( N_1 \), and \( B_1 \) will still have the desired properties for our new D-A \( g \).

E. Stable and unstable manifolds for \( b \circ (g \times b) \). Let \( f = b \circ (g \times b) \). \( f \) is a \( C^\infty \) diffeomorphism of \( T^3 \), and \( f \) respects the foliation \( \tilde{F} = [W^u(x, A) \times S^1] \) since \( b \) preserves \( \tilde{F} \). To obtain \( U \), the open set of diffeomorphisms in the statement of our theorem, we will construct a ball about \( f \) in \( \text{Diff}^r(T^3) \).

Since a study of the orbit structure of maps near \( f \) is parallel to such a study of \( f \), we will try to understand the stable and unstable manifolds for \( f \) in this section. First, note that since we did not alter \( g \times b \) near \( \Omega(g \times b) \) and periodic points are dense in \( \Omega(g \times b) \), \( \Omega(g \times b) \subset \Omega(f) \) with the same hyperbolicity constants there for \( f \) as for \( g \times b \).
We will make frequent use of the following simple lemma:

**Lemma 1.** Let \( f, f_1 \) be diffeomorphisms of compact manifold \( M \). Let \( \Sigma \) be a hyperbolic compact invariant subset in \( \Omega(f) \) with periodic points of \( \Sigma \) dense in \( \Sigma \). Let \( N \) be a subset of \( M \ni f \neq f \) outside \( N \) and \( N \cap \Sigma = \emptyset \).

(a) For \( x \in \Sigma \), let \( W^s_{\text{loc}}(x, f) \) be a subset of \( W^s(x, f) \). If \( f^n W^s_{\text{loc}}(x, f) \cap N = \emptyset \) for all \( n \geq 0 \), then \( x \in \Omega(f) \) and \( W^s_{\text{loc}}(x, f) \subset W^s(x, f_1) \).

(b) If \( f_1 = b \circ f \) where \( \text{supp} \ b \subset N \) and \( f^n W^s_{\text{loc}}(x, f) \cap N = \emptyset \) for all \( n \geq 1 \), then \( x \in \Omega(f) \) and \( W^s_{\text{loc}}(x, f_1) \subset W^s(x, f_1) \).

(c) Let \( W^u_{\text{loc}}(x, f) \) be a subset of \( W^u(x, f) \). If \( f^{-n} W^u_{\text{loc}}(x, f) \cap N = \emptyset \) for all \( n \geq 0 \), then \( x \in \Omega(f) \) and \( W^u_{\text{loc}}(x, f) \subset W^u(x, f_1) \).

(d) If \( f_1 = b \circ f \) where \( \text{supp} \ b \subset N \) and \( f^{-n} W^u_{\text{loc}}(x, f) \cap N = \emptyset \) for all \( n \geq 1 \), then \( x \in \Omega(f) \) and \( b W^u_{\text{loc}}(x, f) \subset W^u(x, f_1) \).

**Proof of Lemma 1.** Let \( x \in \Sigma \). As in [9] and [21], \( W^s(x, f) = \{ y \in M: d(f^n x, f^n y) \to 0 \text{ as } n \to +\infty \} \).

Let \( y \in W^s_{\text{loc}}(x, f) \). \( x, y \not\in N \Rightarrow f x = f_1 x \) and \( f y = f_1 y \). In fact, \( f^n x, f^n y \not\in N \) for all \( n \geq 0 \) \( \Rightarrow f^n x = f_1^n x \) and \( f^n y = f_1^n y \) for all \( n \geq 0 \). So,

\[
d(f^n x, f_1^n y) = d(f^n y, f_1^n x) \to 0 \text{ as } n \to -\infty.
\]

\( x \) is nonwandering for \( f_1 \) since \( \Sigma \cap N = \emptyset \) and periodic points are dense in \( \Sigma \). \( y \in W^s_{\text{loc}}(x, f) \), proving (a).

If \( f_1 = b \circ f \) and \( y \in W^s_{\text{loc}}(x, f) \) (possibly in \( N \)), \( f y \not\in N \) by hypothesis and therefore \( f_1 y = b \circ f y = f y \). Then, argue as in the proof of (a) to obtain (b). (c) follows, since \( W^u(x, f_1) = W^s(x, f_1) \).

(d) \( f^{-1} W^u_{\text{loc}}(x, f) \) is a subset of \( W^u(f^{-1} x, f) \) and \( f^{-n} [f^{-1} W^u_{\text{loc}}(x, f)] \cap N = f^{-n} f_1 W^u_{\text{loc}}(x, f_1) \cap N = \emptyset \) for \( n \geq 0 \) by hypothesis. By (c), \( f^{-1} W^u_{\text{loc}}(x, f) \subset W^u(f^{-1} x, f_1) \); therefore,

\[
 f^{-1} W^u_{\text{loc}}(x, f) \subset f_1 W^u(f^{-1} x, f_1) = W^u(x, f_1).
\]

But, \( f_1 \circ f^{-1} = b \circ f^{-1} \circ f = b \). This proves Lemma 1.

Let \( f = b \circ (g \times b) \) be as defined above. As above, for \( x \in \Sigma \), let \( W^s_L(x, f) \), the local stable manifold for \( x \), be the path component of \( W^s(x, f) \cap B \) that contains \( x \); and let \( W^u_L(x, f) \), the local unstable manifold for \( x \), be the path component of \( W^u(x, f) \cap B \) that contains \( x \).

**Lemma 2.** For \( x \in \Sigma \),

(a) \( W^s_L(x, f) = W^s_L(x, g \times b) \),

(b) \( W^s_L(\Sigma, f) = W^s_L(\Sigma, g \times b) = B_1 \),

(c) \( W^u_L(x, f) = W^u_L(x_0, f) = b[W^u_L(x_0, g \times b)] \).
Proof of Lemma 2. We will use Lemma 1, with $\Sigma = \text{solenoid in } T^2$ and $N$ as constructed in $\S$D. $N \cap \Sigma = \emptyset$ and $f = g \times b$ outside $N$. By (b) in definition of $N_1$ in $\S$D, $(g \times b)^{-n} N \cap B_1 = \emptyset$ for all $n > 0$. Since $B_1 = W^s_L(\Sigma, g \times b)$, $N \cap (g \times b)^n W^u_L(x, g \times b) = \emptyset$ for $n \geq 1$. (a) and (b) follow now from Lemma 1 and the definition of $W^s_L$. For (c), recall that $\Sigma \subset W^u(x_0, g \times b)$ and so $W^u_L(x, g \times b) = W^u_L(x_0, g \times b)$ for all $x \in \Sigma$. $W^u_L(x_0, g \times b)$ meets $N$ only in $T^2 \times N_3$. Since $b^{-n} N_3 \cap N_3 = \emptyset$ for all $n \geq 1$ and $W^u_L(x_0, g \times b)$ is invariant under $(g \times b)^{-1}$, $(g \times b)^{-n} W^u_L(x_0, g \times b) \cap N = \emptyset$ for $n \geq 1$. By (d) of Lemma 1 and the definition of $W^u_L$, $b[W^u_L(x_0, g \times b)] = W^u_L(x_0, f)$.

The local stable and unstable manifolds for $f$ around $x_0$ are pictured in Figure 5.

![Figure 5](image)

It will be helpful to have some notation for the three-dimensional local unstable manifold of $\theta$. Considering $\theta$ first as a source for $g_+ : T^2 \to T^2$, let $W^u_{\text{loc}}(\theta, g_+)$ be a 2-disk in its unstable manifold, with $\{v_0\} \times f^u$ in its interior, as in Figure 2. $W^u_{\text{loc}}(\theta, g_+)$ can be constructed so that

1. interior $N_1 \cap \text{interior } W^u_{\text{loc}}(\theta, g_+) = \emptyset$,
2. boundary $N_1 \cap \text{boundary } W^u_{\text{loc}}(\theta, g_+) \neq \emptyset$,
3. $g_+^{-n} W^u_{\text{loc}}(\theta, g_+) \subset W^u_{\text{loc}}(\theta, g_+)$ is disjoint from $B_1$ for all $n > 0$,
4. $g$ reduces lengths on stable manifolds outside $W^u_{\text{loc}}(\theta, g_+)$ by at least one-third. ($g$ does so near $\Sigma$ and away from $Q$.)

Define $W^u_{\text{loc}}(\theta, g \times b) = W^u_{\text{loc}}(\theta, g_+) \times N_2$ in $T^2 \times S^1$. Since $N \cap (T^2 \times N_2) = N_1 \times N_2$, we can define $W^u_{\text{loc}}(\theta, f) = W^u_{\text{loc}}(\theta, g \times b)$ by Lemma 1 and property (3) above.

Since $f$ respects foliation $\mathcal{F}$, $x \in T^3$ is periodic under $f$ only if leaf $F(x)$ is periodic under $f$. Since the leaves for $F$ are products of the stable manifolds of $\Sigma$ and $S^1$, $x$ must lie on $F(y)$ where $y$ is a periodic point on $\Sigma$. Consequently, a good way to study $\Omega(f)$ is by examining $f^n$ restricted to a leaf of period $n$. 

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Lemma 3. \( f | \Omega(f | F(x_0)) \) is conjugate to the shift automorphism on the bisequence space of 3 symbols, i.e. \( 3^Z \).

Since Lemma 3 is superfluous to the proof of Theorem 1, we merely sketch its proof. One constructs by the methods of \( \text{§I} \), a closed rectangle \( R \) in \( \tilde{F}(x_0) \) such that \( f: R \to F(x_0) \) looks like the standard geometric realization of the shift on \( 3^Z \), as in [22]. See Figure 6.

To show the conjugacy to the shift, one easily applies the methods of [18]. Finally, by using the properties of the subsets constructed in \( \text{§D} \), one shows that \( \Omega(f | \tilde{F}(x_0)) \subseteq R \).

Let \( z \) be a periodic point in \( \Sigma \cap N_0 \) of least period \( k \). Suppose \( f^i z \in \Sigma \cap N_0 \) only for \( i = 0, 1, \cdots, i_s \) if \( i < k \). Then, an analysis like that of Lemma 3 will show that \( f^k | \Omega(f^k | F(z)) \) is conjugate to the shift map acting on a quotient space of \( (3^5)^Z \).

F. Normally-hyperbolic foliations.

Definition. Let \( f \) be a diffeomorphism of compact \( C^\infty \)-manifold \( M^q \) that respects a foliation \( \mathcal{F} \) on \( M \). We call \( f \) r-normally-hyperbolic (with respect to \( \mathcal{F} \)) if \( \mathcal{F} \) a continuous splitting \( TM = E_+ \oplus E_- \oplus T^\mathcal{F} \) invariant under \( T^f \) such that the following conditions hold: for some Riemannian metric on \( M \) constants \( \lambda, \mu \) with \( 0 < \lambda < 1 < \mu \) such that if \( 0 \neq X \in TM \),

\[
\begin{align*}
|TfX| &\leq \lambda|X| & \text{if } X \in E_-, \\
|TfX| &\geq \mu|X| & \text{if } X \in E_+, \\
\lambda|X| &< |Tf^iX| < \mu|X| & \text{for } i = 0, 1, \cdots, r, \text{ if } X \in T^\mathcal{F}.
\end{align*}
\]

Intuitively, this condition means that the contracting (expanding) effect of \( f \) normal to the leaves of the foliation is at least \( r \) times greater than the contracting (expanding) effect of \( f \) on the leaves.
Definition. For foliation $\mathcal{F}$ on $M$, let $Q(\mathcal{F})$ be the quotient space obtained by identifying leaves of $\mathcal{F}$ to points. If $f$ respects $\mathcal{F}$, $f : Q(\mathcal{F}) \to Q(\mathcal{F})$ is well defined. If $g$ respects foliation $\mathcal{G}$ on $M$, $(\mathcal{F}, f)$ is conjugate to $(\mathcal{G}, g)$ if there is homeomorphism $h : Q(\mathcal{F}) \to Q(\mathcal{G})$ such that the following diagram commutes:

\[
\begin{array}{ccc}
Q(\mathcal{F}) & \xrightarrow{f} & Q(\mathcal{F}) \\
h \downarrow & & \downarrow h \\
Q(\mathcal{G}) & \xrightarrow{g} & Q(\mathcal{G})
\end{array}
\]

Theorem (Hirsch-Pugh-Shub [10]). Let $1 < r < \infty$ and $M$ be a compact $C^\infty$-manifold. Let $f$ be a $C^r$ diffeomorphism of $M$ that is $r$ normally-hyperbolic with respect to some foliation $\mathcal{F}$ where the leaves of $\mathcal{F}$ are $C^r$-manifolds. Then, there exists an open set $U$ in $\text{Diff}'(M)$ about $f$ such that if $g \in U$, then $g$ respects a foliation $\mathcal{G}$ whose leaves are $C^r$-manifolds. $(\mathcal{F}, f)$ is conjugate to $(\mathcal{G}, g)$.

Remark 1. As constructed in §B, the D-A map $g$ is $r$ normally-hyperbolic with respect to the foliation $\mathcal{F} = \{W^s(x, A) : x \in T^2\}$. In fact, one can construct an invariant foliation $\mathcal{G}$, everywhere transverse to $\mathcal{F}$ and containing the path components of $\Sigma$ as leaves. $g$ is expanding on leaves of $\mathcal{G}$, but contracting on leaves of $\mathcal{F}$ except near $\theta$ where by proper choice of $\phi$ the expansion can be made arbitrarily slow compared to the expansion along leaves of $\mathcal{G}$. Take $E^+(x)$ to be the tangent space to the leaf of $\mathcal{G}$ through $x$ and $E^-(x)$ to be empty. $\mathcal{G}$ is tangent to a "Denjoy vector field" on $T^2$.

Remark 2. $g \times h : T^2 \to T^2$ is $r$ normally-hyperbolic with respect to $\mathcal{F} = \{W^s(x, A) \times S^1\}$. To see this, one constructs a one-dimensional invariant foliation $\mathcal{G}^+$ on $T^2$, expanding under $f$ and everywhere normal to $\mathcal{F}$, by putting the foliation $\mathcal{G}$ of Remark 1 on each $T^2 \times \{s\}$ for all $s \in S^1$.

Remark 3. $f = b \circ (g \times h)$ is $r$ normally-hyperbolic with respect to $\mathcal{F} = \{W^s(x, A) \times S^1\}$. It is not as simple a task to construct the invariant subbundle $E^+$ for $b \circ (g \times h)$ as it was for $g$ and $g \times h$. However, $b$ takes each leaf of $\mathcal{F}$ into itself and expands lengths by a factor $\leq P$ (as defined in §D) while expansion normal to leaves under $b \circ (g \times h)$ remains greater than $[4(P + 1)]^r$. The stability of foliation $\mathcal{F}$ follows then from the methods of §2 of an expanded version of [10] where Hirsch, Pugh, and Shub characterize normal hyperbolicity by comparing the spectrum of $f_\#$ restricted to $T\mathcal{F}$ (where $f_\#(v) = Tf \circ v \circ f^{-1}$ for sections $v$ of $TM$) to the spectrum of $f_\#$ restricted to the formal normal bundle of $T\mathcal{F}$. Furthermore, in [6, esp. §VI], Fenichel proves a similar perturbation theorem using only the asymptotic behavior of such a map $f$ without assuming any invariant splitting of $TM$. 
G. The open set $U$ in Theorem 1.

1. If the $r$ in Theorem 1 is finite, the last section indicated how $g$ can be chosen so that $f = b \circ (g \times b)$ is $r$ normally-hyperbolic with respect to $\mathcal{F}$. If $r = \infty$, choose $g$ so that $f$ is at least 2 normally-hyperbolic. Then, let $U$ in each case be as in the conclusion of the Hirsch-Pugh-Shub Theorem.

2. Part of the $\Omega$-stability theorem [25] states that if $\Lambda$ is a hyperbolic basic set (as defined in §B) for $f$, then each $g$ close enough to $f$ has an invariant basic set $\Lambda'$ that is near to and conjugate to $\Lambda$. So, we can choose $U$ so that for $f' \in U$, there is a one-dimensional set $\Sigma'$ with $f' \Sigma' \cong \Sigma$. For all $f' \in U$, let $x_0$ denote the fixed point corresponding to the fixed point $x_0$ for $f$. Let $\tilde{F}'(x) = F'(x) \cap B$ for $x \in \Sigma' \cap B$ where $F' \in \mathcal{F}'$, the foliation of $f'$. For $f' C^r$ close to $f$, $\tilde{F}'(x)$ is $C^r$ close to $\tilde{F}(x)$, where again for notation's sake, we are assuming the conjugacy between $f' \Sigma$ and $f' \Sigma'$ is the identity.

Let $W_{L}^{s}(x_0, f')$ for $x \in \Sigma'$ be the path component of $W^{s}(x, f') \cap B$ containing $x$. By the Hirsch-Pugh Stable Manifold Theorem [9], for $f' C^r$ near $f$, $W_{L}^{s}(x, f')$ is $C^r$ near $W_{L}^{s}(x, f)$ and $W_{L}^{u}(x, f')$ is $C^r$ near $W_{L}^{u}(x, f)$.

3. $W_{L}^{u}(x_0, f')$ intersects $W_{L}^{s}(x, f')$ transversally in two points in $N_1 \times N_2$. Choose $U$ so that this is true for all $f' \in U$. In particular, we can demand that, for $f' \in U$, $f'(\Omega(f' \tilde{F}'(x_0)))$ is conjugate to the standard 3-shift since this open condition ([18], [22]) is true for $f$.

4. If $f'$ is $C^r$ near $f = b \circ (g \times b)$, $f' = b' \circ (g \times b)$ where $b'$ is $C^r$ near $b$. Choose $U$ so that, for $f' \in U$, $T^2 \times \{1\}$ intersects $W_{L}^{u}(x_0, f')$ transversally in $N_1 \times N_2$.

5. Let $N_4 \subset N_2 \subset S^1$ be as in §D. Using [9], choose $U$ so that, for all $f' \in U$, $W_{L}^{s}(x, f') \subset B_1 \times N_4$ for all $x$ in interior $\Sigma'$.

6. By (b) of §D, $f(N_1) \cap N_1 = \emptyset$. Choose $U$ so that this holds for all $f' \in U$.

7. $W_{L}^{u}(x_0, f')$ is transverse to the boundary of $B$. Choose $U$ so that this is true for all $f' \in U$. In particular, $\tilde{\Sigma}'$ will be an interval for all $f'$.

8. Choose $U$ so that $W_{L}^{u}(x_0, f') \cap (N_1 \times N_2) \subset \text{interior of } N_1 \times N_2$ for all $f' \in U$.

9. Using stable manifold theory again, choose $U$ so that $f'^{-1} W_{L}^{u}(x_0, f') \cap N = \emptyset$ for all $f' \in U$.

10. Using (4) in construction of $N$, choose $U$ so that $B \cap f'^{-1} N \subset B_1 \times b^{-1}(B_2 - N_3)$.

11. Since $b = \text{identity}$ on $gB_1 \times \{c\}$ (cf. (5) in construction of $N$) and $c \notin N_3$, $b[gB_1 \times \{c\}] \cap T^2 \times N_2 = \emptyset$. Choose $U$ so that for $b'$ as in (4) above, $b'[gB_1 \times \{c\}] \cap T^2 \times N_2 = \emptyset$, i.e. $f'[B_1 \times b^{-1}c] \cap T^2 \times N_2 = \emptyset$.

12. $B$ is the union of 2-disks $\tilde{F}(x)$ for $x \in \tilde{\Sigma}$. For $f' \in U_1$ demand that either $\tilde{F}(x)$ is a 2-disk whose "interior" lies in $B$ or $f'(\tilde{F}(x)) \cap B = \emptyset$.
13. Consider $W^u_{\text{loc}} (\theta, f)$ described in §D. Choose $U$ so that for $f' \in U$, $W^u_{\text{loc}} (\theta, f') \subseteq W^u (\theta, f')$, and $f^{-1} W^u_{\text{loc}} (\theta, f)$ lies in the interior of $W^u_{\text{loc}} (\theta, f)$ and in the complement of $B$.

14. Since $h(N_2) \supset B_2$, one can choose $U$ so that $f'(T^2 \times N_2) \supset T^2 \times B_2$ and $f'([\mathcal{C}(T^2 \times B_2)]) \subseteq [\mathcal{C}(T^2 \times B_2)]$.

15. Let $\partial N_2 = \{a_1, a_2\} \subseteq \mathcal{S}^1$ with $a_1 \in S_4$. In $[T^2 \times N_2] \setminus f^{-1} N$, $f$ increases distances normal to $T^2 \times \{a_1, a_2\}$ by a factor greater than 3 by construction of $h$. Choose $U$ so that this holds for all $f' \in U$.

16. For $x \in T^2 \times \{a_1, a_2\}$, let $K(x)$ be distance measured along $F(x)$ from $x$ to $T^2 \times \{a_1\}$ if $x \in T^2 \times \{a_2\}$ or to $T^2 \times \{a_1\}$ if $x \in T^2 \times \{a_1\}$. For $f = b \circ (g \times b)$, $K(x) = \text{length of } N_2$, for all $x \in T^2 \times \partial N_2$. Choose $U$ so that for all $f' \in U$ and all $x$ as above, $K(x) < 3 \times \text{length of } N_2 = K$.

17. If $f'$ is $C'$ near $b \circ (g \times b)$, $f' = b \circ k$, where $k$ is $C'$ near $(g \times b)$. $g \times b$ satisfies Axiom A and strong transversality condition. Therefore, by [17], it is structurally stable. Choose $U \ni f'$ if $f' \in U$, $f' = b \circ k$ where $k$ is topologically conjugate to $g \times b$.

H. Perturbing maps in $U$.

Notation. If $f'$ is in $U$, let $\mathcal{F}_t$ be the foliation on $T^3$ as in (1) in §G; let $\Sigma_t$ or $\Sigma(f'_t)$ denote the important solenoid as in (2) in §G; let $\Sigma'$ be the path component of $\Sigma_t \cap B$ containing $x_0$; $\Sigma'$ is an interval by (7) in §G. Let $W^s_t (x, f'_t)$, $W^u_t (x, f'_t)$ and $\tilde{F}_t (x)$ be as defined in (2) of §G. In this section, we want to prove

Lemma 4. Given $f_0 \in U$, there is a point $z \in \Sigma'(f_0)$ and a one-parameter family of maps in $U$, $\{f'_t\}$, $0 \leq t \leq 1$, such that the following hold:

1. $\Sigma(f'_t) = \Sigma(f'_0)$ for all $t \in [0, 1]$.

2. $W^s_t (z, f'_0)$ and $W^u_t (z, f'_0)$ have linking number 0 in $N_1 \times N_2$; in fact, they intersect but $W^u_t (z, f'_0)$ lies on one side of $W^s_t (z, f'_0)$.

3. $W^s_t (z, f'_1)$ and $W^u_t (z, f'_1)$ have linking number at least 2 in $N_1 \times N_2$.

Figures 7 and 8 describe the difference between (2) and (3).

![Figure 7. $\tilde{F}_0 (z)$](image1)

![Figure 8. $\tilde{F}_1 (z)$](image2)
Let $f_0$ be an arbitrary map in $\text{Diff}^r(T^3)$. Let $\Lambda_0 = \{x \in \Sigma_0^\infty W^s_L(x, f_0) \cap W^u_L(x_0, f_0) \neq \emptyset \}$ in $N_1 \times N_2$. $\Lambda_0$ is a nonempty proper closed subset of $\Sigma_0^\infty$ by (3) and (8) of Section 8, since $W^u_L(\Sigma_0^\infty)$ is a two-dimensional topological disk, a result of the stable manifold theorem. Ordering the points in the interval $\Sigma_0^\infty$ naturally, there is a unique $z$ in $\Lambda_0$ such that if $z' > z$ in $\Sigma_0^\infty$, then $z' \notin \Lambda_0$. So,

$$W^u_L(z, f_0) \cap W^s_L(z', f_0) \neq \emptyset \quad \text{in} \quad N_1 \times N_2$$

$$W^u_L(z', f_0) \cap W^s_L(z', f_0) = \emptyset \quad \text{in} \quad N_1 \times N_2 \quad \text{for} \quad z' > z.$$

Recall that since $z, z' \in \Sigma_0^\infty \subset W^u_L(x_0, f_0)$, $W^u_L(x_0, f_0) = W^u_L(z, f_0) = W^u_L(z', f_0)$. Since the zero linking number is a closed condition, $W^u_L(x_0, f_0)$ and $W^s_L(z, f_0)$ have linking number zero in $N_1 \times N_2$. However, they do intersect there, as in Figure 7.

We now construct our one-parameter family of maps. Let $y \in W^s_L(z, f_0) \cap W^u_L(x_0, f_0) \cap N_1 \times N_2$, as in Figure 7. Choose $y$ to be the furthest such point on $W^s_L(z, f_0)$ from $z$. $T^0_w(z, f_0) \subset T^0_w(x_0, f_0)$. Choose nonzero vector $X(y)$ normal to $T^0_w(x_0, f_0)$, tangent to $F^0_0(z)$, and pointing in the $S_-$-direction, i.e., away from $W^u_L(x_0, f_0)$. Extend $X(y)$ to a $C^\infty$ constant vector field on $T^3$. Now select an open set $V$ in $N_1 \times N_2$ around $y$ with $f_0(V) \cap V = \emptyset$ and with $W^u_L(x_0, f_0)$ dividing $V$ into two parts. Let $k: T^3 \to \mathbb{R}$ be a $C^\infty$ Urysohn function that is 1 near $y$ but 0 outside $V$; and consider vector field $Y(x) = k(x)X(x)$ for $x \in T^3$ with flow $\alpha_t$.

Defining $F_t = \alpha_t \circ f_0$, let $t_1 > 0$ be such that, for all $t \in [0, t_1]$, $F_t$ is in the open set $U$ in $\text{Diff}^r(T^3)$. By Lemma 1, $\Sigma_0^\infty = \Sigma_0^\infty$, $W^s_L(x, f_t) = W^s_L(x_0, f_0)$ for all $x \in \Sigma_0^\infty$, and $W^u_L(x_0, f_t) = f_t^{-1}W^u_L(x_0, f_0)$.

All one need show now is that, for $t > 0$, $W^s_L(z, f_t)$ and $W^u_L(x_0, f_t)$ have linking number greater than zero in $N_1 \times N_2$. $W^u_L(x_0, f_0)$ divides $V$ into two parts, with $W^s_L(z, f_0) \cap V$ lying in the lower ($S_-$) part. Since $W^s_L(z, f_0)$ is tangent to $W^u_L(x_0, f_0)$ at $y$ and $\alpha_t$ pushes $W^u_L(x_0, f_0)$ in the normal direction, there is $t_2$ with $0 < t_2 \leq t_1$ so that, for $t \in (0, t_2)$, some of $W^s_L(z, f_0)$ lies above $\alpha_t W^u_L(x_0, f_0)$ in $\cap V$ and some lies below. Thus, the linking number of $W^s_L(z, f_0)$ and $\alpha_t W^u_L(x_0, f_0)$ is greater than 0 in $V$ for $t \in (0, t_2)$. Now reparameterize $[0, t_2]$ to $[0, 1]$. Since $W^s_L(z, f_0) = W^s_L(z, f_t)$ and $\alpha_t W^u_L(x_0, f_0) = W^u_L(x_0, f_t)$, the proof of Lemma 4 is complete.

If $r < \infty$ in the statement of Theorem 1, the $F_t(x)$ are $C^r$ manifolds by (1) in Section 8 and $X$ can be chosen everywhere tangent to $F_0$ (e.g., using foliation charts of [8]). In this case, $F_t = F_0$ and one merely pulls $W^u_L(x_0, f_0) \cap F_0(z)$ down along $F_0(z)$ to proceed from the situation of Figure 7 to that of Figure 8.

**Lemma 5.** If $f_0 \in U$, $f_0$ does not satisfy Smale's Axiom A, i.e., $f_0$ has a non-hyperbolic nonwandering point.
Proof. The point \( y \) of Lemma 4 is nonhyperbolic yet nonwandering.
\( y \in \Omega(f_0) : y \in W^s(z, f_0) \cap W^u(x_0, f_0) \). But \( W^u(z, f_0) \) and \( W^s(x_0, f_0) \) intersect transversally since \( W^u(z, f_0) = W^u(x_0, f_0) \) and \( x_0 \in W^u(x_0, f_0) \cap W^s(x_0, f_0) \). By "Cloud Lemma," [24, (7.2)] or [2], \( y \in \Omega(f_0) \).

\( y \) not hyperbolic: \( y \in W^s(z, f_0) \) and \( y \in W^u(x_0, f_0) \). If \( y \) were hyperbolic, \( W^s(y, f_0) = W^s(z, f_0) \), \( W^u(y, f_0) = W^u(x_0, f_0) \) and \( y \in W^u(y, f_0) \cap W^s(y, f_0) \). But \( W^s(z, f_0) \) and \( W^u(x_0, f_0) \) do not meet transversally at \( y \). Q.E.D.

1. Construction of special 2-disks in the \( F(x)'s \). In this section, \( f \) will denote an arbitrary element of \( U \), not necessarily \( b \circ (g \times b) \) as in previous sections. For each \( f \in U \) and periodic point \( x \) in \( \Sigma(f) \cap [N_0 \times N_2] \), we construct a "rectangular" 2-disk \( R(x) \subset F(x) \), which will have roughly the same purpose as the \( R \) in Figure 6. If \( x \in \Sigma(f) \cap [N_0 \times N_2] \) and \( x' \) is the corresponding point in \( \Sigma(f') \), \( R(x, f) \) will be \( C^0 \) close to \( R(x', f') \).

Lemma 6. Let \( f \in U \) and let \( s \) be a path in \( \tilde{F}(x, f) \cap T^2 \times N_2 \) for \( x \in \Sigma \cap [N_0 \times N_2] \). Suppose \( s \cap \Sigma = \emptyset \). If \( f^i s \subset T^2 \times N_2 \) for \( 0 \leq i \leq k \), then \( f^i s \cap N = \emptyset \) for \( 1 \leq i \leq k \). If also \( f^m s \cap W^u_L(x_0, f) = \emptyset \), \( f^n s \cap W^u_L(x_0, f) = \emptyset \) for all \( n \geq m \).

Proof. Last sentence follows from \( f^{-1} W^u_L(x_0, f) \subset W^u_L(x_0, f) \). The geometric reason for \( s \cap N = \emptyset \) is that \( f \) sends points in \( T^2 \times N_2 \) closer to \( W^u_L(x_0, f) \) and away from \( N \). To send \( s \) back to \( N \), \( f \) would have to map some of \( s \) out of \( T^2 \times N_2 \). Suppose \( s \cap N \neq \emptyset \). Since \( s \subset B \), \( s \cap \{ f^{-1} N \cap B \} \neq \emptyset \). By (10) of \( \hat{S}G \), \( s \) contains a path from \( \Sigma \) to \( B_1 \times b^{-1}(B_2 - N_2) \) and so must intersect \( B_1 \times \{ b^{-1}c \} \) where \( c \) is in Figure 3. So, \( s \cap \{ f[B_1 \times b^{-1}c] \} \neq \emptyset \). By (11) of \( \hat{S}G \), \( s \) has a point outside \( T^2 \times N_2 \). This contradicts the hypothesis and shows \( s \cap N = \emptyset \).

For \( j = 2 \), argue as for \( j = 1 \) if \( s \cap B \). Otherwise, \( f^2 s \cap B = \emptyset \) by (12) in \( \hat{S}G \); and \( N \subset B \). Let \( i \) be the first integer \( \geq 2 \) with \( f^i s \cap B \neq \emptyset \) but \( f^{i-1} s \cap B = \emptyset \). If \( f^i s \) met \( N \), it would have to do so in \( N_1 \times N_2 \) since \( N \subset T^2 \times N_2 \). Then, \( f^i s \) would join \( N_1 \times N_2 \) to \( \Sigma \) by a curve in \( F(f^i s) \subset T^2 \times N_2 \). In \( F(f^i s) \cap T^2 \times N_2 \), \( W^u_L(x_0, f) \) separates \( \Sigma \) from \( N_1 \times N_2 \). But \( f^i s \cap W^u_L(x_0, f) = \emptyset \) since \( W^u_L(x_0, f) \) lies in \( B \) and is invariant under \( f^{-1} \). So, \( f^i s \) must leave \( F(f^s) \) and intersect \( W^u_L(\theta, f) \). But then, \( s \cap \{ f^{-i} W^u_L(\theta, f) \} \neq \emptyset \). Since \( s \) lies in \( B \), this contradicts (13) of \( \hat{S}G \). So, \( f^i s \cap N = \emptyset \). An inductive argument then finishes the proof of this lemma.

Now let \( q \) be a periodic point for \( f \), say of (least) period \( m \), in \( \Sigma(f) \cap N_0 \times N_2 \) with \( W^u_L(q, f) \cap N_1 \times N_2 \neq \emptyset \). We are going to construct rectangle \( R(q) \subset \tilde{F}(q) \). Let \( s_1 \) be a closed interval in \( W^s_L(q, f) \) with endpoints \( q \) and \( w_1 \) that is maximal in that \( s_1 \subset s' \subset W^s_L(q, f) \) and \( s_1 \neq s' \) implies \( \partial s' \cap N = \emptyset \). So \( w_1 \) is the point on \( \partial(N \cap \tilde{F}(q)) \cap W^s_L(q, f) \) furthest from \( q \). Let \( s_2 \) be the path along \( \partial(N \cap \tilde{F}(q)) \) from \( w_1 \) to \( w_2 \) where \( w_2 \in T^2 \times \{ a \} \) (cf. (15) of \( \hat{S}G \)). Let \( s_3 \) be the path in \( T^2 \times \{ a \} \cap \tilde{F}(q) \) from \( w_2 \) to \( w_3 \), a point on \( f^{-1} W^u_L(q, f) \cap \tilde{F}(q) \). Let \( s_4 \) be the path in \( f^{-1} W^u_L(q, f) \)
$\cap \tilde{F}(q)$ from $w_3$ to $q$, $s_1 \cup s_2 \cup s_3 \cup s_4$ encloses a rectangle $R_0(q) \subset \tilde{F}(q)$. [R in Figure 6 is $R_0(x_0)$.]

Let $R_1(q)$ be the component of $\partial R_0 \cap T^2 \times N_2$ containing $f s_1$. [In Figure 6, $R_1(x_0)$ is $R \cap \partial R$.] Define inductively $R_j(q) = \text{component of } \partial R_{j-1}(q) \cap T^2 \times N_2$ containing $f^j s_1$. By Lemma 6, $R_j(q) \cap N = \emptyset$ for $1 \leq j < m$; and so $R_j(q)$ is component of $\partial R_q \cap T^2 \times N_2$ containing $f^j s_1$. Finally, recalling $f^m q = q$, define $R_m(q) = f(R_{m-1}(q)) \subset \tilde{F}(q)$ and denote $f^{-m} R_m(q) = f^{-m-1} R_{m-1}(q)$ as $R(q)$ or $R(q, f)$. See Figure 9.

Note that for $0 \leq i < m$ each $R_i(q)$ is a "rectangle" with one side, viz. $f^i s_1$, lying in $f^i w_L(q, f)$; and each $f^{-i} R_i(q)$ is a rectangle in $R_0(q)$ with $s_1$ as one of its sides. For notation's sake, label the sides of $R_i(q)$ as $s_{i1}, s_{i2}, s_{i3}, s_{i4}$ and the sides of $R(q)$ as $s_1', s_2', s_3', s_4'$ where $s_{ij}$ and $s_{ij}'$ correspond to $s_j$ in $R_0(q)$, $j = 1, 2, 3, 4$. For each $i < m$ and each $R_i(q)$, call the maximum distance measured along $F(f^i q)$ from $x$ in $s_{i1} = f^i s_1$ to $s_{i3} \subset T^2 \times \{a_1\}$ the height of $R_i(q)$. For each $i \leq m$ and each $f^{-i} R_i(q)$, call the maximum distance measured along $F(q)$ from $x$ in $s_{i1}$ to the opposite side of $f^{-i} R_i(q)$, viz. $f^{-i} s_{i3}$, the height of $f^{-i} R_i(q)$. By (16) of §G, the height of each $R_i(q) < K$. By (15) of §G, height of $f^{-i} R_i(q) < K/3^i$ for $0 \leq i < m$ and so height of $R(q) < K/3^{m-1}$.

We now describe the sides of $R_m(q)$, $s_{m,i} = f^i s_{m-1,i}$, $s_{m-1,3} \subset T^2 \times \{a_1\}$ and (14) of §G imply that $s_{m,3}$ lies above $T^2 \times B_2$, i.e. above $F(q)$, as in Figure 9.

Since $s_{m,4}$ is the path component of
are disjoint from \( W^u_L(x_0, f) \) as in Figure 9. Putting all this together, one obtains

**Lemma 7.** Let \( q \) be a periodic point for \( f \in U \), of (least) period \( m \), in \( \Sigma(f) \cap [N_0 \times N_2] \) such that \( W^s_L(q, f) \) meets \( P \), the boundary of \( N_1 \times N_2 \) nearest \( \theta \). Then, there is a "rectangle" \( R(q) \) in \( \widetilde{F}(q) \cap T^2 \times N_2 \) with boundary \( s'_1 \cup s'_2 \cup s'_3 \cup s'_4 \) where

- \( s'_1 \) is the arc of \( W^s_L(q, f) \) from \( q \) to \( P \),
- \( s'_2 \subset P \cap \widetilde{F}(q) \),
- \( s'_4 \subset \overline{W^u_L(q, f)} \cap \widetilde{F}(q) \), and
- \( s'_3 \) joins \( s'_2 \) to \( s'_4 \) and is opposite \( s'_1 \).

Height of \( R(q) < K/3^{m-1} \). Let \( R_m(q) = f^m R(q) \) with sides \( s_{mi} = f^m s'_i \), \( i = 1, \ldots, 4 \).

\( s_{m,1} \subset s'_1; s_{m,3} \) lies above \( \widetilde{F}(q) \) in \( F(q) \);
- \( s_{m,4} \subset W^u_L(q, f) \cap F(q) \) and joins \( s_{m,1} \) to \( s_{m,3} \);
- \( s_{m,2} \) lies strictly between \( W^s_L(q, f) \cap \widetilde{F}(q) \) and \( W^u_L(x_0, f) \cap \widetilde{F}(q) \), as in Figure 9.

\( R(q) \) varies continuously with \( f \in U \). If \( f_t \) is a one-parameter family of maps in \( U \) which agree outside \( N \) and respect the same foliation, then one \( R(q) \) works for all the \( f_t \)’s.

Now \( R(q) \) contains at least one point period \( \leq m \), viz. \( q \). In Lemma 8, one constructs another 2-disk \( R^\#(q) \) about \( R(q) \) in \( F(q) \) such that \( f \) has no points of period \( \leq m \) in \( R^\#(q) \cap R(q) \). For Lemma 9, one thickens \( R^\#(q) \) to a 3-disk \( V(q) \) such that \( f \) has no points of period \( \leq m \) in \( V(q) \setminus R(q) \).

**Lemma 8.** Let \( q, m, f \) be as in Lemma 7 with \( R(q) \subset \widetilde{F}(q) \) as constructed in Lemma 7. Then, there is another 2-disk \( R^\#(q) \subset \widetilde{F}(q) \) such that

- (i) \( R^\#(q) \) contains \( R(q) \) in its interior as a 2-disk, and
- (ii) \( f \) has no points of period \( \leq m \) in \( R^\#(q) \setminus R(q) \).

\( R^\#(q) \) varies continuously for \( f \in U \). If \( f \) near \( f \) respects the same foliation and equals \( f \) off \( N \), then \( R^\#(q, f') = R^\#(q, f) \).

**Proof.** The proof is simple but a little tedious. So, we will sketch it geometrically, using Figure 9. Let \( s'_1, s'_2, s'_3, s'_4 \) be the edges of \( R(q) \) as in Lemma 7.

There are no points of period \( \leq m \) in \( \widetilde{F}(q) \) below \( W^s_L(q, f) \). To see this, write \( f \) as \( b \circ k \) where \( k \) is topologically conjugate to \( g \times b \) as in (17) of §G. \( k \) has two invariant tori, \( T^2_4(k) \) and \( T^2_2(k) \), with \( \Sigma(f) \) and the \( W^s_L(x, f) \) contained in \( T^2_4(k) \) by Lemma 1. \( k \) and \( b \) send all points "below" \( T^2_4(k) \) toward \( T^2_2(k) \). Thus, there are no nonwandering points "below" \( T^2_2(k) \), and hence below \( W^s_L(q, f) \) for \( f = b \circ k \).

There are no nonwandering points to the right of \( s'_2 \) in \( \widetilde{F}(q) \) since by its construction in §E and by (13) of §G such points are in the three-dimensional \( W^u_L(\theta, f) \). There are no points of period \( \leq m \) to the left of \( s'_4 \) in \( \widetilde{F}(q) \). One way
to see this is to extend $R_0(q)$ to a rectangle $R'_0(q)$ with boundary $s_1', s_2', s_3', s_4'$ where

$$s_1' \subset s_1 \subset W^s_L(q, f), \quad s_2' = s_2, \quad s_3' \subset s_3 \subset T^2 \times \mathbb{R}_+,$$

and $s_4' \subset$ left boundary of $\tilde{F}(q)$.

Define $R_1(q)$ inductively as above and let $R(q) = f^{-m} R_m(q)$, an extension of $R(q)$ to the left. If $x \in R(q) \setminus R(q)$, $f^i(x) \in (T^2 \times N_q') \setminus N$ for $i = 0, \ldots, m - 1$, and so $f^i(x) = k^i(x)$ where $k$ is conjugate to $g \times h$. $x$ cannot have period $\leq m$ for $f$ since $q$ is the only point of period $\leq m$ for $k$ in $\tilde{F}(q)$.

Finally, we need to see that we can extend $R(q)$ beyond $s_3'$. $f^ms_3' \cap T^2 \times B_2 = \emptyset$ by Lemma 7 and, by (14) and (17) of §G, $\Omega(f) \cap \tilde{C}(T^2 \times B_2) \subset T^2 \setminus \Omega(k)$. So $f^ms_3' \cap \Omega(f) = \emptyset$ and there is a 2-disk $V_3$ about $f^ms_3'$ but missing closed set $\Omega(f)$. $f^{-m} V_3$ is disjoint from $\Omega(f)$ and extends $R(q)$ above $s_3'$. This finishes our sketch of the construction of $R^h(q)$.

We want to thicken $R(q)$ to a 3-disk $V(q)$ such that all points of period $\leq m$ in $V(q)$ actually lie in $R(q)$.

**Lemma 9.** Let $q, m, f$ be as in Lemma 7. Let $R(q)$ and $R^h(q)$ be as constructed in Lemmas 7 and 8. Then, there is a 3-disk $V(q)$ in $T^3$ such that $R(q) \subset R^h(q) \subset V(q)$. If $x \in V(q)$ with $f^jx = x$ and $0 < j \leq m$, then $x \in R(q)$. $V(q) \cap \tilde{F}(q) = R^h(q)$ and $V(q)$ varies continuously with $f \in U$.

**Proof.** We first show that points of period $j$ not on $R^h(q)$ do not accumulate on $R^h(q)$. Suppose the contrary, i.e. suppose there exists a sequence of points $\{x_n\}$ such that

1. $x_n \notin R^h(q)$ for all $n$,
2. $f^jx_n = x_n$ for all $n$, with $0 < j \leq m$, and
3. the sequence $\{x_n\}$ accumulates on $R^h(q)$.

By compactness and since $\text{Fix}(f^j)$ is a closed set, there is a point $\bar{x} \in R^h(q)$ s.t. $x_n \to \bar{x}$, where $\{x_n\}$ is now a subsequence of the original sequence and $f^j(\bar{x}) = \bar{x}$.

Therefore, $j = m$. Otherwise, $f^jF(q) = F(q)$ and $f^jW^s(q, f) \cap f^jW^s(q, f) = \emptyset$ for $0 < j < m$.

Choose chart $R^3$ about $F(q)$ where $R^2 \times [0, 1]$ contains $\tilde{F}(q)$ and $R^2 \times \{t\} \subset$ leaf of foliation. Let $\pi_3 : R^3 \to R^1 \times R^1$ be the projection on the third factor. Using [10], we can choose our chart so that for $f'$ near $f$:

1. new chart $R^2 \times R^1$ is close to the original one,
2. $R^2 \times \{t\} \subset$ leaf of foliation for $f'$,
3. $R^2 \times [0, 1] \subset \tilde{F}(q, f')$,
4. $\pi_3$ for $f'$ is $C^2$-close to $\pi_3$ for $f$.

Now, $\pi_3(\pi_2^{-1}(0)) : R^3 \to R^1$ with $\pi_2(\pi_2^{-1}(0)) = 0$. $\partial \pi_3 / \partial x_3(\bar{x}) \neq 0$ since $f$ is expanding in the $x_3$ direction, i.e. normal to the foliation. By the implicit function theorem, $\pi_3(\pi_2^{-1}(0))^{-1}(0)$ forms a two-dimensional submanifold.
through \( \bar{x} \) in our chart. \( f^m R^u(q) \subseteq F(q) \). So, \( y \in R^u(q) \) implies \( y \in R^2 \times 0 \) and \( f^m y \in R^2 \times 0 \). \( f^m y - y \in R^2 \times 0 \) or \( \pi_2 o (f^m - \text{id}) y = 0 \). Therefore, \( R^u(q) \subseteq [\pi_2 o (f^m - \text{id})]^{-1}(0) \).

Since \( f^m x_n = x_n \) for all \( n \), all \( x_n \in [\pi_2 o (f^m - \text{id})]^{-1}(0) \). But by the submanifold property, the \( x_n \) cannot accumulate to \( R^u(q) \) without being on \( R^u(q) \). So, points of period \( j \) not on \( R^u(q) \) do not accumulate on \( R^u(q) \) and, consequently, there is an open neighborhood \( V(q) \) about \( R^u(q) \) as in the conclusion of this lemma. As \( f \) varies, \( \pi_2 \) and \( f^m \) vary smoothly; so \( [\pi_2 o (f^m - \text{id})]^{-1}(0) \) and \( V(q) \) vary continuously with \( f \in U \).

J. Comparison of \( /_0 \) and \( /_1 \). As in the statement of Theorem 1, let \( /_0 \) be an arbitrary map in \( U \) and \( U_0 \) an arbitrary neighborhood of \( /_0 \) in \( U \). For convenience, we can without loss of generality consider \( U \) or \( U_0 \) since every open subset of \( U_0 \) has the properties in §G. Let us now use the 2-disk \( R(q) \) constructed in §1 to study the one-parameter family of maps \( /_t \) discussed in §H. Recall that for all \( f \in U \), \( \Sigma(f) \) is locally the product of a Cantor set and an interval ([27], [30]). For \( z \in \widetilde{\Sigma}(f) \), \( W_L^s(z, f) \cap \Sigma(f) \) is a Cantor set and so points of \( W_L^s(z, f) \cap \Sigma(f) \) accumulate on \( z \).

Let \( /_t \) and \( z \) be as in Lemma 4. Choose \( z' > z \) in \( \widetilde{\Sigma}(f) \) \( \subseteq \widetilde{\Sigma}(f_0) \), using the order in §H, such that \( W_L^s(z', f_t) \) and \( W_L^u(z', f_t) \) \( \subseteq W_L^u(x_0, f_t) \) have nonzero linking number in \( N_1 \times N_2 \). By choice of \( z \) in \( \widetilde{\Sigma}(f_0) \), \( W_L^u(z', f_0) \) and \( W_L^u(z', f_0) \) do not intersect in \( N_1 \times N_2 \). Using the stable manifold theorem [9], the openness of nonzero linking number and of nonempty intersection, and \( \Sigma(f_0) = \Sigma(f_1) \), one can choose a neighborhood \( H \) of \( z' \) in \( \Sigma(f_0) \) such that, for all \( y \in H \),

(a) \( W_L^s(y, f_0) \cap W_L^u(y, f_0) \) intersect in \( N_1 \times N_2 \),

(b) \( W_L^s(y, f_0) \cap W_L^u(x_0, f_0) \) and consequently \( W_L^s(y, f_0) \cap W_L^u(y, f_0) \) is empty in \( N_1 \times N_2 \).

Let \( H, \) be a compact nbd of \( z \) satisfying (a) and (b) and homeomorphic to the product of a Cantor set and an interval. Since \( H \) is closed, there is an \( \epsilon > 0 \) such that for \( y \in H \) the distance (measured along \( F(y) \)) between \( W_L^u(y, f_0) \) and \( W_L^u(y, f_0) \) in \( N_1 \times N_2 \) is at least \( \epsilon \), using (b).

Since periodic points are dense in \( \Sigma(f) \) and there are finitely many points of each period [27], there are periodic points in \( H \) of arbitrarily high period. Choose \( q \in H_1 \) of (least) period \( m \) where \( K/3^m < \epsilon \). Construct \( R(q, f_0) \) and \( R(q, f_1) \) as in Lemma 7. So \( f^m R(q, f_t) \) \( = R_m(q, f_t) \) lies in \( F(q) \) and is bounded by \( W_L^u(q, f_t) \), \( W_L^u(q, f_t) \cap F(q) \), and \( W_L^u(x_0, f_t) \cap F(q) \) as in Figure 9. Also, in \( N_1 \times N_2 \), \( f^m R(q) \) lies below \( W_L^u(q, f_t) \cap F(q) \) and above \( W_L^u(x_0, f_t) \cap F(q) \). In the \( C^r \) case, \( r < \infty \), \( R(q, f_0) = R(q, f_1) \) by Lemma 4 and the last sentence of Lemma 7.

However, the height of \( R(q) \) \( < K/3^m \) \( < \epsilon \), while the distance between \( W_L^u(x_0, f_0) \) and \( W_L^u(q, f_0) \) is at least \( \epsilon \) in \( F(q) \cap N_1 \times N_2 \). So we have exactly the situation of Figure 9 with \( R(q) \) and \( f^m R(q) \) not intersecting in \( N_1 \times N_2 \). On the other hand, since \( W_L^u(q, f_1) \) has nonzero linking number with \( W_L^u(q, f_1) \) in \( N_1 \times N_2 \), Figure 10
would more accurately describe the situation for $R(q, f_1)$.

**Figure 10.** $\tilde{F}(q, f_1)$

**Lemma 10.** $q$ is the only point of period $\leq m$ for $f_0$ in $R(q, f_0)$. However, $f_1^m R(q, f_1) \cap R(q, f_1)$ has at least three components each of which contains a fixed point of $f_1^m$.

**Proof.** Since $q$ has least period $m$, $f_i^j F(q, f_i) \cap F(q, f_i) = \emptyset$ for $0 < i < m$ and so there are no points of period $< m$ in $R(q, f_i)$. Let $x \in R(q, f_0)$ with $f_0^m x = x$. Since $R \cap f_0^m R \cap N_1 \times N_2 = \emptyset$, $x \in N$. By construction of $R$, $f_0^j x \notin N$ for $j < m$.

Using (17) of §6, $f_0 = b \circ k_0$ where $k_0$ is conjugate $g \times b$ and support $b \subset N$. So $f_j^1 x = k_0^j x$ for $j = 0, 1, \ldots, m$ and $x \in \text{Fix}(k_0^m) \cap \tilde{F}(q, f_0)$. Therefore, $x = q$ and $q$ is the only point of period $m$ in $R(q, f_0)$.

The situation is different for $f_1$. Let $s_1', s_2', s_3'$ be the sides of $R(q, f_1)$ as in Lemma 7. As in Figure 10, $f_1^m s_4'$ and $f_1^m s_2'$ cut across $R(q)$ in $N_1 \times N_2$, dip below $R(q)$, and then cross it again. More precisely, there exist closed subintervals $I_{11}', I_{12}'$ of $f_1^m s_4'$ and closed subintervals $I_{21}, I_{22}$ of $f_1^m s_2'$ such that

- (a) there is $x_1'$ between $I_{11}'$ and $I_{12}'$ on $f_1^m s_4'$ lying below $R(q)$,
- (b) there is $x_2'$ between $I_{21}$ and $I_{22}$ on $f_1^m s_2'$ lying below $R(q)$,
- (c) each $I_{1j}$ has one endpoint on $s_1'$ and the other on $s_3'$, e.g., the points $a, b, c, d$ in Figure 10 where $ad$ is $I_{12}'$ and the subinterval $bc$ is $I_{22}'$.

Choose $I_{21}'$ and $I_{22}'$ so that the subinterval $\overline{ab}$ of $s_3'$ and the subinterval $\overline{cd}$ of $s_1'$ have minimal length. Similarly, choose $I_{11}'$ and $I_{12}'$. Let $M_1 \subset R \cap f_1^m R$ be the 2-disk bounded by $I_{11}'$ and $I_{12}'$ and let $M_2 (= abcd$ in Figure 10) $\subset R \cap f_1^m R$ be
the 2-disk bounded by $T_2^4$ and $T_2^2$.

Claim. $f_1^m$ has a fixed point in $M_1$ and another one in $M_2$. We will work on $M_2$; the proof for $M_1$ is isomorphic, modulo a change in orientation.

To facilitate the analysis of $M_2$, one introduces a coordinate system on $F(q) \cap T^2 \times N_2$ with $W^s_L(q, f)$ the x-axis, $q$ the origin and the positive direction toward $N_1 \times N_2$, i.e. to the right in Figure 10. Let $s'_1$ be the y-axis with positive direction toward $s_1$, i.e. "up" in Figure 10. Now $f_1^{-m}ab \subset s'_2$ and thus lies to the right of $M_2$ and $f_1^{-m}bc \subset s'_1$ and lies to the left of $M_2$; $f_1^{-m}ab$ lies in $R(q)$ below $s'_2$ and $f_1^{-m}cd$ lies in $R(q)$ above $s'_1$.

Williams has shown me the following simple technique for exhibiting a fixed point for $f_1^{-m}|M_2$ given the above situation. The set $E = \{z \in M_2: f_1^{-m}z \text{ and } z \text{ have the same y-coordinate}\}$ separates $M_2$ into two disjoint open sets, $\{z: f_1^{-m} \text{ increases y-coordinate of } z\}$ containing $cd$ and $\{z: f_1^{-m} \text{ decreases y-coordinate of } z\}$ containing $ab$. Similarly, $E_x = \{z \in M_2: f_1^{-m}z \text{ and } z \text{ have the same x-coordinate}\}$ separates $M_2$ into two disjoint open sets, one containing $bc$ and the other containing $ad$. Since $M_2$ is closed, $E \cap E_x = \emptyset$ by point-set topology arguments. But $E \cap E_y = \{z \in M_2: f_1^{-m}z = z\}$, proving this lemma.

Summarizing, we have a one-parameter family of diffeomorphisms in $U$: $f_0^m|R(q, f): R(q, f) \to R(q, f)$. $R(q, f)$ varies continuously with $t$ and in the $C^r$ case, $r < \infty$, do not vary at all. $f_0^m$ has exactly one fixed point in $R(q, f_0)$, while $f_1^m$ has at least three fixed points in $R(q, f_1)$. The set of $f$ in $U$ that have $q$ as the only fixed point of $f^m$ in $R(q, f)$ is open. So, there is a $T$ with $0 < T < 1$ such that $f_1^m$ has more than one fixed point in $R(q, f_t)$ but $f_1^m$ has $q$ as its only fixed point in $R(q, f_t)$ for all $t$ with $0 < t < T$.

K. Three perturbations of $f_t$ in $U$. In this section $U$ will mean $B^m = \bigcup_{j=0}^{m-1}$. First, one makes hyperbolic all periodic points of $f_T$ of period $\leq m$ not in the orbit of $R(q, f_T)$. From Lemma 9, there is a 3-disk $V$ such that $R \subset \text{interior } V$ and all points of $V$ of period $\leq m$ are actually in $R$. Choose $V$ small enough so that $V$, $f_TV, \ldots, f_T^{m-1}V$ are mutually disjoint. By Peixoto's proof of the Kupka-Smale Theorem [16], one can choose $f_T$ so that

1. $f_T = f_T$ in $Y = \bigcup_{j=0}^{1} f_T^jV$,
2. if $f_T^n z = z$, $0 < n \leq m$, and $z \notin Y$, then $z$ is a hyperbolic fixed point of $f_T^n$,
3. $f_T \notin U$.

Since $f_T = f_T$ in $Y$, $f_T$ has at least $3m$ points of period $m$ in $U f_T^i R$. Now, perturbing $f_T$ in $Y$ to make all points in $Y$ of period $m$ hyperbolic, one obtains, via [16] again, $g_T$ where

1. $g_T = f_T$ outside $Y$,
2. $g_T \notin U$,
3. $g_T$ has at least $3m$ points of period $m$ in $U f_T^i R$ and all points of...
period \( \leq m \) are hyperbolic.

We now want to perturb \( f_T \) in another way to \( \bar{g}_T \in U \) where

(i) \( \bar{g}_T = \tilde{f}_T \) outside \( Y \),

(ii) \( \bar{g}_T = f_T \) on \( R \) for some \( T_1 < T_0 \),

(iii) \( \bar{g}_T \) has exactly \( m \) points of period \( \leq m \) in \( Y \), all of which are hyperbolic.

Then, we will have \( g_T \) and \( \bar{g}_T \) in \( U \) such that

(a) \( g_T \) and \( \bar{g}_T \) have all points of period \( \leq m \) hyperbolic,

(b) \( N_m(g_T) \geq N_m(\bar{g}_T) \).

One of \( \{g_T, \bar{g}_T\} \) must satisfy the conclusion of Theorem 1, i.e. \( N_m(f_0) \neq N_m(g_T) \) or \( N_m(f_0) \neq N_m(\bar{g}_T) \).

So, we need only construct \( \bar{g}_T \) as above. Let \( Y_i = \bigcup f_i^j V_i, i = 0, 1, 2, 3 \) where \( V_0 \supset V \supset V_1 \supset V_2 \supset V_3 \) are all closed 3-disks with the properties that all points of \( V \) of period \( \leq m \) for \( f_T \) lie in \( R \), int \( V_3 \supset int R \), and \( \bar{V}_3 \supset int V_{i+1} \) for each \( i \).

Let \( \phi: T^3 \to R \) be \( C^r \) with the property that \( \phi = 0 \) outside \( Y \) but \( \phi = 1 \) inside \( Y_1 \) and consider the one-parameter family of maps of \( T^3 \), \( k_t = (1 - \phi) \bar{f}_T + \phi f_T \) for all \( t \). \( k_T = f_T \) since \( \phi / 0 \), i.e. in \( Y \). \( k_T = \bar{f}_T \) outside \( Y \) for all \( t \) since \( \phi = 0 \).

Let \( R_t \) be the 2-disk \( R(q, f_{t}) \) in \( \tilde{F}(q, f_t) \). \( R_t \) varies continuously with \( t \) so, there is an open interval \( (t_1, t_2) \) about \( T \) such that \( R_t \subset V_2 \) for \( t \in (t_1, t_2) \). Choose \( V_1 \) and \( (t_1, t_2) \) so that all points of period \( \leq m \) for \( f_t \) in \( V_1 \) lie on \( R_t \) when \( t \in (t_1, t_2) \). Since \( \bigcup f_T^1 V_2 \subset \bigcup \text{int } f_T^1 V_1 \), one can choose an open interval \( (t_3, t_4) \) about \( T \) so that \( \bigcup f_T^1 V_2 \subset \bigcup \text{int } f_T^1 V_1 \) for \( t \in (t_3, t_4) \). Choose an open interval \( (t_5, t_6) \) about \( T \) so that for such \( t \), \( k_t \in U \) for \( t \in (t_5, t_6) \). Choose an open interval \( (t_7, t_8) \) about \( T \) so that for such \( t \), \( k_t \) has no points of period \( \leq m \) in \( \bigcup f_T^1 (V \setminus V_1) \). This is possible since \( k_T = f_T \) has no such periodic points. Finally, choose an open interval \( (t_9, t_10) \) about \( T \) so that \( \bigcup f_T^1 V_3 \subset \bigcup k_T^1 V_2 \) for \( t \in (t_9, t_10) \).

Now choose \( t < T \) with \( t \in \bigcap_{i=1}^5 (t_{2i-1}, t_{2i}) \). Claim \( k_t \) is our desired \( \bar{g}_T \).

So it suffices to show that \( k_t \) has only \( m \) points of period \( \leq m \) in \( Y \). Let \( x \in Y \) with \( k_t^i x = x \) for some \( i \leq m \). \( Y = \bigcup f_T^i V_3 \cup \bigcup f_T^i Y \). \( x \in \bigcup f_T^i V_3 \) since \( t \in (t_7, t_8) \). \( x \in \bigcup k_t^i V_2 \) since \( t \in (t_9, t_10) \). Also, \( k_T = f_T \) in \( \bigcup f_T^i V_2 = \bigcup k_t^i V_2 \subset Y_1 \) since \( t \in (t_3, t_4) \). So \( x \in \bigcup f_T^i V_2 \) and \( f_T^i x = x \) for some \( i \). \( 1 \leq i \leq m \). Because \( t \in (t_1, t_2) \), \( x \in \bigcup f_T^i R_i \). Since \( t < T \), \( q \) is the only point of period \( \leq m \) for \( f_T \) in \( R_t \). Therefore, \( x = f_T^{-q} \) for some \( j \leq m \). So, \( k_t \) can be the \( \bar{g}_T \) needed to finish the proof of this theorem.

**BIBLIOGRAPHY**


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