ABSTRACT. Subsequences of boundedly complete $M$-bases need not be boundedly complete. An example of a somewhat reflexive space is given whose dual and one of whose factors fail to be somewhat reflexive. A geometric description of boundedly complete $M$-bases is given which is equivalent to the definitions of V. D. Milman and W. B. Johnson. Finally, certain $M$-bases for separable spaces give rise to proper complemented subspaces.

A sequence $(x_n)$ in a Banach space is called an $M$-basis (or Markushevich basis) of $E$ if $(x_n)$ is complete in $E$ (i.e., the closed linear span $[x_n]$ of $(x_n)$ is the whole space $E$) and if there exists a total sequence of functionals $(f_n) \subset E^*$ (i.e., $\{x \in E | f_n(x) = 0 \ (n = 1, 2, \ldots)\} = \{0\}$) such that $(x_n, f_n)$ is a biorthogonal system (i.e., $f_i(x_j) = \delta_{ij}$ for $i, j = 1, 2, \ldots$); obviously, $(f_n)$ is uniquely determined. The sequence $(x_n)$ is called a basis of $E$ if for every $x \in E$ there exists a unique sequence of scalars $(\alpha_n)$ such that

$$x = \sum_{i=1}^{\infty} \alpha_i x_i.$$

It is known [2] that every basis is an $M$-basis, with $f_n(x) = \alpha_n \ (x = \sum_{i=1}^{\infty} \alpha_i x_i \in E, \ n = 1, 2, \ldots)$, which are called the coefficient functionals.

A basis $(x_n)$ of a Banach space $E$ is called boundedly complete if the relation

$$\sup_n \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| < \infty$$

implies that $\sum_{i=1}^{\infty} \alpha_i x_i$ converges in $E$. It is known ([18], [3]) that $(x_n)$ is a boundedly complete basis of $E$, with the coefficient functionals $(f_n)$, iff the canonical mapping $\phi$ of $E$ into $[f_n]^*$ (i.e., the mapping defined by $[\phi(x)](f) = f(x)$ for all $x \in E, f \in [f_n]^*$) is an isomorphism of $E$ onto $[f_n]^*$. The notion of boundedly completeness has been extended to $M$-bases in several different ways; most of them equivalent to this property of the canonical mapping $\phi: E \rightarrow [f_n]^*$.
We shall say that an M-basis, \((x_n; f_n)\), of \(E\) is boundedly complete if \(\phi\) is an isomorphism onto \([f_n]^{**}\). In \(\S1\) of the present note we shall disprove a claim of V. D. Milman [21] on boundedly complete M-bases, which, if it had been true, would have provided an affirmative answer to a problem of [4] on boundedly completeness. In \(\S2\) we consider the existence of reflexive subspaces in Banach spaces. We also answer some questions concerning the *somewhat reflexive* spaces of Herman and Whitley [10]. In \(\S3\) we introduce a new geometric characterization of boundedly complete M-bases. Finally, in \(\S4\) we examine complemented subspaces in separable Banach spaces arising from certain M-bases.

1. The existence of boundedly complete basic sequences. In [4] the question was raised whether every separable conjugate Banach space \(E^*\) contains a subspace with a boundedly complete basis (with no assumption of separability on \(E^*\) the question was raised in [25, Problem 3.8] and [22, Problem 3]); by subspace we shall mean, throughout this note, closed linear subspace. Since the first version of this paper, Johnson and Rosenthal [16] have given an affirmative answer to the problem. Earlier, V. D. Milman [21] claimed that the answer to this question is affirmative, via the following argument: Since \(E^*\) is separable, there exists, by a result of Gapoškin and Kadec [7], a biorthogonal system \((x_n; f_n)\) such that \([x_n] = E\) and \([f_n] = E^*\). Then \((f_n)\) is a boundedly complete M-basis of \(E^*\). Since \([x_n] = E\), \((f_n)\) contains \((17), (21)) a subsequence \((f_{n_j})\) which is a basic sequence (i.e., a basis of its closed linear span \([f_{n_j}]\)). Now, Milman claims [21, Proposition 3.4b] that every subsequence of a boundedly complete M-basis is a boundedly complete M-basis of its closed linear span and hence \([f_{n_j}]\) is a subspace of \(E^*\) with the desired property, i.e., having a boundedly complete basis \((f_{n_j})\). Firstly, we want to remark that this claim is false, even when \((f_{n_j})\) is a basic sequence, as shown by Example 1. Let \(E = c_0\) and let

\[
\begin{align*}
x_{2n-1} &= e_{2n-1} - 2^n e_{2n} + 2^{n+1} e_{2n+2} & (n = 1, 2, \ldots), \\
x_{2n} &= 2^n e_{2n} & (n = 1, 2, \ldots), \\
f_{2n-1} &= b_{2n-1} & (n = 1, 2, \ldots), \\
f_2 &= b_1 + \frac{1}{2} b_2, & f_{2n} = - b_{2n-3} + b_{2n-1} + (1/2^n)b_{2n} & (n = 2, 3, \ldots),
\end{align*}
\]

where \((e_n)\) is the unit vector basis of \(c_0\) (i.e., \(e_n = 0, \ldots, 0, 1, 0, \ldots\) and \((b_n)\) the sequence of coordinate functionals on \(E = c_0\) (i.e., \(b_n(x) = \xi_n\) for all \(x = (\xi_j) \in E\), hence \(b_n(e_j) = \delta_{ij}\) (i, j = 1, 2, \ldots). Then \((x_n; f_n)\) is a biorthogonal system. Furthermore, since \(e_{2n} = (1/2^n)x_{2n}\), \(e_{2n-1} = x_{2n-1} + x_{2n} - x_{2n+2}\) (\(n = 1, 2, \ldots\)) and, since \([e_n] = E\), we have \([x_n] = E\). Similarly, since \(b_{2n-1} = \ldots\)
BOUNDEDLY COMPLETE M-BASES

Let \( b_2 = 2(f_2 - f_1), b_{2n} = 2^n(f_{2n} - f_{2n-1} + f_{2n-3}) \) \((n = 2, 3, \ldots)\) and, since \( [b_n] = E^* \), we have \( [f_n] = E^* \), making \( (f_n) \) boundedly complete. However, the subsequence \( (f_{2n}) \) of \( (f_n) \) is a basic sequence which is not boundedly complete. Indeed, for any scalars \( a_1, \ldots, a_n \) we have

\[
\sum_{i=1}^{n} \frac{a_i}{2^i} = \sum_{i=1}^{n} a_i \left( \frac{1}{2} b_1 + \sum_{i=2}^{n} a_i \left( -\frac{b_{2i-2} + b_{2i-1} - \frac{1}{2} b_{2i} \right) \right) = \sum_{i=1}^{n-1} \left( a_i - a_{i+1} \right) b_{2i-1} + \sum_{i=1}^{n} \frac{1}{2^i} a_i b_{2i} \\
= \sum_{i=1}^{n-1} \left| a_i - a_{i+1} \right| + \sum_{i=1}^{n} \frac{1}{2^i} |a_i|,
\]

whence, for any scalars \( a_1, \ldots, a_{n+1}, \)

\[
\sum_{i=1}^{n} \frac{a_i}{2^i} \leq \sum_{i=1}^{n} \left| a_i - a_{i+1} \right| + \sum_{i=1}^{n} \frac{1}{2^i} |a_i| = \sum_{i=1}^{n+1} a_i f_{2i},
\]

and therefore, \( (f_{2n}) \) is a "monotone" basic sequence [3]. Finally, for \( a_1 = \ldots = a_n = 1 \), we have \( \sum_{i=1}^{n} a_i f_{2i} = 1 + \sum_{i=1}^{n} (1/2^i) \) \((n = 1, 2, \ldots)\), whence \( \sup_i \| \sum_{i=1}^{n} f_{2i} \| = 2 \), but \( \sum_{i=1}^{\infty} 1/2^i \) is not convergent, since \( \| f_{2n} \| > 1 \) \((n = 1, 2, \ldots)\), and thus \( (f_{2n}) \) is not boundedly complete.

It is interesting to see what conditions on a biorthogonal system \( (x_n; f_n) \) such that \( [x_n] = E, [f_n] = E^* \) will guarantee that every basic subsequence \( (f_{n_j}) \) of \( (f_n) \) is boundedly complete. Standard arguments show that, if \( [f_{n_j}] \) is \( \sigma(E^*, E) \)-closed, then \( (f_{n_j}) \) is boundedly complete (if \( [f_n] = E^* \)). Hence, since clearly \( [f_{n_j}] \subset [x_{i,j}]_{i \neq n_1, n_2, \ldots} \), and since the latter is \( w^* \)-closed, a rather natural sufficient condition is \( \Sigma_{i,j}^{\infty} 2^{-ij} b_{2i} \) is in \( [x_{2n-1}]^1 \) but is not in \( [f_{2n}] \). These considerations lead to the following problem.

**Problem 1.** If \( E^* \) is separable, is there an M-basis \( (x_n; f_n) \) for which \( [f_n] = E^* \) and such that, for every subsequence of the integers, \( [f_{n_j}] \) is \( \sigma(E^*, E) \)-closed?

**Problem 2.** Does every separable Banach space have an M-basis for which \( [x_{n_j}] = [f_{n_j}] \) for every subsequence \( (n_j) \) of the integers?

An M-basis satisfies the condition of Problem 2 if and only if for every \( x \) in \( E \) there exist multipliers \( \lambda_{n, i}(x) \) \(1 \leq i \leq n < \infty\) such that \( x = \lim_{n \to \infty} \sum_{i=1}^{n} \lambda_{n, i}(x) f_{i} \) \( \lambda_{n, i}(x) \). Such a condition is somewhat weaker than strongly series summable M-bases ([23], [14]) in which the multipliers do not depend on \( x \).

Such M-bases exist in separable complex Banach spaces whose duals have the bounded approximation property [14].

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
2. On somewhat reflexive spaces. Milman [21] has also used the erroneous claim (see §1) that subsequences of boundedly complete $M$-bases are boundedly complete in his proof of the following interesting result: If $E^{**}$ is separable then both $E$ and $E^*$ contain infinite dimensional reflexive subspaces.

W. B. Johnson and H. P. Rosenthal [16] have proven that the above statement is true. Below we still include a proof of this statement in the case that $E^{**}$ has the approximation property, which we obtained independently. These results are used in the example concerning somewhat reflexive spaces in the weaker form we present.

Herman and Whitley [10] have called a space $E$ somewhat reflexive if every infinite dimensional subspace of $E$ contains an infinite dimensional reflexive subspace. They show, for example, that every quasi-reflexive space (i.e., $\dim E^{**}/E < \infty$) is somewhat reflexive and note that the space $(\ell_2 \times \ell_2 \times \cdots)_2$ (with $\ell_2$ the quasi-reflexive space of James [11]) is somewhat reflexive. It should be noted that a proof of the Milman statement (above) will force $E$ (having $E^{**}$ separable) to be somewhat reflexive: If $X$ is an infinite dimensional subspace of $E$, then $X^{**} \sim X^{\perp_{**}}$ makes $X^{**}$ separable (where $\sim$ stands for "isomorphic to"), so that $X$ would contain an infinite dimensional reflexive subspace.

In what follows, we make frequent use of the following theorem of James [12] and Lindenstrauss [19]: For any separable $B$-space, $E$, there is a $B$-space $Y$ with a shrinking basis such that $E$ is a continuous image of $Y^*$ and such that $Y^{**} \sim Y \times E^*$.

**Theorem 1.** If $E^{**}$ is separable and has the approximation property, then both $E$ and $E^*$ are somewhat reflexive.

**Proof.** Let $Y$ be the space of James and Lindenstrauss above which has $Y^{**} \sim Y \times E^*$. Then $Y^{**}$ has the approximation property, and hence a boundedly complete basis by Theorem 1.4 of [15]. By Corollary 5 of [4], every infinite dimensional subspace $G$ of $E$ ($G \subseteq E^{**} \subseteq Y^{**}$) contains a boundedly complete basic sequence, say $(x_n)$. By Proposition 2 of [4], some block basic sequence $(y_n)$ with respect to $(x_n)$ is shrinking. It follows that $(y_n) \subseteq G$ is reflexive as desired. The same argument can now be applied to $E^*$ since, by [8], $E^*$ has the approximation property.

We now turn to some questions raised in [10] concerning somewhat reflexive spaces. There the authors remarked that they did not know whether or not duals and/or quotients of somewhat reflexive spaces are somewhat reflexive. This example answers both questions in the negative:

**Example 2.** Let $Y$ be the space of James and Lindenstrauss above such that $c_0$ is a quotient of $Y^*$ and $Y^{**} \sim Y \times l_1$. $Y$ and $l_1$ both have bases, so by
Theorem 1, $Y^*$ is somewhat reflexive. However, neither $l_1$ nor $c_0$ contains an infinite dimensional reflexive subspace so that this space is somewhat reflexive and has both a quotient and dual which are not somewhat reflexive. A somewhat more remarkable feature of this space $(Y^*)$ is that although it is separable, contains no copy of $c_0$ or $l_1$, its second dual $Y^{**} \sim Y^* \times m$ is nonseparable.

Consideration of this example has led the authors to the following question.

**Problem 3.** If $E$ is a separable somewhat reflexive space, is $E^*$ separable?

3. **Boundedly complete $M$-bases.** Many definitions of boundedly complete $M$-bases appear in the literature. All of these (known to the authors) reduce in the basis situation to the concept studied by Dunford and Morse [6], Alaoglu [1], Karlin [18] and James [11] (the name "boundedly complete" seems to be due to Day [3]).

The original definition asserts that a basis is boundedly complete if the boundedness of the partial sums $\sum_{i=1}^{n} a_i x_i$ forces the convergence of the series $\sum_{i=1}^{\infty} a_i x_i$. The main motivation for such a definition (at least at this point in time) is the fact that boundedly complete bases span dual spaces. The second author (in [25]) has shown that such "boundedness implies convergence" conditions are much too strong for use with $M$-bases. Therefore, for $M$-bases, the definitions in the literature are constructed so that boundedly complete $M$-bases span separable duals ([13], [21]). In this section we present a geometric definition of boundedly complete $M$-basis which is equivalent to the known "soft" definitions in the literature.

**Definition.** Let $(x_n; f_n)$ be an $M$-basis for $E$. We shall call it norm-boundedly-complete if

$$\sup_n \inf_{\|u\| \leq 0} \left\| \sum_{i=1}^{n} a_i x_i + u \right\| < \infty$$

implies the existence of $x$ in $E$ with $f_n(x) = a_n$ for all $n$. (Here $S_n u = \sum_{i=1}^{n}/(a) x_i$.)

The above definition is strongly related to the norming characteristic of the subspace $[f_n]$ of $E^*$ through the following considerations: For any $x$ in $E$ let

$$|x| = \sup_{\|f\| \leq 1} f(x).$$

This always defines a norm on $E$, and in case it is equivalent to the original norm, we say that $[f_n]$ is norming (or of positive characteristic [5]). If $(x_n)$ is a basis for $E$, it follows that $[f_n]$ is norming [24]. If $(x_n; f_n)$ is an $M$-basis for $E$, then it can be shown that
\[ |x| = \sup_n \inf_{u=0} \|(S_n x) + u\|. \]

In what follows, \( \phi: E \rightarrow [f_n]^* \) is to denote the natural map defined by \( (\phi(x))(f) = f(x) \). It is well known (e.g., [5]) that \([f_n]^*\) is norming if and only if \( \phi \) is an isomorphism of \( E \) into \([f_n]^*\). We call an \( M \)-basis \((x_n; f_n)\) of \( E \) boundedly complete if \( \phi \) is an isomorphism of \( E \) into \([f_n]^*\) (equivalently, \( \phi(E) = [f_n]^* \)).

**Theorem 2.** An \( M \)-basis \((x_n; f_n)\) for \( E \) is boundedly complete if and only if it is norm-boundedly complete.

**Proof.** Assume that \( \phi(E) = [f_n]^* \) and let \((a_j)\) be a sequence of scalars with

\[ \inf_{S_n u=0} \left\| \sum_{i=1}^{n} a_i x_i + u \right\| \]

bounded in \( n \). Since \([f_n]^*\) is separable, some subsequence of \( \phi(\sum_{i=1}^{n} a_i x_i + u_n) \) converges weak* to some \( F \) in \([f_n]^*\). \((u_n)\) has been chosen to keep \( (\sum_{i=1}^{n} a_i x_i + u_n) \) a bounded sequence. Then \( F = \phi(x) \) for some \( x \in E \) whence

\[ f(x) = (\phi(x))(f) = F(f) = \lim_{k} \phi \left( \sum_{i=1}^{n_k} a_i x_i + u_{n_k} \right) \]

for all \( f \in [f_n]^* \) and hence \( f(x) = a_i \) (\( i = 1, 2, \ldots \)). For the other direction suppose that \((y_j)\) is a bounded sequence in \( E \) with \( \lim_j f_n(y_j) = a_n \) (existing for each \( n \)). It follows that \( S_n y_j \) converges strongly to \( \sum_{i=1}^{n} a_i x_i \) for each \( n \). Thus, one can choose \((y_j)\) and

\[ z_n = \sum_{i=1}^{n} a_i x_i + (I - S_n) y_{i_n} \]

in such a way that \( \|z_n - y_{i_n}\| < 2^{-n} \) for each \( n \). Since

\[ \inf_{S_n u=0} \left\| \sum_{i=1}^{n} a_i x_i + u \right\| \leq \|z_n\| \leq \|y_{i_n}\| + 2^{-n}, \]

there is \( x \in E \) with \( f_i(x) = a_i \) for all \( i \). This gives the result by Theorem II.5 of [13].

4. \( M \)-bases and complemented subspaces. Milman's Proposition 3.5, Theorem 3.7, and Theorem 3.8 of [21] give conditions which guarantee the complementation in \( E \) of \([x_n]\) where \((x_n)\) is a boundedly complete basic sequence in the
separable Banach space $E$. For example, $[x_n]$ is complemented if there are coefficient functionals $(f_n^*) \subset E^*$ with $[f_n^*]$ norming $[x_n]$. In Theorem 3 below, $(x_{n_k})$ need not be basic or boundedly complete, but must be part of an $M$-basis for $E$. Extensions of $M$-bases of subspaces to $M$-bases of $E$ are treated in [9] and [26].

Theorem 3. Let $(x_n^*; f_n^*)$ be an $M$-basis for $E$, let $(n_k)$ be an infinite subsequence of the integers and let $(n_k')$ denote the complementary subsequence. If $[f_{n_k}^*]$ is norming over $[x_{n_k}]$, then $E = [x_{n_k}] \oplus [x_{n_k}]$. If $[f_n^*]$ is norming over $E$, the converse holds.

Proof. Consider the quotient map $q: E \to E/[x_{n_k}^*]$. Then by the norming hypothesis and by $[f_{n_k}^*] \subset [x_{n_k}]$, there exists $\mu > 0$ such that, for $u \in [x_{n_k}]$,

$$\mu \|u\| \leq \sup_{g \in [f_{n_k}^*]; \|g\| \leq 1} |g(u)| \leq \|q(u)\| \leq \|u\|,$$

whence $q|\ [x_{n_k}]$ is an isomorphism. Further $sp(x_n)$ is dense in $E$, so that $sp(q(x_{n_k}))$ is dense in $E/[x_{n_k}]$. Thus, $[x_{n_k}]$ is isomorphic to $E/[x_{n_k}]$. It is standard (and readily verified) that a projection of $E$ onto $[x_{n_k}]$ along $[x_{n_k}]$ is given by $P = (q|\ [x_{n_k}]^{-1}q$.

For the second assertion, let $[f_n^*]$ be norming over $E = [x_{n_k}] \oplus [x_{n_k}]$. Let $u \in [x_{n_k}]$ and $g \in sp(f_n^*)$ with $\|g\| \leq 1$ such that $g(u) \geq \mu \|u\|$. If $P$ is the projection of $E$ onto $[x_{n_k}]$ along $[x_{n_k}]$, $P^*g \in sp(f_{n_k})$ (because $P^*f_{n_k} = f_{n_k}$, $P^*f_{n_k} = 0$) and $\|P^*g\|P^*P(P^*g)(u) = (P^*g)(u) = g(Pu) = g(u) \geq \mu \|u\|$. Thus, $(P^*g\|P\|P)(u) \geq (\mu \|P\|\|u\|$, so that $[f_{n_k}^*]$ norms $[x_{n_k}]$.

Corollary. Let $(x_n^*; f_n^*)$ be an $M$-basis for $E$ such that for every subsequence $(n_k)$ of the integers, $[f_{n_k}^*]$ norms $[x_{n_k}]$. Then $(x_n)$ is an unconditional basis of $E$.

Proof. For every $(n_k)$, by Theorem 3, $E = [x_{n_k}] \oplus [x_{n_k}]$. The result follows from a result of Lorch [20].

REFERENCES


DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210

INSTITUTE OF MATHEMATICS, ACADEMY OF SCIENCE OF THE SOCIALIST REPUBLIC OF ROMANIA, BUCHAREST, ROMANIA