A CHARACTERIZATION OF $U_3(2^n)$ BY ITS SYLOW 2-SUBGROUP

BY

ROBERT L. GRIESS, JR.

ABSTRACT. We determine all the finite groups having a Sylow 2-subgroup isomorphic to that of $U_3(2^n)$, $n \geq 3$. In particular, the only such simple groups are the $U_3(2^n)$.

1. Introduction. Let $N$ be the normalizer of a Sylow 2-subgroup in the projective special unitary group $U_3(2^n)$, $n \geq 3$. In [1], M. Collins proved

Theorem. Suppose $G$ is a finite simple group with Sylow 2-subgroup $S$. If $N_G(S)/O(N_G(S)) \cong N$, then $G \cong U_3(2^n)$.

In this paper, we remove the hypothesis on the normalizer.

Theorem 1. If $G$ is a finite simple group with Sylow 2-subgroup isomorphic to that of $U_3(q)$, $q = 2^n$, $n \geq 3$, then $G \cong U_3(q)$.

Theorem 2. If $G$ is a finite group with Sylow 2-subgroup isomorphic to that of $U_3(q)$, $q = 2^n$, $n \geq 3$, then either

(i) $G$ is solvable of 2-length one; or
(ii) $G/O(G)$ has a normal subgroup of odd index isomorphic to $U_3(q)$.

These results are a step in the general program of characterizing simple groups by their Sylow 2-subgroups. Using Collins' method as a skeleton for our proof, we analyze the possibilities for the action of $N_G(S)$ on $S$ and $Z(S)$, where $S$ is a Sylow 2-subgroup of $G$, then generalize certain of his arguments. For $q = 4$, the conclusion of Theorem 1 was obtained by R. Lyons [5]. Since the author proved these theorems, he has learned that M. Collins has obtained similar results. Collins' methods and our methods of proof differ significantly, however.

2. Notation and assumed results. Group theoretic notation is standard (e.g., see [3]). For a group $X$, $O(X)$ denotes the largest normal subgroup of odd order. $A_G(X)$ denotes $N_G(X)/C_G(X)$ for $X \subseteq G$. We use the bar convention for denoting

Received by the editors October 26, 1971.


Key words and phrases. Projective special unitary groups, 2-closed centralizers of involutions, Sylow 2-subgroups, connected, balanced group.

Copyright © 1973, American Mathematical Society

181
homomorphic images. The 2-rank \( m(X) \) of a group \( X \) is the minimal number of
generators for an elementary abelian 2-subgroup of maximal order in \( X \).

We need some information about the structure of \( S \). Throughout this paper,
\( q = 2^n, n \geq 3 \).

**Lemma 1.** (i) \( \Omega_1(S) = Z(S) = S' = \Phi(S) \).
(ii) If \( t \in S, t \notin Z(S), \) then \( [t, S] = Z(S) \) and \( C_S(t) \) is abelian of order \( q^2 \).
(iii) If \( H < Z(S), \) then \( Z(S/H) = Z(S)/H; \) for \( t \) above, \( C_{S/H}(tH) \) has order
\( q^2 \) as \( [tH, S/H] = Z(S/H) \).

**Proof.** The assertions follow from inspecting this presentation of \( S \):
\( S = \{x(a, b) \mid a, b \in GF(q^2), aa^\sigma = b + b^\sigma \ \text{where} \ \langle \sigma \rangle = \text{Gal}(GF(q^2)/GF(q)) \)
and \( x(a, b)x(c, d) = x(a + c, b + d + ac^\sigma) \).

We also use, sometimes without comment, the Feit-Thompson theorem on the
solvability of groups of odd order \( [2], \) Walter's classification of groups with
abelian Sylow 2-subgroups \( [8] \), Suzuki's classification of groups with 2-closed
centralizers of involutions \( [7] \), and the following result of Gorenstein-Walter
(see \([4]\) for definitions).

**Theorem C.** If \( G \) is a connected, balanced group, with 2-rank \( m(G) \geq 3 \),
\( O(G) = 1 \), and the centralizer of every involution is 2-generated, then
\( O(C_G(x)) = 1 \), for all involutions \( x \in G \).

Since Theorem 1 follows directly from Theorem 2, we may assume henceforth
that \( G \) is a group satisfying the hypotheses of Theorem 2, and that \( O(G) = 1 \).

3. The proof.

**Lemma 2.** Suppose \( \alpha \in A_G(S), 1 \neq |\alpha| \) is odd, and \( \alpha \) acts trivially on \( Z(S) \).
Then \( \alpha \) acts fixed point freely on \( S/Z(S) \).

**Proof.** Write \( \overline{S} = S/Z(S) \) and suppose \( \overline{x} \in \overline{\alpha} \), \( \overline{x}^\alpha = \overline{x} \). Then, as \( C_S(x) = C_S(y), \)
for any \( x, y \in x, \alpha \) stabilizes \( C_S(x) \), for \( x \in x \). By Fitting's theorem, \( \alpha \) is
trivial on \( C_S(x) \) since \( C_S(x) \) is \( \alpha \)-invariant, abelian, and \( Z(S) = \Omega_1(C_S(x)) \).

Let \( L = L_1 \oplus L_2 \) be the Lie algebra associated with \( S \) and let \( M \) be the
image of \( C_S(x) \) in \( L_1 \). Let \( L_0 = L_1 \otimes L_2, M_0 \) be the above objects tensored
with an algebraically closed field \( k \) of characteristic 2. In \( M_0, \alpha \) has eigen-
 vectors \( \xi_1, \ldots, \xi_n \) for the eigenvalue 1. Let \( \eta_1, \ldots, \eta_n \) be a complementary
set of eigenvectors in \( L_1 \otimes \) for the \( n \) remaining eigenvalues \( \lambda_1, \ldots, \lambda_n \in k \).
Since \( [M_0, N_0] = 0 \) in \( L_2, k \eta_1 \oplus \cdots \oplus k \eta_n = N_0 \) is an \( \alpha \)-invariant complement
 to \( M_0 \) in \( L_1 \) and the pairing \( (n, m) \mapsto \langle n, m \rangle \in L_2, n \in N_0, m \in M_0 \) is non-
dergnerate, or else some element of \( C_S(x) \setminus Z(S) \) has too large a centralizer in \( S \).
So, $L_{20}$ is spanned by all $[\eta_i, \xi_j]$, which are eigenvectors for the values $\lambda_i \cdot 1 = \lambda_i$. Since $\alpha$ is trivial on $Z(S) = L_2$, $\lambda_i = 1$. Thus, $\alpha$ is trivial on $S/Z(S) = L_1$. By 5.3.2 of [3], $\alpha = 1$. This proves the lemma.

**Lemma 3.** $N_G(Z(S))$ is solvable of 2-length 1.

**Proof.** It suffices to prove the statement for $C_G(Z(S))$, since $|N_G(Z(S))/C_G(Z(S))|$ is odd.

Set $D = C_G(Z(S))/C_G(Z(S))/O(C_G(Z(S)))$. Then, $O(D) = 1$, $Z(D) = Z(S)$. Set $E = D/Z(S)$; $E$ has abelian Sylow 2-subgroups.

Suppose $E$ is nonsolvable. Then $E$ has a normal subgroup $F$ of odd index, where $F$ is the direct product of an elementary abelian 2-group, and at least one Janko group, group of Ree type, or $L_2(q)$ ($q \equiv 3, 5$ (mod 8), $q \geq 5$, or $4|q$). Let $N$ be the normalizer in $F$ of a Sylow 2-subgroup $S^*$. By the Frattini argument, $N$ is a quotient of $N_G(S) \cap C_G(Z(S))$. Lemma 2 then implies that any element of $A_F(S^*)$ acts fixed point freely on $S^*$. Hence the only possibility is $F = L_2(q)$. If $F$ is the preimage of $F$ in $D$, $F$ is a nonsplit perfect extension of $F$ by $Z(S)$ because the induced extension $\overline{S}$ of $S^*$ has $Z(S) \subseteq \overline{S}'$. But $Z(S)$ is noncyclic while the multiplier of $L_2(q)$ is always cyclic [6], contradiction.

Thus, $E$ is solvable, and so is $C_G(Z(S))$.

**Definition.** Choose $z \in Z(S)^*$, and set $E_1 = C_G(z)/O(C_G(z))(z)$. Let $\mathcal{E}_i = \{E_1\}$. Define families $\mathcal{E}_i$, $i = 2, \ldots, n$, of sections of $E_1$ as follows: we say $E \in \mathcal{E}_i$ if there is an $F \in \mathcal{E}_{i-1}$ and an involution $\zeta \in Z(T)$, $T$ a Sylow 2-subgroup of $F$, with $E = C_F(\zeta)/O(C_F(\zeta))(\zeta)$.

$E_i$ denotes a typical member of $\mathcal{E}_i$, and $S_i$ denotes the image of $S$ in $E_i$ under the obvious sequence of homomorphisms.

**Proposition.** Each $E_i$ is solvable and 2-closed.

We use downward induction on $i$. The proof goes in a sequence of lemmas which are directed toward using Theorem C. In what follows, the $\zeta$ of the definition may be assumed to lie in $S_{i-1}$.

**Lemma 4.** Let $\nu$ be an involution in $S_i$ not in $Z(S_i)$, for $i < n$. Then $\nu$ is not conjugate in $E_i$ to an element of $Z(S_i)$.

**Proof.** By Lemma 1(iii), $S_i$ is nonabelian. Suppose $\nu$ is conjugate to $\zeta \in Z(S_i)$. By regarding $E_i$ as a section of $C_G(Z)$, consider $S_i$ as a quotient of $S$, and see that the preimage in $S$ of $\langle \nu \rangle$ has exponent 4, while the preimage of $\langle \zeta \rangle$ is elementary, contradiction. The lemma follows.

In the next four lemmas, when $i < n$, $\nu$ has the above meaning, and when $i = n$, $\nu$ is any involution of $S_n$. We may drop the subscript and write $E$ for $E_i$ when confusion is unlikely.
Lemma 5. A Sylow 2-subgroup of $C_E(v)$ is contained in any Sylow 2-subgroup of $E$ in which $v$ lies.

Proof. Use Lemmas 1(iii) and 4.

Lemma 6. $C_E(v)$ is solvable of 2-length 1.

Proof. Express $K = C_E(v)/O(C_E(v))$ as a section $K = A/M$ of $C_G(z)$. The lemma will follow once we show that $A/M$ is covered by a subgroup of $N_G(Z(S))$, by Lemma 3.

Let $w \in A$ represent $v$ with $w^2 = t \in Z(S)$#. Let $T$ be a Sylow 2-subgroup of $M$, $T \subseteq Z(S)$. Let $(\zeta_0, \ldots, \zeta_{i-1})$ be the sequence of involutions defining $E$, i.e., $\zeta_0 = z$, $\zeta_1 \in E_1$, $\ldots$, etc. We may choose an involution $z_j \in T$ representing $\zeta_j$.

We claim $NA(T)$ acts trivially on $T = \langle z_0, \ldots, z_{i-1} \rangle$. For $i = 0$, this is obvious, so assume $i > 0$. Consider $K$ as a quotient of a subgroup $H$ of $C_E(z_i-1)$, and write $H$ as a quotient $A/M$ of $C_G(z)$, with $M \supseteq B$. By the Frattini argument, $A = B \cdot NA(T)$. Since $z_{i-1} \in T$ maps to an element of $Z(A/M)$, $N_A(T)$ stabilizes the normal series $T \supset T_0 \supset 1$, where $T_0 = T \cap B$ is a Sylow 2-subgroup of $B$. Now $N_A(T)$ acts trivially on $T/T_0 \cong Z_2$, and, by induction, is trivial on $T_0$. So, $N_A(T)$ induces a 2-group of automorphisms on $T$, by 5.3.2 of [3]. But $T$ is contained in a Sylow 2-center of $C_G(z)$. Hence, $N_A(T)$ acts trivially, i.e., $N_A(T) = C_A(T)$.

Now, set $C = C_{C_A(T)}(w)$, $C_j = \{x \in C_A(T) | [w, x] \in \langle z_0, \ldots, z_j \rangle \}$ for $j = 0, \ldots, i - 1$. Then, $C_{i-1}$ covers $A/M$, $|C_j : C_{j-1}| = 2$, for $j = 1, \ldots, i - 1$, and $|C_0 : C| = 2$; also, $C$ and $C_0, \ldots, C_{i-1}$ are all normal in $C_{i-1}$, and these groups have common core $O(C)$. Now, $T \subseteq C$ and $U = C_S(w)$ is abelian of exponent 4, order $q^2$. So, $T \subseteq Z(S) = \Phi(U)$. Since $U$ is abelian and $\Omega_1(U) = \Phi(U)$, Walter's classification implies $U \leq \overline{C} \subseteq \overline{C}_i / O(C)$; hence $Z(S) \leq \overline{C}$. Now, $Z(S) \leq S \cap C_{i-1}$, and $C(S \cap C_{i-1}) = C_{i-1}$. Therefore, $Z(S) \leq \overline{C}_i$. This means $A/M$ is covered by subgroup of $N_{G}(Z(S))$, which is solvable of 2-length one. This proves the lemma.

Lemma 7. $C_E(v)$ is 2-generated.

Proof. Set $\Gamma = \Gamma_{C_{i-1}}$, where $C_i = C_{S_i}(v)$. For all $i$, $C_i$ contains a four-group disjoint from $\langle v \rangle$. So, $O(C_E(v)) \subseteq \Gamma$. But $X = C_E(v)/O(C_E(v))$ is 2-closed, and $O_2(X)$ contains a four-group. Hence, the Frattini argument implies that $C_E(v)$ is 2-generated.

Lemma 8. If $t$ is an involution in $C_E(v)$, then $O(C_E(t)) \cap C_E(v) \subseteq O(C_E(v))$.

Proof. Let $D = C_E(v)/O(C_E(v))$ and let $\bar{t} \in S_i$ be the image in $D$ of $t \in S_i$.
Suppose \( i < n \). Then \( O_2(D) \) is a Sylow 2-subgroup of \( D \) and any nonidentity element of odd order in \( D \) acts nontrivially on \( Z(O_2(D)) \), by Lemmas 2, 3, 6. Thus, \( Z(O_2(D)) \) normalizes no subgroup of odd order in \( D \). Since \( Z(O_2(D)) \subseteq C_{E} t = 1 \), \( O(C_{E} E) = 1 \), which implies the lemma.

Suppose \( i = n \). Then \( O_2(D) \) is abelian and every element of odd order in \( D \) acts fixed point freely on \( O_2(D) \). So, the centralizer in \( D \) of any \( t \) is a 2-group. Again, the lemma holds.

**Lemma 9.** The proposition holds.

**Proof.** By construction, each \( O(E_i) = 1 \). We argue by downward induction on \( i \).

Let \( i = n \). Then \( E_n \) has abelian Sylow 2-subgroups and is a section of \( C_G(Z(S)) \) by the proof of Lemma 6 (take \( T = Z(S) \) in that notation). Hence, \( E_n \) is solvable by Lemma 3 and \( E_n = O_{2,2'}(E_n) \). So, for \( i = n \), the lemma holds.

Now, let \( F \in \mathcal{E}_i \), \( i < n \). For any \( \zeta \in Z(S_i) \), \( E = C_F(\zeta)/O(C_F(\zeta)) \) \( \zeta \in \mathcal{E}_{i+1} \) is solvable and 2-closed by induction. For an involution \( \nu \) outside a Sylow 2-center, \( C_F(\nu) \) is solvable of 2-length 1, by Lemma 6.

We wish to show balance holds in \( F \). By Lemma 8, it suffices to prove, for \( t \in S_i \), that the image of \( O(C_F(t)) \) in \( C = C_F(\zeta)/O(C_F(\zeta)) \) is 1. Now, \( E = C_F(\zeta)/O(C_F(\zeta)) \) \( \zeta \) is solvable and 2-closed. Imitating the argument that \( N_A(T) = C_A(T) \) in the proof of Lemma 6, we get that \( C \) is isomorphic to a subgroup \( C^* \) of odd index in \( N^* = (N_G(S) \cap C_G(T))/O(N_G(S))/O(N_G(S))T \), where \( T \subseteq Z(S) \). If \( S^* \) is the image of \( S \) in \( N^* \), we will have \( O(C_{C^*}(t)) = 1 \) for any involution \( t \in C^* \), provided we show \( C_{S^*}(t) \) normalizes no subgroup of odd order in \( C^* \).

If \( t \in Z(S^*) \), clearly \( S^* = C_{S^*}(t) \) normalizes no subgroup of odd order in \( N^* \). If \( t \) is an involution in \( S^* \) not in \( Z(S^*) \) with \( O(C_{N^*}(t)) \neq 1 \), then \( Z(S^*) \) normalizes, hence centralizes (as \( N^* \) is 2-closed), a nontrivial subgroup of odd order. Let \( x \in O(C_{N^*}(t)) \) and let \( y \in N_G(S) \cap C_G(T) \) represent \( x \), \( |y| \) odd. Lemma 2 implies that \( y \) acts nontrivially on \( Z(S) \) since \( y \) is nontrivial on \( S \) and fixes the coset of \( Z(S) \) in \( S \) corresponding to \( t \). But \( y \) centralizes \( T \), hence must act nontrivially on \( Z(S)/T \simeq Z(S^*) \), as \( |y| \) is odd. So, \( x \) acts nontrivially on \( Z(S^*) \), a contradiction. This gives \( O(C_{C^*}(t)) = 1 \) in all cases. Therefore, balance holds in \( F \).

Next, we show that \( C_F(t) \) is 2-generated for every involution \( t \) of \( F \). If \( t \) is not 2-central, this is proven in Lemma 7. Let \( t \) be 2-central. There is a four-group in \( C_F(t) \) disjoint from \( t \). So, \( O(C_F(t)) \subseteq \Gamma = \Gamma_{S_i} \). Consider \( C_F(t)/O(C_F(t)) \). Since this group is 2-closed, it is 2-generated because, for \( i < n - 1 \), a Sylow 2-center has rank at least 2, and for \( i = n - 1 \), a Sylow 2-subgroup is extra special of order \( 2^{n+1} \), hence contains a four-group. The Frattini argument now shows that \( C_F(t) \) is 2-generated.
Now, we show $F$ is connected. For $i < n - 1$, a Sylow 2-center is noncyclic, whence connectivity. For $i = n - 1$, the extra special Sylow 2-subgroup contains an elementary abelian normal subgroup of order $2^n > 2^3$. By the remark on p. 4 of [4], $F$ is connected in this case, too.

Since $O(F) = 1$, $m(F) \geq 3$, Theorem C implies $O(C_F(t)) = 1$ for every involution $t \in F$. Our previous arguments then imply that every involution of $F$ has 2-closed centralizer. By Suzuki’s classification, $S_i$ does not occur as a Sylow 2-subgroup of a simple group. So, $F$ is not simple. We want to show $F$ solvable.

If $O_2(F) \neq 1$, then $W = Z(S_i) \cap O_2(F) \neq 1$. Lemma 4 implies that $W$ is strongly closed in $O_2(F)$ with respect to $F$. Hence $W \lhd F$. Since $|F: C_F(W)|$ is odd, $F$ is solvable and 2-closed because $C_F(W)/\langle \zeta \rangle$, $\zeta \in W^C$, is contained in some $E \in S_{i+1}$ and $E$ is solvable and 2-closed by induction. If $O_2(F) = 1$, then Theorem 2 of [7] implies that $F$ has cyclic, quaternion, or semidihedral Sylow 2-subgroups, contradicting $m(F) \geq 3$. Thus, $F$ is solvable, and the proposition is proven.

Lemma 10. Let $z$ be an involution of $S$. Then $C_G(z) \subseteq N_G(S)$.

Proof. We know $C_G(z)$ is solvable of 2-length 1. Since $m(S) \geq 3$, Theorem C implies $O(C_G(z)) = 1$ as $O(G) = 1$. Thus $O_G(z)$ is 2-closed, i.e., $C_G(z) \subseteq N_G(S)$.

Lemma 11. Theorem 2 holds.

Proof. For each involution $z$ of $G$, $C_G(z)$ is 2-closed. Thus, Suzuki’s classification implies Theorem 2.

REFERENCES