AN APPROXIMATION THEOREM
FOR BIHOLOMORPHIC FUNCTIONS ON $D^n$

BY
JOSEPH A. CIMA

ABSTRACT. Let $F$ be a biholomorphic mapping of the polydisk $D^n$ into $C^n$. We construct a sequence of polynomial mappings $\{P_j\}$ such that each $P_j$ is subordinate to $P_{j+1}$, each $P_j$ is subordinate to $F$ and the $P_j$ converge uniformly on compacta to $F$. The polynomials $P_j$ are biholomorphic.

Introduction. Let $D$ be the disk in the complex plane $C$ with center at the origin and radius $r > 0$ ($D_1 = D$). MacGregor [1] has shown that if $f$ is a schlicht mapping of $D$ into $C$ then there exists a sequence of schlicht polynomials $\{P_j\}_{j=1}^\infty$, each $P_j$ has degree $j$, such that $P_j$ converges uniformly to $f$ on compacta and such that $P_j$ is subordinate to $P_{j+1}$ for each $j = 1, 2, \ldots$. A second result is that if $f$ is a convex schlicht mapping of $D$ into $C$, then the $\{P_j\}$ in the above result can be chosen to be convex schlicht polynomials. A close scrutiny of the proofs of these results show that they depend principally upon the facts that $C$ is a normed linear space and the fact that $f$ is a homeomorphism. We extend these results to the following case. Let $D^n$ be the $n$-fold product of $D$ and assume $F$ is a biholomorphic mapping of $D^n$ into $C^n$, $F(0) = 0$. Then there exists a sequence of polynomial mappings $\{P_j\}$, which are biholomorphic, and which converge uniformly to $F$ on compacta. Further, each $P_j$ is subordinate to $P_{j+1}$ for $j = 1, 2, 3, \ldots$. Using a result of T. J. Suffridge we can also show that if $F(D^n)$ is convex in $C^n$ then each $P_j$ can be chosen so that $P_j(D^n)$ is convex.

Notation and definition. Let $D_r$ denote the disk in the complex plane with center at the origin and radius $r > 0$, $D^n_r$ is the $n$-fold product of such disks and $\overline{D}_r$ is the closure of such a disk. If $r = 1$ we omit the subscript. A point $Z$ in $C^n$ will be written as $Z = (z_1, \ldots, z_n)$, $z_j \in C$, and a mapping $F$ from $D^n$ into $C^n$ as $F(Z) = (f_1(Z), \ldots, f_n(Z)) = W$. If $f_j(Z)$ is holomorphic on $D^n$, then $f_j(Z) = \sum_{k=0}^\infty b_{j,k}(Z)$ which $b_{j,k}$ are homogeneous polynomials of degree $k$. For $N$ a positive integer we let $f_j,N(Z) = \sum_{k=0}^N b_{j,k}(Z)$ and $F_N(Z) = (f_1,N(Z), f_2,N(Z), \ldots, f_n,N(Z))$. Whenever a sequence of mappings $F_N$ converge uniformly on compacta to a mapping $F$ we will write $F_N(Z) \rightarrow F(Z)$. A mapping $F$ (from $D^n$ into $C^n$)

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where each \( f_j(Z) \) is a polynomial will be called a polynomial mapping, and the degree of \( F \) is \( N(j) = \max \text{ degrees } f_j \) (\( 1 \leq j \leq n \)). The norm of \( Z \) is written as \( |Z| = \max_{1 \leq j \leq n} |z_j| \). The boundary of a set \( B \) is denoted by \( \partial B \) and the distance from \( A \) to \( B \) is \( d(A, B) \). Finally, if \( 0 < r < 1 \) and \( F \) is a mapping of \( D^n \) into \( C^n \) then \( F(r) \) denotes the image of the polydisk \( D^n_r \) under the mapping \( F \). Given two biholomorphic mappings \( F \) and \( G \) with \( F(0) = G(0) \) then \( F \) is subordinate to \( G \) if \( \Re F \subset \Re G \) and this is written \( F < G \).

Two lemmas. We shall need two lemmas. The proofs follow from standard properties of analytic functions and straightforward computation.

**Lemma 1.** Let \( f \) be a holomorphic mapping of \( D^n \) into \( C \), \( f(Z) = \sum_{k=0}^{\infty} P_k(Z) \), where the \( P_k \) are homogeneous polynomials. The partial sums of \( f \) are \( f_N = \sum_{k=0}^{N} P_k(Z) \). Let \( \{C_j\} \) be a strictly increasing sequence of positive integers and \( 0 < r_j \leq 1 \). Then \( f_{C_j}(r_jZ) \) tends uniformly to \( f \) on compacta, written \( f_{C_j}(r_jZ) \to f(Z) \).

Before stating Lemma 2 we need a definition. Assume \( F \) is a holomorphic mapping on \( D^n \) into \( C^n \). Let \( Z \) and \( W \) be in \( C^n \). Define the matrix \( A_F(Z, W) = (a_{ij}(Z, W)) \) as follows:

\[
a_{ij}(Z, W) = \left( \frac{\partial f_i}{\partial z_j}(z_1, \ldots, z_j, W_{j+1}, \ldots, W_n) \right) \quad \text{if } z_j = W_j
\]

\[
= \frac{[f_i(z_1, \ldots, z_j, W_{j+1}, \ldots, W_n) - f_i(z_1, \ldots, z_{j-1}, W_j, \ldots, W_n)]}{|z_j - W_j|} \quad \text{if } z_j \neq W_j,
\]

and define the nonnegative function

\[
\phi_F(Z, W) = |\det(a_{ij}(Z, W))| + \sum_{j=1}^{n} |f_j(Z) - f_j(W)|.
\]

The Jacobian of \( F \) is written \( J_F(Z) \).

**Lemma 2.** Assume \( F \) is a holomorphic mapping of \( D^n \) into \( C^n \), \( F(Z) = (f_1(Z), \ldots, f_n(Z)) \) and assume that \( J_F \) is nonsingular for \( Z \in D^n \). Then \( \phi_F(Z, W) = 0 \) if and only if \( F(Z) = F(W), Z \neq W \).

The subordination theorems in \( C^n \). We will use the following lemmas to prove our results.

**Lemma 3.** Let \( F \) be a biholomorphic mapping of \( D^n \) into \( C^n \), \( F(0) = 0 \) and let \( 0 < r_0 < 1 \) be given. There exists an integer \( N_1(r_0) \) such that \( F_N \) is a biholomorphic mapping on \( D^n_{r_0} \) if only \( N \geq N_1(r_0) \).

**Proof.** For any \( r < 1 \) define the nonnegative function
Let \( f_i(Z) \) have an expansion in terms of its homogeneous polynomials 
\[
 f_i(Z) = \sum_{k=1}^{\infty} A_k^i(Z)
\]
and then
\[
 F_N(Z) = (f_{1,N}(Z), \ldots, f_{n,N}(Z)) = \left( \sum_{k=1}^{N} A_k^1(Z), \ldots, \sum_{k=1}^{N} A_k^n(Z) \right)
\]
We claim that the entries of the matrix for \( A_{F_N}(Z, W) \) tend uniformly to the entries of the matrix \( A_F(Z, W) \) if \( |Z| < r \) and \( |W| < r \). For the terms of the form \( \partial f_i^N(z)/\partial z_j^N \) this is a consequence of the Cauchy integral formula (and the convergence is uniform on compacta). Also if \( |z_j - W_j| > \eta > 0 \) it is clear that the terms
\[
 [f_{i,N}(z_1, \ldots, z_j, W_{j+1}, \ldots, W_n) - f_{i,N}(z_1, \ldots, z_{j-1}, W_j, \ldots)]/[z_j - W_j]
\]
can be made uniformly close to the corresponding term in the \( A_F \) matrix. For \( r \) fixed and \( \rho = (r + 1)/2 \) we have a positive constant \( P \) such that \( |F_N(Z)| < P \) for all \( |Z| < \rho \) and all \( N = 1, 2, 3, \ldots \). An application of the Cauchy integral theorem shows that if \( 0 < |z_j - W_j| \leq \eta = (1 - r)/4 \) we have another constant \( M \) (depending on \( r \) but not \( N \)) such that
\[
 |f_{i,N}(z_1, \ldots, z_j, W_{j+1}, \ldots, W_n) - f_{i,N}(z_1, \ldots, z_{j-1}, W_j, \ldots)| < |z_j - W_j| \cdot M.
\]
Hence, an application of the triangle inequality will yield that for \( (i, j) \) fixed there is an \( N \) such that the \( (i, j) \) entry of \( A_{F_N}(Z, W) \) is within \( \epsilon \) of the \( (i, j) \) entry of the \( A_F(Z, W) \) matrix on \( |Z| \leq r, |W| \leq r \). This means for \( r_0 < 1 \) and \( 0 < \epsilon < m_F(r_0)/2 \) we can find an integer \( N_0 \) so that
\[
 |m_{F_N}(r) - m_F(r)| < \epsilon
\]
for \( 0 \leq r \leq r_0 \). Also the entries in the Jacobian matrix for the \( F_N \) converge uniformly on \( \{|Z| \leq r_0\} \) to the entries in the Jacobian matrix for \( F \). Applying Lemma 2 to the \( F_N \) mappings we can find an \( N_1(r_0) \) so that the result of Lemma 3 will hold on \( |Z| \leq r_0 \) for \( N \geq N_1(r_0) \).

**Lemma 4.** Let \( r_0 < 1 \) and assuming that \( F \) is as in Lemma 3 define \( d(r_0) = d(\mathcal{R}F(r_0), \mathcal{R}F(1)). \) Then \( 0 < d(r_0) < \infty \) and there is an \( N_2(r_0) \) such that \( d(\mathcal{R}F_N(r_0), \mathcal{R}F(1)) > 0 \) whenever \( N \geq N_2(r_0) \).

**Proof.** \( F \) is assumed to be a homeomorphism, hence \( d(r_0) > 0 \). If \( d(r_0) = \infty \) this would imply that \( \mathcal{R}F(1) = \mathbb{C}^n \). Let \( G(W) = F^{-1}(W) \) be the inverse mapping to
$F$ from $\mathbb{C}^n$ into $D^n$. Using one variable theorems one shows that $g_i(W_1, 0, \ldots, 0) = g_i(0, \ldots, 0) = 0$. For $W_1$ fixed (say $W_1 = W_0$) we have

$$g_i(W_0, W_2, 0, \ldots, 0) = g_i(W_1, 0, \ldots, 0) = g_i(0, \ldots, 0) = 0.$$  

This will show each $g_i(Z) = 0$ and this implies $F$ is not biholomorphic. Since $F_N \Rightarrow F$ we can find $N_2 = N_2(r_0)$ so that if $N \geq N_2$ then $|F_N(Z) - F(Z)| < a(r_0)/2$ on $|Z| \leq r_0$. This yields the result of our lemma.

Having established the above lemmas we can proceed to the principal theorem.

**Theorem 1.** Assume $F$ is a biholomorphic mapping of $D^n$ into $\mathbb{C}^n$ ($F(0) = 0$), $F(Z) = (f_1(Z), \ldots, f_n(Z))$. Then there exist polynomial mappings $P_k(Z)$ which are biholomorphic (on $D^n$) and of degree $k$ ($k = 1, 2, 3, \ldots$) such that $P_1 < P_2 < P_3 \ldots$ ($P_k < F$) and $P_k(Z) \Rightarrow F(Z)$.

**Proof.** We assume first that some component of $F$ (i.e. some $f_j$) is not a polynomial. Let then $0 < r_0 < 1$ be given. For this $r_0$ we select an $N_1(r_0)$ as in Lemma 3 and an $N_2(r_0)$ as in Lemma 4 and we set $N(r_0) = N_1(r_0) + N_2(r_0)$. Choose an $N_1 > N(r_0)$ so that $F_{N_1}$ has degree $N_1$. Let $E_N(r)$ be the preimage of $\mathcal{R}F_N(r)$ under the mapping $F$, $E_N(r) = F^{-1}(\mathcal{R}F_N(r))$. By the openness of the mapping $F$ we can find a number $1 > p_1 > r_0$ so that $E_N(r_0) \subseteq D^{n_1}$. We continue now by setting $1 > r_1 > (1 + p_1)/2$. We can select an $N(r_1)$ as in Lemma 3. Let $a$ be the number which is the minimum of $d(r_1)$ and $d(\mathcal{R}F_N(r_0), \partial \mathcal{R}F(r_1))$. Again uniform convergence of $F_N$ to $F$ on compacta allows us to find $N_2(r_1)$ so that $|F(Z) - F_N(Z)| < a/2$ whenever $N \geq N_2(r_1)$ and $|Z| \leq r_1$. These last comments imply that

$$d(\mathcal{R}F_{N_1}(r_0), \partial \mathcal{R}F_{N_1}(r_1)) > 0$$

if $N \geq N_2(r_1)$. We have arcwise connected sets $\mathcal{R}F_{N_1}(r_0)$ and $\mathcal{R}F_{N_1}(r_1)$ with a common point and (1) implies $\mathcal{R}F_{N_1}(r_1) \supseteq \mathcal{R}F_{N_1}(r_0)$ if $N \geq N_2(r_1)$. We can set $N = N_1(r_1) + N_2(r_1) + N_1$ and choose $N_2 \geq N$ so that $F_{N_2}$ has degree $N_2$. We define the polynomial mappings

$$P_{N_1}(Z) = F_{N_1}(r_0Z), \quad P_{N_2}(Z) = F_{N_2}(r_1Z).$$

$P_{N_1}$ and $P_{N_2}$ are biholomorphic on $D^n$ and are polynomial maps of degree $N_1$ and $N_2$ respectively. They are also subordinate. One can now proceed to construct polynomial mappings $P_N(Z)$ ($N_1 < N_2 < \ldots$) which are biholomorphic on $D^n$ and satisfy $P_{N_1} < P_{N_2} < P_{N_3} \ldots$. An application of Lemma 1 shows that $P_{N_j}(Z) \Rightarrow F(Z)$.

The remaining part of this proof consists of filling in the polynomial mappings of appropriate degrees between the $P_{n_k}$ and $P_{n_{k+1}}$. Assume then $k < j$ and that $P$ and $Q$ are polynomial mappings of degrees $k$ and $j$ respectively. Further we have
$P$ and $Q$ biholomorphic on open sets containing $D^n$ and $P < Q$ on $D^n$. There is an $\overline{R} > 1$ so that $P$ is a biholomorphic polynomial mapping on $D^n_R$. Choose $1 < \rho < \overline{R}$ and define $P^*_{\rho}$ on $D^n$ as $P^*_\rho(Z) = P(\rho Z)$. Note that $\rho$ can be chosen so that 

\[ d(\overline{R} P^*_{\rho}(1), \overline{R} P^*_{\rho}(1)) > 0 \text{ and } d(\overline{R}^2 P^*_{\rho}(1), \overline{R}^2 Q) > 0, \]

and $P^*_\rho$ is a biholomorphic polynomial mapping on $D^n$ of degree $k$. Let $(b) = (b_1, \cdots, b_n) \in C^n$ and define

\[ Q^*(Z) = P^*_\rho(Z) + (b_1 z_1^{k+1}, b_2 z_2^{k+1}, \cdots, b_n z_n^{k+1}). \]

It is clear that $Q^*$ can be made uniformly close to $P^*_\rho$ on $D^n$ if only $|b|$ is small. Hence $|\det J_{Q^*}(Z)| \neq 0$ on $D^n$ and since $\phi_{Q^*}$ can be made arbitrarily close to $\phi_{P^*_\rho}$ by a suitable choice of $b$ we conclude from Lemma 2 that there is an $\eta > 0$ such that if $|b| < \eta$ then $Q^*$ is a biholomorphic polynomial of degree $k + 1$. Since 

\[ d(\overline{R} P^*_{\rho}, \overline{R} Q^*) \]

can be made small with $|b|$ and since $P < P^*_\rho < Q$ we have for $0 < \eta$ sufficiently small that $P < Q^* < Q$.

The pair $Q^*, Q$ are now biholomorphic polynomial mappings on an open set containing $D^n$ of degrees $k + 1$ and $j$ respectively and so we can find such polynomial mappings for $k + 1, \cdots, k + j - (k + 1)$. We have now the chain of polynomial, biholomorphic mappings $P_j$ of degree $j$ which satisfy the subordination relation $P_1 < P_2 < P_3 < \cdots, P_n < F$, and such that $P_n \Rightarrow F$. Assume then that $P_{n_j}$ and $P_{n_{j+1}}$ are successive members of the sequence $\{P_{n_j}\}$ and the degree of $P_{n_j}$ is $m$ and the degree of $P_{n_{j+1}}$ is $k$. We have chosen numbers $1 < \rho_\nu$ and $b_\nu \in C^n$, $\nu = m + 1, m + 2, \cdots, k - 1$, and successively defined polynomials $P_{\nu+1}(Z) = P_\nu(P_\nu Z) + (b_1^{\nu+1} z_1^{\nu+1}, b_2^{\nu+1} z_2^{\nu+1}, \cdots, b_n^{\nu+1} z_n^{\nu+1})$. Since $P_{n_j}(Z) = P_m(Z)$ is biholomorphic on $D^n$ we can choose the $1 < \rho_\nu$ so close to one that $P_m(P_m P_{m+1}, \cdots, P_{\nu-1} Z)$ is arbitrarily close to $P_m(Z)$ on all of $D^n$. The recursive definition of the $P_\nu$ will yield the estimate

\[ |P_\nu(Z) - P_m (\rho_m \rho_{m+1} \cdots \rho_{\nu-1} Z)| \]

\[ = \left| \sum_{j=1}^{\nu-m} (b_{m+j} (\rho_{\nu-m+1} \rho_{\nu-m+2} \cdots \rho_{\nu-m-(j-1)})^{m+j} z_1^{m+j}, \right| \]

\[ \cdots, b_n^{m+j} (\rho_{\nu-m+1} \cdots \rho_{\nu-m-(j-1)})^{m+j} z_n^{m+j}) \right| \]

\[ \leq |(\rho_{m+1} \cdots \rho_{k-1})|^{k-m} \sum_{j=1}^{m-1} |b^{m+j}|. \]

Now if we are given $\epsilon > 0$ we choose the $|\rho_\nu|$ so close to one that $|P_m(Z) - P_m (\rho_m \rho_{m+1} - \rho_{\nu-1} Z)| < \epsilon/2$ for $\nu = m + 1, \cdots, k - 1$ and so that if $|b^{m+j}| < \eta$ for $j = 1, 2, \cdots, k - (m - 1)$ then

\[ |P_\nu(Z) - P_m (\rho_m \rho_{m+1} \cdots \rho_{\nu-1} Z)| < \epsilon/2. \]

Hence, $|P_\nu(Z) - P_m(Z)| < \epsilon$ for $z \in D^n$ and $\nu = m + 1, \cdots, k - 1$. This proves that
one can choose the full sequence \( \{P_n(Z)\} \) so that it satisfies the subordination chain relation and so that \( P_n(Z) \Rightarrow F(Z) \).

It remains to consider only the case where \( F \) is a biholomorphic polynomial mapping of \( D^n \) of say degree \( N \). Choose \( 0 < r_j / 1 \) and define \( F_j(Z) = F(r_j Z) \) for \( j = 1, 2, 3, \ldots \). We can choose \( b^1 = (b^1_1, \ldots, b^1_n) \in \mathbb{C}^n \) of small norm so that

\[
(b^1_1 z^1_1, \ldots, b^1_n z^1_n) = P_1(Z) < F_1(Z).
\]

Now the first part of the proof allows us to assert the existence of biholomorphic polynomials \( P_1, \ldots, P_N \) satisfying \( P_1 < P_2 < \cdots < P_N = F_1 \), where each \( P_j \) has degree \( j \) and the degree of \( F_1 \) is \( N \). Again choose a \( b^2 \in \mathbb{C}^n \) with small modulus so that

\[
(b^2_1 z^{N+1}_1, \ldots, b^2_n z^{N+1}_n) = P_{N+1}(Z) < F_2(Z) + ib^2_2 z^{N+1}_2, \ldots, b^2_n z^{N+1}_n)
\]

is univalent in \( D^n \) and so that the image of \( D^n \) under the mapping \( P_N \) is properly contained in the image of \( D^n \) under the mapping \( P_{N+1} \) which in turn is properly contained in the image of \( D^n \) under the mapping \( F_1 \). The proof is now finished.

**Theorem 2.** Let \( F(Z) \) be a biholomorphic mapping \((F(0) = 0)\) of \( D^n \) into \( \mathbb{C}^n \) such that the image of \( D^n \) under \( F \) is a convex domain. Then there exists a sequence of biholomorphic polynomial mappings \( P_k, \) of degree \( k \), from \( D^n \) into \( \mathbb{C}^n \) with convex range and such that \( P_k < P_k+1 \) for \( n = 1, 2, 3, \ldots \) and \( P_k \Rightarrow F \).

**Proof.** The proof is a sandwiching together of two known results. The first is that of T. J. Suffridge [2] which states that if \( F \) satisfies the hypothesis of Theorem 2 then there exist convex univalent mappings \( g_j: D \rightarrow \mathbb{C} \) and a nonsingular linear transformation \( T \) of \( \mathbb{C}^n \) into \( \mathbb{C}^n \) such that \( F \) has a decomposition

\[
F(Z) = T \circ G(Z) = T(g_1(z_1), g_2(z_2), \ldots, g_n(z_n)).
\]

The second result is that of T. H. MacGregor [1] which states that there exist convex univalent polynomials \( P_{k,i}(z_i) \) so that \( P_{k,i}(z_i) < P_{k+1,i}(z_i) \) for \( k = 1, 2, 3, \ldots \), and such that \( P_{k,i}(z_i) \Rightarrow g_i(z_i) \), on \( D \). The degree of \( P_{k,i} \) is \( k \). Defining the functions

\[
T_k(Z) = T(P_{k,1}(z_1), \ldots, P_{k,n}(z_n)) = T \circ P_k(Z)
\]

we see that \( T_k \) is a polynomial mapping of degree \( k \) and that \( T_k \) is biholomorphic. If \( T_k(Z) \equiv T \circ P_k(Z) = T(W) = X \), then \( W = P_k(Z) = (P_{k,1}(z_1), \ldots, P_{k,n}(z_n)) \). But \( P_{k,i} < P_{k+1,i} \) implies the existence of a \( Z^1 = (z_1^1, \ldots, z_n^1) \in D^n \) such that \( P_{k+1,i}(z_j^1) = P_{k,i}(z_j) \). This implies that \( T_{k+1}(Z^1) = T_k(Z) \) and so \( T_k < T_{k+1} \) for all \( k = 1, 2, \ldots \). If now \( Z \in D^n \) we have

\[
|F(Z) - T_k(Z)| = |T \circ G(Z) - T \circ P_k(Z)| \leq \|T\| |G(Z) - P_k(Z)|
\]

and so \( T_k(Z) \Rightarrow F(Z) \).
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506

Current address: Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27514