QUASI-COMPLEMENTED ALGEBRAS

BY

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ABSTRACT. In this paper we introduce a class of algebras which we call quasi-complemented algebras. A structure and representation theory is developed. We also study the uniformly continuous quasi-complementors on $B^*$-algebras.

1. Introduction. Complemented Banach algebras were introduced in [11] and have been studied by various authors. The present work is an attempt to generalize these algebras.

The concept of quasi-complemented algebra is introduced in §2. Let $A$ be a semisimple quasi-complemented algebra in which every maximal modular right ideal is closed. We show that the socle of $A$ is dense in $A$. This enables us to establish a structure theorem for $A$ if $A$ has the property $x \in \text{cl}(xA)$ for all $x \in A$.

We also give a representation theorem for a primitive Banach algebra in which every maximal closed right ideal is modular and $x \in \text{cl}(xA)$ for all $x \in A$. In §5, we study quasi-complementors induced by given quasi-complementors.

We introduce the concept of continuous quasi-complementors in §6. Then we show that if $A$ is a $B^*$-algebra which has no minimal left ideals of dimension less than three, then every uniformly continuous quasi-complementor on $A$ is a complementor.

As we observed above, many fundamental properties of a complemented algebra hold for a quasi-complemented algebra. However a quasi-complemented algebra, in general, is not complemented as shown by the examples in §2.

2. Notation and preliminaries. For any subset $S$ in an algebra $A$, let $\ell(S)$ and $r(S)$ denote the left and right annihilators of $S$ in $A$, respectively. Let $A$ be a topological algebra. Then $A$ is called an annihilator algebra if, for every closed left ideal $J$ and for every closed right ideal $R$, we have $r(J) = \{0\}$ if and only if $J = A$ and $\ell(R) = \{0\}$ if and only if $R = A$. If $\ell(r(J)) = J$ and $r(\ell(R)) = R$, then $A$ is called a dual algebra.

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Let \( A \) be a topological algebra and let \( L_r \) be the set of all closed right ideals in \( A \). Then \( A \) is called a right quasi-complemented algebra if there exists a mapping \( q: R \rightarrow R^q \) of \( L_r \) into itself having the following properties:

(2.1) \( R \cap R^q = (0) \) (\( R \in L_r \));
(2.2) \( (R^q)^q = R \) (\( R \in L_r \));
(2.3) if \( R_1 \supset R_2 \), then \( R_2^q \supset R_1^q \) (\( R_1, R_2 \in L_r \)).

We call the mapping \( q \) a right quasi-complementor on \( A \) and \( R^q \) the right quasi-complement of \( R \) in \( A \). It is clear that the concept of quasi-complementation extends that of orthogonal complementation when \( A \) is a Hilbert algebra.

A right quasi-complemented algebra \( A \) is called a right complemented algebra if it satisfies:

(2.4) \( R + R^q = A \) (\( R \in L_r \)).

In this case, the mapping \( q \) is called a right complementor on \( A \) (see [11, p. 615, Definition 1]). A right quasi-complemented algebra may not be right complemented as shown by the following examples:

Example 2.1. Let \( B \) and \( p \) be given in [1, p. 396, Example 1]. Then \( p \) is a right quasi-complementor on \( B \). But \( p \) is not a right complementor. However \( B \) is a right complemented algebra under the right complementor \( R \rightarrow l(R)^* \) (see [3, p. 463, Theorem 3.6]).

Example 2.2. Let \( G \) be the compact group of real numbers mod 1 and \( A = L_p(G, \mu) \), where \( 1 < p < \infty \) and \( p \neq 2 \). It is well known that \( A \) is a commutative dual \( A^* \)-algebra which is not an ideal in the completion of its auxiliary norm (see [9, p. 35]). By Theorem 6.5, the mapping \( q: R \rightarrow l(R)^* \) is the only right quasi-complementor on \( A \). It follows from [4, p. 233, Theorem 3.8] and [9, p. 35, Theorem 23] that \( p \) is not a right complementor on \( A \). Since \( A \) has a unique right quasi-complementor, \( A \) is not a right complemented algebra.

Analogously we define left quasi-complemented algebras. In this paper, we limit our attention to right quasi-complemented algebras with the remark that similar properties hold for left quasi-complemented algebras. From now on a quasi-complemented (resp. complemented) algebra will always mean a right quasi-complemented (resp. right complemented) algebra.

Let \( X \) be a topological space and \( S \) a subset in \( X \). Then \( \text{cl}(S) \) will denote the closure of \( S \) in \( X \).

In this paper, all algebras and linear spaces under consideration are over the complex field \( C \). Definitions not explicitly given are taken from Rickart's book [10].

We shall need the following result.

Lemma 2.1. Let \( A \) be a semisimple dual algebra in which every maximal modular right ideal is closed. Then for each nonzero closed right ideal \( R \) of \( A \),
we have \( R = \text{cl}(\sum e_\alpha A) \), where \( \{e_\alpha\} \) is the family of all minimal idempotents of \( A \) contained in \( R \).

**Proof.** By [5, p. 569, Theorem 4.2], \( \{e_\alpha\} \) is not an empty set. Let \( J = \text{cl}(\sum e_\alpha A) \). By a similar argument in the proof of [5, p. 570, Theorem 5.1], we have \( \mathcal{I}(J)R = (0) \) and so \( R \subseteq \mathcal{I}(\mathcal{I}(J)) = J \). Therefore \( R = J \). This completes the proof.

### 3. A structure theorem.

**Lemma 3.1.** Let \( A \) be a quasi-complemented algebra with a quasi-complementor \( q \). Then

(i) For any family of closed right ideals \( \{R_\lambda\} \) in \( A \), we have \( \text{cl}(\sum R_\lambda) = (\bigcap R_\lambda^q)^q \).

(ii) For every closed right ideal \( R \) of \( A \), \( R + R^q \) is dense in \( A \).

**Proof.** (i) follows from the proof of [3, p. 461, Lemma 2.1].

(ii) Since \( A^q = A^q \cap A = (0) \), we have \( (0)^q = A \). Therefore it follows from (i) that

\[
\text{cl}(R + R^q) = (R^q \cap R)^q = (0)^q = A.
\]

Therefore \( R + R^q \) is dense in \( A \).

**Corollary 3.2.** A finite dimensional quasi-complemented normed algebra is a complemented algebra.

**Proof.** This follows easily from Lemma 3.1 (ii).

**Lemma 3.3.** Let \( A \) be a semisimple quasi-complemented algebra in which every maximal modular right ideal is closed. Then the socle of \( A \) is dense in \( A \).

**Proof.** Let \( \{R_\lambda : \lambda \in \Lambda\} \) be the family of all maximal modular right ideals of \( A \). By the semisimplicity of \( A \), \( \bigcap R_\lambda = (0) \) and therefore by Lemma 3.1, \( A = \text{cl}(\sum R_\lambda^q) \). Clearly \( R_\lambda^q \neq (0) \); for otherwise \( R_\lambda = (R_\lambda^q)^q = (0)^q = A \), a contradiction. Since \( R_\lambda + R_\lambda^q \) is a right ideal which contains \( R_\lambda \) properly, it follows that \( R_\lambda + R_\lambda^q = A \). Therefore by Lemma 3.1 in [7], \( R_\lambda^q \) is a minimal right ideal. Hence \( R_\lambda^q \) is contained in the socle \( S \) of \( A \) and therefore \( S \) is dense in \( A \). This completes the proof.

**Lemma 3.4.** Let \( A \) be a semisimple quasi-complemented algebra such that \( x \in \text{cl}(xA) \) for all \( x \in A \). Then each closed two-sided ideal \( J \) in \( A \) is a quasi-complemented algebra.

**Proof.** Let \( R \) be a closed right ideal in \( J \). Since \( \mathcal{I}(J) = r(J) \) (see [14, p. 37]) and \( J^q J \subseteq J \cap J^q = (0) \), it follows that \( J^q \subseteq \mathcal{I}(J) = r(J) \neq (0) \). Therefore
by the proof of [10, p. 99, Lemma (2.8.11)], \( R \) is a closed right ideal in \( A \). Let \( q \) be a given quasi-complementor on \( A \) and let \( R^q_J = R^q \cap J \). We show that \( q_J \) is a quasi-complementor on \( J \). By Lemma 3.1, we have

\[
(R^q_J)^q_J = (R^q \cap J)^q \cap J = \text{cl}(R + J^q) \cap J.
\]

Let \( x \in (R^q_J)^q_J \) and write \( x = \lim \alpha (a_\alpha + \beta_\alpha) \) with \( a_\alpha \in R \) and \( b_\alpha \in J^q \). Since \( x \in J \), it follows from Lemma 3.1 that

\[
x_A = x\text{cl}(J + J^q) \subseteq \text{cl}(x(J + J^q)) = \text{cl}(x_J).
\]

Since \( x \in \text{cl}(x_A) \), we have \( x \in \text{cl}(x_J) \). Therefore we can write \( x = \lim \beta xy_\beta \) with \( y_\beta \in J \). Since

\[
xy_\beta = \lim \alpha (a_\alpha y_\beta + b_\alpha y_\beta) = \lim \alpha a_\alpha y_\beta,
\]

it follows that \( xy_\beta \in R \) and consequently \( x \in R \). Therefore \( (R^q_J)^q_J \subseteq R \). Since \( R^q \cap J \subseteq R^q \), we have \( R \subseteq (R^q_J)^q_J \) and hence \( (R^q_J)^q_J = R \). It is easy to see that the mapping \( q_J \) satisfies the conditions (2.1) and (2.3). Therefore it is a quasi-complementor on \( J \) and this completes the proof.

We shall need the following result in \( \S 7 \).

**Corollary 3.5.** Let \( A, J \) and \( q_J \) be as in Lemma 3.4. If \( M \) is a closed right ideal in \( A \), then \( M^q \cap J = (M \cap J)^q_J \).

**Proof.** By Lemma 3.1, we have

\[
(M^q \cap J)^q_J = (M^q \cap J)^q \cap J = \text{cl}(M + J^q) \cap J.
\]

Hence by the proof of Lemma 3.4, we have \( (M^q \cap J)^q_J \subseteq M \cap J \) and so \( M^q \cap J \subseteq (M \cap J)^q_J \). Since \( M \cap J \subseteq M \), it follows that \( (M \cap J)^q_J \supseteq M^q \cap J \). Hence \( M^q \cap J = (M \cap J)^q_J \).

Now we have the following structure theorem.

**Theorem 3.6.** Let \( A \) be a semisimple quasi-complemented algebra in which every maximal modular right ideal is closed and \( x \) belongs to \( \text{cl}(xA) \) for all \( x \in A \). Then \( A \) is the direct topological sum of its minimal closed two-sided ideals, each of which is a simple quasi-complemented algebra.

**Proof.** By Lemma 3.3, the socle of \( A \) is dense in \( A \). Therefore by [14, p. 31, Lemma 3.11], \( A \) is the topological direct sum of its minimal closed two-sided ideals. By Lemma 3.4, each minimal closed two-sided ideal of \( A \) is a quasi-complemented algebra and this completes the proof.

**Remark.** Let \( A \) be an algebra. The condition that \( x \in \text{cl}(xA) \) for all \( x \in A \) is automatically satisfied if \( A \) has an approximate identity or \( A \) is a semisimple complemented algebra. Also, if \( A \) is a semisimple dual algebra, it has this property.
4. A representation theorem. The following lemma is implicit in [2, p. 40, Proposition 1].

**Lemma 4.1.** Let $A$ be a semisimple Banach algebra and $I$ a minimal left ideal in $A$. Then

(i) For each closed right ideal $R$ in $A$, $RI = R \cap I$.

(ii) For each closed subspace $E$ in $I$, $E = \text{cl}(EA) \cap I$.

**Proof.** (i). We can write $I = Ae$, where $e$ is a minimal idempotent of $A$ (see [14, p. 37]). Let $R$ be a closed right ideal in $A$ and let $x \in R \cap I$. Since $x = xe \in RI$, we have $R \cap I \subseteq RI$. But $RI \subseteq R \cap I$ and so $RI = R \cap I$. This proves (i).

(ii) Let $E$ be a closed subspace in $I$ and let $R = \text{cl}(EA)$. Since $Ee = E$, we have $E \subseteq R \cap I$. It follows from (i) that

$$R \cap I = \text{cl}(EA)I \subseteq E(eAe) = Ee = E.$$ 

Therefore $E = \text{cl}(EA) \cap I$ and this completes the proof.

Let $A$ be a primitive quasi-complemented Banach algebra and $I$ a minimal left ideal of $A$. Then $I = Ae$ for some minimal idempotent $e$ of $A$. By [10, p. 68, Corollary (2.4.16)], the left regular representation $a \mapsto T_a$ of $A$ is a faithful, continuous, strictly dense representation on $I$. Let $A' = \{T_a : a \in A\}$. Then by [10, p. 67, Theorem (2.4.12)], the image of the socle of $A$ is the set of all operators of finite rank in $A'$. Since by Lemma 3.3, the socle of $A$ is dense in $A$, it follows that $A$ is a simple algebra (see [10, p. 65]).

**Lemma 4.2.** Let $A$ be a primitive Banach algebra with a quasi-complementor $q$ such that $x \in \text{cl}(xA)$ for all $x \in A$. For each closed subspace $E$ in $I$, let $E' = [\text{cl}(EA)]^q \cap I$. Then an inner product $(x, y)$ can be introduced in $I$ having the following properties:

(i) $I$ becomes a Hilbert space under $(x, y)$.

(ii) The norm $|x| = (x, x)^{1/2}$ is equivalent to the given norm $\|x\|$ in $I$.

(iii) If $A$ is infinite-dimensional, then $E'$ is the orthogonal complement of $E$ in $I$.

**Proof.** Let $R = [\text{cl}(EA)]^q$. Since $A$ is a simple algebra and since $IA$ is a two-sided ideal in $A$, $IA$ is dense in $A$. Let $x \in R$. Then

$$xA = x \text{cl}(IA) \subseteq \text{cl}(xIA) \subseteq \text{cl}(RIA).$$

Since $x \in \text{cl}(xA)$, it follows that $x \in \text{cl}(RIA)$ and so $R \subseteq \text{cl}(RIA)$. Clearly $R \cap \text{cl}(RIA) \subseteq \text{cl}(RIA)$. Hence $R = \text{cl}(RIA)$. Therefore by Lemma 4.1(i), $R = \text{cl}(R \cap I)A = \text{cl}(E'A)$. Hence it follows from Lemma 4.1(ii) that

$$E'' = [\text{cl}(E'A)]^q \cap I = R^q \cap I = \text{cl}(EA) \cap I = E.$$ 

If $x \in E \cap E'$, then by Lemma 4.1(ii) $x \in \text{cl}(EA) \cap [\text{cl}(EA)]^q$ and so $x = 0$. 

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Therefore $E \cap E' = (0)$. If $E_1$ and $E_2$ are closed subspaces of $I$ such that $E_1 \subseteq E_2$, then clearly $E_1 \supset E'$. Therefore by [8, p. 731, Theorem 2], an inner product $(x, y)$ can be introduced in $I$ having properties (i) and (ii). If $A$ is infinite dimensional, then so is $I$. Hence (iii) follows from [8, p. 729, Theorem 1].

We have the following representation theorem.

**Theorem 4.3.** Let $A$ be a primitive quasi-complemented Banach algebra in which every maximal closed right ideal is modular and $x \in \text{cl}(xA)$ for all $x \in A$. Then there exists a continuous isomorphism of $A$ onto an algebra $A'$ of completely continuous operators on a Hilbert space. Also $A$ is a dual algebra.

**Proof.** Let $I$ be a minimal left ideal in $A$. By Lemma 4.2, $I$ is a Hilbert space. Let $a \rightarrow T_a$ be the left regular representation of $A$ on $I$ and $A' = \{T_a^*; a \in A\}$. Then $a \rightarrow T_a$ is a continuous isomorphism of $A$ onto $A'$. Letting $A'$ have the given norm of $A$, we can identify $A$ with $A'$. Let $q$ be a given quasi-complementor on $A$ and $R$ a proper closed right ideal of $A$. Since the socle of $A$ is dense in $A$, by [14, p. 37, Lemma 3.1], $R^q$ contains a minimal right ideal $M$. It is easy to see that $M^q$ is a maximal closed right ideal and so modular by the assumption. Therefore by [14, p. 38, Lemma 3.3], $I(M^q) \neq (0)$. Since $R \subset M^q$, it follows that $I(R) \neq (0)$. Therefore by the proof of [10, p. 101, Lemma (2.8.20)], $A$ contains all operators of finite rank on $I$. Hence $A$ is an algebra of completely continuous operators on $I$ (see the proof of [11, p. 657, Theorem 7]). By [10, p. 104, Theorem (2.8.23)] $A$ is an annihilator algebra. Since $I$ is reflexive and since $x \in \text{cl}(xA)$, it follows from the proof of [10, p. 105, Theorem (2.8.27)] that $A$ is a dual algebra.

**Corollary 4.4.** Let $A$ be a semisimple quasi-complemented Banach algebra in which $x \in \text{cl}(xA)$ for all $x \in A$. Then $A$ is an annihilator algebra if and only if every maximal closed right ideal of $A$ is modular.

**Proof.** Suppose every maximal closed right ideal of $A$ is modular. Let $I$ be a minimal closed two-sided ideal of $A$ and $M$ a maximal closed right ideal of $I$. By the proof of Lemma 3.4, $M^q$ is a minimal right ideal of $I$ and $A$. Therefore $N = (M^q)^q$ is a maximal modular right ideal of $A$. Since $M^q \oplus N = A$, by [3, p. 462, Lemma 3.1] $N = (1 - e)A$ and $M^q = eA$, where $e$ is a minimal idempotent. Since $e \in I$, $M = (M^q)^q = N \cap I = (1 - e)I$. Therefore $M$ is modular. By the proof of Lemma 3.4, we have $x \in \text{cl}(xl)$ for all $x \in I$. Hence by Theorem 4.3, $I$ is an annihilator algebra and so is $A$ by [10, p. 106, Theorem (2.8.29)]. The converse of the corollary follows from [10, p. 98, Corollary (2.8.7)].

**Theorem 4.5** (we use the notation in Theorem 4.3.). If $A'$ is a two-sided ideal of $B(I)$, the set of all continuous linear operators on $I$, then every quasi-complementor $q$ on $A$ is a complementor.
Proof. By Corollary 3.2, we can assume that $A$ is infinite dimensional. In this proof, we identify $A$ with $A'$. Let $R$ be a closed right ideal in $A$. To complete the proof, it suffices to show that $R + R^q$ is closed by Lemma 3.1. Let $E = R \cap I$ and let $E' = [cl(EA)]^q \cap I$. By Lemma 4.2(iii), $E'$ is the orthogonal complement of $E$ in $I$. Denote the orthogonal projection on $E$ by $P$. Let $a \in cl(R + R^q)$ and write $a = \lim_{n} (b_n + c_n)$ with $b_n \in R$ and $c_n \in R^q$. Since $b_n I \subseteq R I = R \cap I = E$, we have $(P b_n)(b) = b_n(b)$ for all $b \in I$. Hence $P b_n = b_n$. Since $c_n I \subseteq R^q \cap I = [cl(RA)]^q \cap I = cl( EA )^q \cap I = E'$, we have $P c_n = 0$. By the proof of [2, p. 41, Theorem 3], we have $\| P a - b_n \| \leq k \| a - b_n - c_n \|$, where $k$ is a constant. Hence we have $P a \in R$ and so $a - P a \in R^q$. Therefore $a = P a + (a - P a) \in R + R^q$. Hence $R + R^q$ is closed and this completes the proof.

5. Induced quasi-complementors. In this section, unless otherwise stated, $A$ will be a semisimple Banach algebra with norm $\| \cdot \|$ which is a dense subalgebra of a semisimple Banach algebra $B$ with norm $| \cdot |$. Further $A$ and $B$ have the following properties:

(5.1) There exists a constant $k$ such that $k \| x \| \geq | x |$ for all $x \in A$, i.e., $\| \cdot \|$ majorizes $| \cdot |$.

(5.2) Every proper closed left (right) ideal in $B$ is the intersection of maximal modular (right) ideals in $B$.

Notation. For any subset $E$ of $A$, $cl^A (E)$ (resp. $cl (E)$) will denote the closure of $E$ in $A$ (resp. $B$) and $l^A (E)$ and $r^A (E)$ (resp. $l (E)$ and $r (E)$) the left and right annihilators of $E$ in $A$ (resp. $B$).

Lemma 5.1. Let $A$ be an annihilator algebra. Then

(i) For each closed right ideal $R$ of $A$, we have $cl(R) \cap A = r_A (l_A (R))$.

(ii) If $M$ is a closed right ideal of $B$, then $M = cl(M \cap A)$.

Proof. First we note that $B$ is a dual algebra [13, p. 81] and $A$ and $B$ have the same socle $S$ (Lemma 4.1 in [7]).

(i) Let $\{ e_a \}$ be the family of all minimal idempotents of $B$ contained in $l(R)$. Since $B$ is a dual algebra, it follows from Lemma 2.1 that $cl( \Sigma_a B e_a ) = l(R)$. Since $e_a \in l(R) \cap S \subseteq l(R) \cap A = l_A (R)$, we have $cl(l_A (R)) \supseteq l(R)$. Clearly $l(R) \supset cl(l_A (R))$ and therefore $cl(l_A (R)) = l(R) = l(cl(R))$. Hence by the duality of $B$, we have

$$r_A (l_A (R)) = r(l_A (R)) \cap A = r(cl(l_A (R))) \cap A = r(l(cl(R))) \cap A = cl(R) \cap A.$$

This proves (i).

(ii) Let $\{ e_\beta \}$ be the family of all minimal idempotents of $B$ contained in $M$. 

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By Lemma 2.1, \( M = \text{cl}(\Sigma \alpha e_{\alpha}B) \). Since each \( e_{\beta}B \subseteq M \cap S \subseteq M \cap A \), we have \( \Sigma \alpha e_{\alpha}B \subseteq M \cap A \). It is now easy to see that \( M = \text{cl}(M \cap A) \). This completes the proof.

**Lemma 5.2.** Let \( A \) be an annihilator algebra. Then the following statements are equivalent:

(i) \( A \) is a dual algebra.

(ii) For each element \( x \in A \), we have \( x \in \text{cl}_{A}(xA) \cap \text{cl}_{A}(Ax) \).

(iii) For each closed right (left) ideal \( R \) of \( A \), we have \( R = \text{cl}(R) \cap A \).

**Proof.** (i) \( \Rightarrow \) (ii). This follows immediately from [10, p. 97, Corollary (2.8.2)].

(ii) \( \Rightarrow \) (iii). Suppose (ii) holds. Let \( S \) be the socle of \( A \). By Lemma 4.1 in [7], \( S \) is also the socle of \( B \). Let \( R \) be a closed right ideal of \( A \). We show that \( \text{cl}(R)S \subseteq R \). In fact, let \( x \in \text{cl}(R), \ y \in A \) and \( e \) a minimal idempotent in \( A \). Let \( \{x_n\} \) be a sequence in \( R \) such that \( x_n \to x \) in \( | \cdot | \). By the proof of [13, p. 82, Lemma 3.2], the norms \( \| \cdot \| \) and \( | \cdot | \) are equivalent on \( A \). Hence it follows easily that \( x_nye \to xye \) in \( \| \cdot \| \). Therefore \( xye \in R \) and so \( \text{cl}(P) \subseteq R \). Let \( a \in \text{cl}(R) \cap A \). Then we have

\[
a \in \text{cl}_{A}(aA) = \text{cl}_{A}(aS) \subseteq \text{cl}_{A}(\text{cl}(R)S) \subseteq R.
\]

Hence \( \text{cl}(R) \cap A \subseteq R \). Clearly \( R \subseteq \text{cl}(R) \cap A \) and so \( R = \text{cl}(R) \cap A \). This proves (iii).

(iii) \( \Rightarrow \) (i). Suppose (iii) holds. Let \( R \) be a closed right ideal of \( A \). By Lemma 5.1, we have \( R = \text{cl}(R) \cap A = \text{cl}(A) \). Similarly we can show that \( J = \text{cl}(J) \) for all closed left ideals \( J \) of \( A \). Therefore \( A \) is a dual algebra and the proof is complete.

**Theorem 5.3.** Let \( A \) be a dual algebra. Then for every quasi-complementor \( r \) on \( B \), the mapping \( q: R \to [\text{cl}(R)]^p \cap A \) on the closed right ideals \( R \) of \( A \) is a quasi-complementor on \( A \).

**Proof.** Let \( R \) be a closed right ideal of \( A \). Since \( A \) is a dual algebra, by Lemma 5.2, \( R = \text{cl}(R) \cap A \). Therefore

\[
R \cap R^q = \text{cl}(R) \cap [\text{cl}(R)]^p \cap A = 0.
\]

By Lemma 5.1, we have \( [\text{cl}(R)]^p = \text{cl}([\text{cl}(R)]^p \cap A) \). Therefore it follows that \( (R^q)^q = [\text{cl}([\text{cl}(R)]^p \cap A)]^p \cap A = [\text{cl}(R)]^pp \cap A = \text{cl}(R) \cap A = R \).

If \( R_1 \) and \( R_2 \) are closed right ideals of \( A \) such that \( R_1 \supset R_2 \), then clearly \( R_1^q \subseteq R_2^q \). Therefore \( q \) is a quasi-complementor on \( A \).

We now establish the converse of Theorem 5.3.

**Theorem 5.4.** Let \( A \) be a dual algebra. Then for every quasi-complementor \( q \)
on $A$, the mapping $p: M \to \text{cl}(\{M \cap A\}^q)$ on the closed right ideals $M$ of $B$ is a quasi-complementor on $B$.

Proof. Let $M$ be a closed right ideal of $B$. Then it follows from Lemma 5.2 that $M \cap M^p \cap A = \{M \cap A\} \cap \{M \cap A\}^q = (0)$. Hence it follows from Lemma 5.1 that $M \cap M^p = \text{cl}(M \cap M^p \cap A) = (0)$. We also have

$$(M^p)^p = \text{cl}(\{\text{cl}(\{M \cap A\}^q) \cap A\}^q) = \text{cl}(\{M \cap A\}^q) = M.$$ 

If $M_1$ and $M_2$ are closed right ideals of $B$ such that $M_1 \supset M_2$, then clearly $M_1^p \subset M_2^p$. Therefore $p$ is a quasi-complementor on $B$ and this completes the proof.


Theorem 6.1. Let $A$ be a dual $A^*$-algebra. Then $A$ is a quasi-complemented algebra under the quasi-complementor $q: R \to \mathcal{I}(R)^*$.

Proof. Let $R$ be a closed right ideal of $A$. Since $\mathcal{I}(R)^* = r(R^*)$, by the duality of $A$, we have $(R^q)^q = R$. It is easy to see that $q$ has properties (2.1) and (2.3). Therefore $q$ is a quasi-complementor on $A$ and this completes the proof.

It is known that a $B^*$-algebra is complemented if and only if it is dual (see [3, p. 463, Theorem 3.6]). A similar result is true for quasi-complemented algebras. In fact we have the following:

Corollary 6.2. Let $A$ be an $A^*$-algebra which is a dense two-sided ideal of a $B^*$-algebra $B$. Then $A$ is a dual algebra if and only if $A$ is quasi-complemented and $x \in \text{cl}(xA)$ for all $x \in A$.

Proof. Suppose $A$ is quasi-complemented and $x \in \text{cl}(xA)$ for all $x \in A$. Let $e$ be a minimal idempotent of $A$. Clearly $Ae = Be$. Therefore by Lemma 3.3 and Theorem 4.3 in [7], $A$ is an annihilator algebra. Hence by Lemma 5.2, $A$ is a dual algebra. The converse of the corollary follows from Theorem 6.1 and Lemma 5.2.

Corollary 6.3. Let $A$ be a $B^*$-algebra. Then $A$ is a dual if and only if $A$ is quasi-complemented.

Proof. Since a $B^*$-algebra has an approximate identity, it follows that $x \in \text{cl}(xA)$. Therefore Corollary 6.3 follows immediately from Corollary 6.2.

Lemma 6.4. Let $A$ be an annihilator semisimple Banach algebra with a quasi-complementor $q$. Then for every maximal closed right ideal $R$ of $A$, there exists a unique minimal idempotent $f$ such that $R^q = fA$ and $R = (1 - f)A$.

Proof. By [10, p. 97, Theorem (2.8.5)], $R$ is a maximal modular right ideal of $A$. Since $R + R^q = A$, by [3, p. 462, Lemma 3.1] we have the desired result.
Definition. Let $A$ be a quasi-complemented Banach algebra. A minimal idempotent $f$ in $A$ is called a $q$-projection if $(fA)^q = (1-f)A$.

We now introduce the concept of continuous quasi-complementor on annihilator $A^*$-algebras. This is similar to the concept of continuous complementor on $B^*$-algebras (see [3, p. 463, Definition 3.7]).

Definition. Let $A$ be an annihilator $A^*$-algebra with a quasi-complementor $q$. Let $E$ denote the set of all hermitian minimal idempotents and $E_q$ the set of all $q$-projections in $A$. For each $e \in E$, let $Q(e)$ be the unique element of $E_q$ such that $Q(e)A = eA$ (Lemma 6.4). The mapping $Q: e \rightarrow Q(e)$ is called the $q$-derived mapping of $E$ into $E_q$. The quasi-complementor $q$ is said to be continuous if $Q$ is continuous in the relative topologies of $E$ and $E_q$ induced by the given norm on $A$.

Remark 1. Since by [10, p. 261, Lemma (4.10.1)] every minimal right ideal of $A$ is of the form $eA$ with a unique $e \in E$, it follows that $Q$ maps $E$ onto $E_q$.

Remark 2. Let $A$ and $q$ be as in Theorem 6.1. Then $E = E_q$ and so the $q$-derived mapping $Q$ of $q$ is the identity mapping. Hence $q$ is uniformly continuous.

For commutative dual $A^*$-algebras, the study of quasi-complementor becomes very trivial.

Theorem 6.5. Let $A$ be a commutative dual $A^*$-algebra. Then there is only one quasi-complementor $q$ on $A$; $q$ is uniformly continuous.

Proof. Let $B$ be the completion of $A$ in an auxiliary norm. We use the notation introduced in §5. The existence of a quasi-complementor on $A$ is given by Theorem 6.1. Let $q$ be any given quasi-complementor on $A$. By Theorem 5.6, $q$ induces a quasi-complementor $p$ on $B$. Let $M$ be a closed ideal in $B$. Since $M \cap M^q = (0)$, it follows from [10, p. 259, Corollary (4.9.22)] that $M + M^q$ is a closed ideal in $B$. Therefore, by Lemma 3.1, $M + M^q = B$. Since $MM^q \subseteq M \cap M^q = (0)$, $M^p \subseteq \mathcal{K}(M) = r(M)$. Since $M + \mathcal{K}(M) = B$, it follows that $M^p = \mathcal{K}(M)$. Let $R$ be an ideal in $A$. Then we see that $R = R^*$ and $R^q = [cl(R)]P \cap A = \mathcal{I}_A(R)$. Therefore $q$ is uniquely determined. By Remark 2, $q$ is uniformly continuous and this completes the proof.

Corollary 6.6. Let $A$ be a commutative dual $A^*$-algebra which is a dense two-sided ideal of a $B^*$-algebra. Then there is a unique complementor $q$ on $A$; $q$ is uniformly continuous.

Proof. This follows easily from Theorem 6.5, [4, p. 233, Theorem 3.8] and [9, p. 30, Theorem 16].

7. Quasi-complementors on $B^*$-algebras. In this section, unless otherwise stated, $A$ will be a $B^*$-algebra with a quasi-complementor $q$. By Corollary 6.3, $A$ is a dual algebra.
Let $H$ be a Hilbert space with inner product $(\ ,\ )$. If $x$ and $y$ are elements of $H$, then $x \otimes y$ will denote the operator on $H$ given by the relation $(x \otimes y)(h) = (b, y)x$ for all $b \in H$. $LC(H)$ will denote the algebra of all completely continuous linear operators on $H$. If $A$ is a simple dual $B^*$-algebra, then it is well known that $A = LC(H)$ for some Hilbert space $H$. $H$ can be chosen as a minimal left ideal in $A$ with the inner product given in [10, p. 261, Theorem (4.10.3)].

**Lemma 7.1.** Let $A$ be a simple $B^*$-algebra. Then every quasi-complementor $q$ on $A$ is a complementor.

**Proof.** Since $A$ has the form $LC(H)$, it follows from Theorem 4.5 that $q$ is a complementor on $A$.

**Notation.** Let $A = LC(H)$. For every closed subspace $X$ of $H$, let $J(X) = \{a \in A: a(H) \subset X\}$. For every closed right ideal $R$ of $A$, let $S(R)$ be the smallest closed subspace of $H$ that contains the range $a(H)$ of each operator $a$ in $R$.

Let $A = LC(H)$. For each closed right ideal $R$ of $A$, by Lemma 7.1, the projection $P_R$ on $R$ along $R^q$ is continuous. Let $P_R'$ be the projection on $S(R)$ along $S(R^q)$. Since by [3, p. 464, Lemma 4.1], $S(R) \oplus S(R^q) = H$, it follows that $P_R'$ is continuous.

**Lemma 7.2.** Let $R$ be a closed right ideal of $A = LC(H)$. Then $\|P_R\| = \|P_R'\|$.

**Proof.** Let $k > 0$ be given. Choose $x \in A$ such that $\|x\| \leq 1$ and $\|P_R(x)\| \geq \|P_R\| - k/2$. Hence there exists some $b \in H$ such that $\|b\| \leq 1$ and $\|(P_R(x))(b)\| > \|P_R\| - k$. Write $x = y + z$ with $y \in R$ and $z \in R^q$. Then $y(b) \in S(R)$ and $z(b) \in S(R^q)$ and so

$$\|P_R'(x(b))\| = \|y(b)\| = \|(P_R(x))(b)\| > \|P_R\| - k.$$  

Since $\|x(b)\| \leq 1$ and $k$ is arbitrary, it follows that $\|P_R'\| \geq \|P_R\|$. By using [3, p. 464, Lemma 4.1] and a similar argument, we can show that $\|P_R'\| \geq \|P_R\|$. Therefore $\|P_R\| = \|P_R'\|$.

**Lemma 7.3.** Suppose $A = LC(H)$ with $\dim H \geq 3$, $q$ a continuous quasi-complementor on $A$ and $R$ a closed right ideal of $A$. If $\|P_R\| > k$ for some constant $k$, then there exists a $q$-projection $f \in R$ such that $\|f\| > k$.

**Proof.** By Lemma 7.2, $\|P_R'\| > k$. Hence there exists an element $b \in H$ such that $\|b\| = 1$ and $\|P_R'(b)\| > k$. Write $b = u + v$ with $u \in S(R)$ and $v \in S(R^q)$. It is clear that $u \neq 0$. Let $Q$ be a $q$-representing operator on $H$ (see [3, p. 467, Definition 5.4]) and put $f = (u \otimes Qu)/(u, Qu)$. Then $f$ is a $q$-projection (see [3, p. 467]). Since $u \in S(R), f \in R$. Let $(x, y) = (x, Qy)$ for all $x, y \in H$. Since $q$ is a continuous complementor, by the proof of [3, p. 473, Theorem 6.11], $S(R)$ is the orthogonal complement of $S(R^q)$ in $H$ relative to the inner product $(x, y)$. Since
Let $A$ be a $B^*$-algebra with a quasi-complementor $q$. Let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of $A$. Since $A$ is a dual $B^*$-algebra, $A = (\sum \lambda I_\lambda)_0$, the $B^*(\infty)$-sum of $\{I_\lambda : \lambda \in \Lambda\}$. Since each $I_\lambda$ is a simple dual $B^*$-algebra, $I_\lambda = LC(H_\lambda)$ for some Hilbert space $H_\lambda (\lambda \in \Lambda)$. By Corollary 3.5, $q$ induces a quasi-complementor $q_\lambda$ on each $I_\lambda$. By Lemma 7.1, $q_\lambda$ is a complementor on $I_\lambda$.

Let $E$ (resp. $E_\lambda$) be the set of all hermitian minimal idempotents in $A$ (resp. $I_\lambda$) and let $E_q$ (resp. $E_q^\lambda$) be the set of all $q$-projections in $A$ (resp. $I_\lambda$). Clearly $E_\lambda = E \cap I_\lambda$ and $E_q^\lambda = E_q \cap I_\lambda (\lambda \in \Lambda)$.

**Lemma 7.4.** A quasi-complementor $q$ on $A$ is continuous if and only if each $q_\lambda$ is continuous.

**Proof.** By a similar argument in [3, p. 464, Theorem 3.9], we have the desired result.

**Lemma 7.5.** Let $A$ be a $B^*$-algebra which has no minimal left ideal of dimension less than three and $q$ a quasi-complementor on $A$. If $E_q$ is a closed and bounded subset of $A$, then $q$ is a complementor on $A$.

**Proof.** For each closed right ideal $R_\lambda$ of $LC(H_\lambda)$, let $P_{R_\lambda}$ be the projection on $R_\lambda$ along $R_\lambda^a$. Let

$$k_\lambda = \sup \{\|P_{R_\lambda}\| : R_\lambda \subseteq LC(H_\lambda)\} \quad (\lambda \in \Lambda),$$

and let

$$k = \sup \{k_\lambda : \lambda \in \Lambda\}.$$

We show that $k$ is finite. Suppose this is not so. Then for each positive integer $n$, there exists some $k_\lambda \in \{k_\lambda : \lambda \in \Lambda\}$ such that $k_\lambda > n$. Hence there exists a closed right ideal $R_n \subseteq LC(H_n)$ such that $\|P_{R_n}\| > n$. Since $E_q^n = E_q \cap I_n$, it follows immediately from the assumption that $E_q^n$ is a closed and bounded subset of $I_n$. Since $q_\lambda$ is a complementor on $I_n$, by [12, p. 257, Theorem 3], $q_\lambda$ is continuous. Since $\|P_{R_n}\| > n$, it follows from Lemma 7.3 that there exists some $f_n \in E_q^n \subseteq E_q$ such that $\|f_n\| > n (n = 1, 2, \cdots )$. This contradicts the boundedness of $E_q^n$ and shows that $k$ is finite.

Let $M$ be a closed right ideal of $A$ and let $M_\lambda = M \cap I_\lambda (\lambda \in \Lambda)$. Since $A = (\sum \lambda I_\lambda)_0$, we see that $M = (\sum \lambda M_\lambda)_0$. Since by Corollary 3.5, $M_q \cap I_\lambda = M_q^\lambda$, we have

$$M_q = (\sum \lambda M_q^\lambda \cap I_\lambda)_0 = (\sum \lambda M_q^\lambda)_0.$$
Let $x = (x_\lambda) \in A$ and write $x_\lambda = y_\lambda + z_\lambda$, where $y_\lambda \in M_\lambda^A$ and $z_\lambda \in M^qA$. Then $\|y_\lambda\| = \|P_{M_\lambda^A}x_\lambda\| \leq k\|x_\lambda\|$ ($\lambda \in \Lambda$). Since $k$ is finite, it follows that $(y_\lambda) \in (\sum_\lambda M_\lambda^A)q = M$. Similarly we have $(z_\lambda) \in M^qA$. Therefore $A = M + M^q$ and so $q$ is a complementor on $A$.

We can now prove the main result of this section.

**Theorem 7.6.** Let $A$ be a $B^*$-algebra which has no minimal left ideal of dimension less than three and $q$ a quasi-complementor on $A$. If $q$ is uniformly continuous, then it is a complementor.

**Proof.** By Lemma 7.5, it suffices to show that $E_q^A$ is a closed and bounded subset of $A$. By Lemma 7.4 and [12, p. 257, Theorem 3] each $E_q^A$ is closed and bounded. Hence it follows that $E_q^A$ is closed. It remains to show that $E_q^A$ is bounded. Suppose this is not so. Then we can choose a sequence of $q$-projections $f_n$ such that $f_n \in E_q^A$ and $\|f_n\| > n$ ($n = 1, 2, \ldots$). Let $T_n$ be a $q$-representing operator on $H_n$. Then by [3, p. 470, Theorem 6.4], $T_n$ is a continuous positive linear operator with inverse $T_n^{-1}$. We may assume that $\|T_n^{-1}\| = 1$ for all $n$ (see [3, p. 472, Corollary 6.10]). We can write

\[ f_n = (u_n \otimes T_n u_n)/(u_n, T_n u_n), \]

where $u_n \in H_n$ and $\|u_n\| = 1$ ($n = 1, 2, \ldots$) (see [3, p. 467]). Since $$ \inf\{(b_n, T_n b_n) : \|b_n\| = 1 \text{ and } b_n \in H_n\} = \|T_n^{-1}\|^{-1} = 1,$$

if follows from (*) that $\|T_n u_n\| > n$ ($n = 1, 2, \ldots$). Let $Q$ be the $q$-derived mapping of $q$. By using the argument in [3, p. 477, Theorem 7.4], we can find minimal idempotents $a_n, b_n \in E$ such that $\|a_n - b_n\| \to 0$ and $\|Q(a_n) - Q(b_n)\| \to \infty$. This contradicts the uniform continuity of $Q$. Therefore $E_q^A$ is bounded and this completes the proof.

**Remark.** Let $B$ and $p$ be given in [1, p. 396, Example 1]. Then $p$ is a continuous quasi-complementor on $B$. But $p$ is not a complementor. Therefore a continuous quasi-complementor may not be uniformly continuous by Theorem 7.6. However a continuous complementor on a $B^*$-algebra is uniformly continuous (see [1] and [3]).

**Corollary 7.7.** Let $A$ be as in Theorem 7.6. Then a quasi-complementor $q$ on $A$ is uniformly continuous if and only if $E_q^A$ is a closed and bounded subset of $A$.

**Proof.** The corollary follows immediately from Theorem 7.6 and [12, p. 257, Theorem 3].
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