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INFINITE MATROIDS(1)

BY

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ABSTRACT. *Matroids* axiomatize the related notions of dimension and independence. We prove that if S is a set with k matroid structures, then S is the union of k subsets, the ith of which is independent in the ith matroid structure, iff for every (finite) subset A of S, $|A|$ is not larger than the sum of the dimensions of A in the k matroids.

A matroid is *representable* if there is a dimension-preserving imbedding of it in a vector space. A matroid is constructed which is not the union of finitely many representable matroids. It is shown that a matroid is representable iff every finite subset of it is, and that if a matroid is representable over fields of characteristic p for infinitely many primes p, then it is representable over a field of characteristic 0. Similar results for other kinds of representation are obtained.

1. **Introduction.** We sometimes use x for $\{x\}$ if the meaning is clear. We use $|S|$ for the cardinality of the set S.

Definition. A *matroid* is a set S together with a distinguished collection of subsets of S, called *independent* subsets of S, such that

I_1: There is an independent subset of S.

I_2: A subset A of S is independent iff every finite subset of A is independent.

I_3: If A and E are independent subsets of S with $|A| < |E|$, then there is an x in $E - A$ such that $A \cup x$ is independent.

Note that in the presence of I_2, I_1 can be replaced by

I'_1: ϕ is independent.

Note also that for finite matroids, I_2 simply says

I'_2: Every subset of an independent set is independent.

The collection of independent sets of a matroid is called a *matroid structure* on the set S. S may have several matroid structures on it, but when we are considering only one of them we will call S the matroid. We use *dependent* to mean not independent.

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The *dimension* or *rank* $d(A)$ of a subset A of S is

$$d(A) = \max\{|E|;\ E \subset A,\ E \text{ independent}\}.$$

A subset A of S is *finite dimensional* if $d(A)$ is finite. A *basis* for S is a maximal independent subset, and a *circuit* is a minimal dependent subset.

Matroids were introduced by H. Whitney [18] in an attempt to axiomatize the related notions of dimension and independence for vector spaces. R. Rado [11] has studied dimension in infinite matroids. He begins with an infinite set satisfying Whitney's axioms for a finite matroid, defined independence for infinite subsets via I_2, and then defines dimension as we define it. Then he proves I_3 for infinite sets, and the existence and equicardinality of bases.

There are many well-known examples of matroids: modules over an integral domain (dependence means linear dependence), field extensions (independence means algebraic independence), purely inseparable field extensions of exponent 1 (see [10] and [6]), the set of edges of a graph (a set is dependent if it contains the edge set of a 1-cycle), chain groups (see [14], [15], and [16]), and projective spaces (see [9]).

If T is a subset of a matroid S, then T inherits a matroid structure from S in a natural way ($A \subset T$ is independent iff A is independent as a subset of S). We call T a *submatroid* of S.

The direct sum of disjoint matroids S_α $(\alpha \in J)$ is $\bigoplus_\alpha S_\alpha = \bigcup_\alpha S_\alpha$, where $A \subset \bigoplus_\alpha S_\alpha$ is independent iff, for every α, $A \cap S_\alpha$ is independent in S_α.

Some authors call our direct sum the "union" of matroids, but we say S is the union of A and B when we mean simply that $S = A \cup B$ and A, B are submatroids of S.

2. **A combinatorial theorem.** Suppose we have a set with several matroid structures on it. The following theorem characterizes those subsets which are the union of sets, each independent in one of the matroid structures. In the special case where the matroid structures are identical, it characterizes the subsets which are the union of a certain number of independent sets.

Theorem 1. *Let k be a positive integer and S be a set with k matroid structures on it. Let d_i $(i = 1, \cdots, k)$ be the respective dimension functions. Then the following are equivalent:*

(a) $S = \bigcup_1^k S_i$, *where S_i is independent in the ith matroid structure.*

(b) *For every subset A of S,*

$$|A| \leq \sum_1^k d_i(A) \tag{1}$$

(c) (1) *holds for every finite dimensional subset* A *of* S.

(d) (1) *holds for every finite subset* A *of* S.

(*Note. In* (c), *a subset* A *of* S *is finite dimensional if* $d_i(A)$ *is finite for every* i.)

This theorem was proved for submatroids of vector spaces by Rado [13], and for finite matroids by Edmonds and Fulkerson [2].

Proof. It is trivial that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). By [2], (d) $\Rightarrow$ (a) for finite matroids. It suffices to prove (d) $\Rightarrow$ (a) for infinite matroids. Let S be an infinite matroid and let $\mathfrak{D}$ be the diagram of the statement that S is a set with the k given matroid structures and that part (a) is true for S. Any finitely axiomatized part of $\mathfrak{D}$ involves only a finite number of elements of S, so it is true by (d) $\Rightarrow$ (a) for finite matroids, and hence has a model. By the Compactness Theorem, $\mathfrak{D}$ has a model and so part (a) is true for S.

In [17], we use Rado's lemma [11] to prove the infinite case of (d) $\Rightarrow$ (a), just as Rado did in [13] for infinite vector spaces.

Corollary 1. *Let* V *be a left vector space over a skew-field* F. *Let* $F_1, \cdots, F_k$ *be sub-skew-fields of* F. *Then* V *is a left vector space over each* F_i. *Let* d_i *be the associated dimension function. Let* S *be a subset of* V. *Then the following are equivalent*:

(a) $S = \bigcup_1^k S_i$, *where* S_i *is linearly independent over* F_i.

(b) (1) *holds for every subset of* S.

(c) (1) *holds for every finite subset of* S.

Corollary 2. *Let* $K_1, \cdots, K_k$ *be subfields of a field* L *and* S *be a subset of* L. *Then the following are equivalent*:

(a) $S = \bigcup_1^k S_i$, *where* S_i *is algebraically independent over* K_i.

(b) *For every subset* A *of* S,

$$|A| \leq \sum_1^k \text{tr.d.}\ (K_i(A)/K_i). \tag{2}$$

(c) (2) *holds for every finite subset* A *of* S.

The case of identical matroid structure in Theorem 1 extends to infinite cardinals as follows:

Theorem 2. *Let* ω *be a cardinal number and* S *be a matroid. Consider the following two statements*:

(a) $S = \bigcup_{\alpha<\omega} S_\alpha$, *where each* S_α *is independent.*

(b) *For all* $A \subset S$, $|A| \leq \omega \cdot d(A)$.

Then (a) *always implies* (b). *If there are no cardinal numbers* γ *such that* $\aleph_0 \leq \omega < \gamma < |S|$, *then* (b) *implies* (a).

Proof. (a) $\Rightarrow$ (b). Let A be a subset of S. Then

$$|A| \leq \sum_{\alpha<\omega} |S_\alpha \cap A| = \sum_{\alpha<\omega} d(S_\alpha \cap A) \leq \sum_{\alpha<\omega} d(A) = \omega \cdot d(A).$$

(b) $\Rightarrow$ (a). If $\aleph_0 > \omega$, apply Theorem 1. If $\omega \geq |S|$, use singleton S_α. Thus we can assume that $\aleph_0 \leq \omega < |S|$ and that there are no cardinals between ω and $\sigma = |S|$.

By elementary properties of cardinal numbers, (b) implies that if $A \subset S$ and $|A| > \omega$, then $d(A) = |A|$. Therefore $d(S) = \sigma$. Let $B = \{\xi_\alpha | \alpha < \sigma\}$ be a basis for S. Thus (see [17, Theorem 2.5]) for every $x \in S$ there is a unique smallest $A_x \subset B$ such that x depends on(2) A_x. Also A_x is finite and if x depends on $A \subset B$, then $A_x \subset A$. For $x \in S$ define $a(x)$ = the largest $a < \sigma$ such that $\xi_a \in A_x$. For $a < \sigma$, define $R_a = \{x \in S;\ a(x) = a\}$ and $B_a = \{\xi_\gamma\ ;\ \gamma \leq a\}$. Therefore $|B_a| < \sigma$, so by our hypothesis, $|B_a| \leq \omega$. If $|R_a| > \omega$, then $d(R_a) \leq d(B_a) = |B_a| \leq \omega < |R_a|$, a contradiction. Therefore $|R_a| \leq \omega$, so we can write $R_a = \{x_{a,b};\ b < \omega\}$. For $b < \omega$, let $S_b = \{x_{a,b};\ a < \sigma\}$. Note that for $x \neq y \in S_b$ we have $a(x) \neq a(y)$. Now $S = \bigcup_{a<\sigma} R_a = \bigcup_{b<\omega} S_b$, so it suffices to show that each S_b is independent.

If S_b were dependent, then it would contain a circuit D. Let $x \in D$ be such that $a = a(x) > a(y)$ for all $y \in D$, $y \neq x$. D is a circuit, so x depends on $D - x$. Also, for all $y \in D - x$, y depends on A_y. Therefore x depends on $E = \bigcup_{y \in D-x} A_y \subset B$. Therefore $A_x \subset E$, so $\xi_a \in E$. Therefore $\xi_a \in A_y$ for some $y \neq x$. Therefore $a(y) \geq a$, a contradiction. Therefore S_b must be independent, and the proof is complete.

We obtain a result of Erdös and Kakutani [22] as a corollary to Theorem 2.

Corollary 1 (Erdös and Kakutani). *The Continuum Hypothesis implies that the set of nonzero real numbers is the countable union of Hamel bases.*

Proof. Let $\omega = \aleph_0$ and S be the nonzero reals with matroid structure as a subset of the vector space of the reals over the rationals. By the Continuum Hypothesis and Theorem 2 we have (b) implies (a). It is easy to prove (b), and the conclusion follows easily from (a).

Similarly one can prove

Corollary 2. *The Continuum Hypothesis implies that the set of transcendental real numbers is the countable union of transcendence bases for the reals over the rationals.*

3. **The representation problem.** Now we consider the problem of representing or imbedding one matroid in a second one, where the second matroid is of some

(2) We say x *depends on* A if either $x \in A$ or A has an independent subset E such that $E \cup x$ is dependent.

special form such as a vector space. We will define several kinds of representability according to the form of the second matroid.

From now on, we assume that our matroids have no singleton dependent sets, that is, no x such that $d(x) = 0$. Thus, the zero vector is excluded from a vector space matroid, a field extension matroid consists only of the transcendental elements, etc. This is really no restriction, since the excluded elements are of no interest in representing matroids.

Definition. A matroid S with dimension function d_S is *representable* in a matroid T with dimension function d_T if there is a function $f: S \to T$ (called a representation of S in T) such that $d_S(A) = d_T(f(A))$ for every subset A of S. If f is $1:1$, this is equivalent to saying that A is independent in S iff $f(A)$ is independent in T. If f is $1:1$ and onto, we say that S and T are *isomorphic* matroids.

The composition of representations is a representation. From the finite character of independence we see that in order to verify that a map f is a representation, it suffices to check that f preserves dimension for finite subsets A of S. Isomorphic matroids are representable in the same matroids.

Definition. Let S be a matroid. S is *LVS representable* in a left vector space V over a skew-field F if V has the usual left vector space matroid structure, and S is representable in V.

In a similar manner we define VS, TR, IE, DM, and G representability to mean representability in a vector space, a transcendental extension of a field, a purely inseparable field extension of exponent 1, a module over an integral domain, and the edge set of a graph, respectively, where these objects have the matroid structures mentioned earlier.

For field characteristics p, we say that S is *LVS-p representable* (*DM-p representable*) if S is LVS representable (DM representable) over some skew-field (integral domain) of characteristic p. We define VS-p, TR-p, and IE-p representability similarly.

The significance of the characteristic will become clear in §5 when we prove results on characteristic changing. One could also define chain group representation, but it is easy to see [17] that this is the same as DM representation via a $1:1$ function.

W. T. Tutte [16] shows that every finite graph matroid is VS representable, but that some (finite) vector spaces are not G representable. It follows (from Theorem 5(a)) that G representability implies VS representability but not conversely.

It is easy to show that VS and DM representabilities are equivalent:

Proposition. *A module M over an integral domain R is VS representable*

over the quotient field F of R. Hence DM and VS representabilities are equivalent.

Proof. $M \otimes_R F$ is an F module, i.e., a vector space over F, and $x \mapsto x \otimes 1$ is a representation of M in this vector space.

Theorem 3. *A vector space V over a field F is TR representable over F. If the characteristic of F is nonzero then V is IE representable over F. Thus, if a matroid is VS-p representable, then it is TR-p representable, and if $p \neq 0$, then it is IE-p representable, too. (See also Corollary 3 of Theorem 5.)*

Proof. Let B be a vector space basis for V over F. Let $X = \{x_b;\ b \in B\}$ be algebraically independent transcendentals over F. Let $E = F(X)$. Define $f: B \to E$ by $f(b) = x_b$. Extend f linearly to $f: V \to E$. Let $V' = f(V)$. Then V' is isomorphic to V and, since f is $1:1$, it suffices to show that $A \subset V'$ is linearly independent over F iff A is algebraically independent over F. Note that since X is both a vector space basis for V' and a transcendence basis for E, any automorphism of V' extends uniquely to an automorphism of E. If A is linearly dependent, then clearly it is algebraically dependent. If A is linearly independent, extend A to a vector space basis X' for V'. Now $|X| = |X'|$, so there is a $1:1$ correspondence $h: X \to X'$. h extends to an automorphism of V', and hence to one of E. $X = h^{-1}(X')$ is algebraically independent over F so X' must be, too. Therefore A is algebraically independent.

Now let char $F = p \neq 0$ and let B and X be as above. Let $Y = \{x^{1/p};\ x \in X\}$, $L = F(Y)$, and $K = F(X)$. Then L is a purely inseparable extension of exponent 1 of K, and Y is a p-basis for L over K. Define $f: B \to Y$ by $f(b) = x_b^{1/p}$, and extend f linearly to V. Let $V' = f(V)$. The rest of the proof is just like the TR case.

Thus we have the following implications for representability:

$$\begin{tikzcd} G \arrow[r] & DM \arrow[r] & VS \arrow[r] \arrow[dr] \arrow[dr, shift left] \arrow[dr, shift right] & TR \\ & & & IE \\ & & & LVS \end{tikzcd}$$

A. W. Ingleton [5] gives an example of an LVS representable matroid which is not VS representable. We will show in §5 that IE representability implies VS representability, and that TR-0 representability implies VS-0 representability. These results require Theorem 5.

We mention here two easy results proved in [17]. If a matroid S is VS representable in some vector space over the field F, then S is VS representable in every vector space of dimension at least $d(S)$ over F. If S is a matroid of dimension not greater than 2 and $|S| \leq |F| + 1$, then S is VS representable in F^2.

4. Some pathological matroids. T. Lazarson [7] gives the following example of a matroid which is not LVS representable:

Theorem A (Lazarson). *Let p be prime, r a positive integer, and $n = pr + 1$. Let $e_1, \cdots, e_n$ be linearly independent vectors in a left vector space over a skew-field of characteristic p. Let $S = \{e_1, \cdots, e_n, u - e_1, \cdots, u - e_n, u\}$, where $u = e_1 + \cdots + e_n$. Suppose the left vector space matroid S is LVS representable over a skew-field D. Then* char $D \mid pr$. *In particular, if $r = 1$, then* char $D = p$.

Corollary B (Lazarson). *There exists a finite matroid which is not LVS representable (and therefore also not VS representable).*

We give the proof because it is short and we use the same idea in the proof of Theorem 4.

Proof. Let S and T be the matroids we get when we use Theorem A for $n = 3$ $(p = 2)$, and $n = 4$ $(p = 3)$. Any representation of the direct sum $S \oplus T$ gives rise to a representation of S and one of T. Therefore $S \oplus T$ is representable only over skew-fields whose characteristic is both 2 and 3, and there are none. Therefore $S \oplus T$ is not LVS representable.

We have seen that a matroid need not be VS representable. One might ask whether it is always possible to partition a matroid S into the union of two VS representable submatroids. That is, can we find submatroids T and U which are VS representable, and such that $S = T \cup U$? (Note that we do not require $S = T \oplus U$.) If partitioning into two matroids is not enough, then what about some other finite number? Is every matroid the union of finitely many VS representable matroids? We shall show that these questions have negative answers in general.

We need a result from a recent paper of Graham, Leeb and Rothschild [3]. Theorem C below, the finite vector space analog of Ramsey's theorem, was conjectured by Gian-Carlo Rota and appears as Corollary 2 of [3].

Theorem C (Graham, Leeb and Rothschild). *There is a number $N = N(k, r, l_1, \cdots, l_r; q)$ depending only on the integers $k, r, l_1, \cdots, l_r, q$ such that, for any $m \geq N$ and any partition of the set of k-dimensional subspaces of the vector space* $\mathrm{GF}(q)^m$ *into r classes, there is an i, $1 \leq i \leq r$, and an l_i-dimensional subspace all of whose k-dimensional subspaces lie in one class.*

We will use the following lemma to prove Theorem 4.

Lemma. *Given an integer r and a prime p, there exists an integer $N(r, p)$ such that if $n \geq N(r, p)$ and the vector space matroid* $\mathrm{GF}(p)^n - (0)$ *is partitioned into r classes $S_1, \cdots, S_r$, then there is an i such that S_i is LVS representable only over skew-fields of characteristic p.*

Proof. We will apply Theorem C with $k = 1$, $q = p$ and $l_1 = \cdots = l_r = p + 1$. Let $N(r, p) = N(1, r, p + 1, \cdots, p + 1; p)$. Let the ith class of 1-dimensional subspaces include all those subspaces which intersect S_i. By Theorem C there is an i and a $(p + 1)$-dimensional subspace V' of $\mathrm{GF}(p)^n$ all of whose 1-dimensional subspaces intersect S_i. Let S be the matroid of Theorem A with $r = 1$. Then V' contains a submatroid isomorphic to S, and therefore so does S_i. But S is LVS representable only over skew-fields of characteristic p, so S_i has the same property. Of course S_i is VS representable over $\mathrm{GF}(p)$.

Theorem 4. *For every positive integer r there exists a finite matroid M_r which is not the union of r LVS representable matroids.*

Proof. Let P be a set of $r + 1$ distinct rational primes (e.g., the first $r + 1$ primes). Let $M = \bigoplus_{p \in P}(\mathrm{GF}(p)^{N(r,p)} - (0))$, where the direct sum is the matroid direct sum of §1, and $N(r, p)$ is as in the Lemma. Suppose $M = \bigcup_{i=1}^{r} S_i$ is a partition of M. We find i such that S_i is not LVS representable. The partition of M induces a partition of $\mathrm{GF}(p)^{N(r,p)} - (0) = \bigcup_{i=1}^{r} S_i^p$ for each $p \in P$. Also $S_i = \bigoplus_{p \in P} S_i^p$.

By the Lemma, for each $p \in P$ there exists an i_p such that $S_{i_p}^p$ is LVS representable only over skew-fields of characteristic p. But $1 \leq i_p \leq r$ and $|P| = r + 1$, so there exists $p \neq q \in P$ such that $i_p = i_q = i$, say. Thus S_i contains submatroids S_i^p and S_i^q which are LVS representable only over skew-fields of characteristic p and characteristic q, respectively. Therefore S_i is LVS representable only over skew-fields of characteristic both p and q (where $p \neq q$). But, no such skew-fields exist.

Corollary. *There exists a countable matroid which is not the union of finitely many LVS representable matroids.*

Proof. Using just the theorem above, we see that $\bigoplus_{r=1}^{\infty} M_r$ works. But if we examine the proof, we see that the smaller matroid

$$\bigoplus_{p \text{ prime}} (\mathrm{GF}(p)^{N(p,p)} - (0))$$

does the job just as well.

Of course Theorem 4 and its corollary are true with LVS replaced by VS. Thus by the equivalence of VS, DM, and IE representabilities, these results remain true with LVS replaced by DM or IE. Also, since TR-0 representability implies VS-0 representability, the matroid S of Theorem A is not TR-0 representable.

5. **Application of logic to the representation problem.** For certain kinds of representability, the statement that a matroid is representable is equivalent to

the statement that a certain theory has a model. It is this fact which links Model Theory to the representation problem. We will use it to translate the Compactness Theorem into theorems about representability. Let S^* denote the set of all finite subsets of the set S.

Theorem 5. *Let S be a matroid and C be a nonempty set of field characteristics which either is finite or contains* 0 *(or both). Then*

(a) *S is VS representable iff every $T \in S^*$ is VS representable*;

(b) *S is VS-k representable for some $k \in C$ iff every $T \in S^*$ is VS-k representable for some $k \in C$*;

(c) *if S is VS-p representable for arbitrarily large primes p, then S is VS-0 representable*;

(d) *statements* (a)–(c) *remain true if we substitute one of DM, LVS, for VS throughout*;

(e) *statement* (b) *remains true if we replace VS by IE—of course, in this case, $0 \notin C$, and therefore C is finite*;

(f) *S is G representable iff every $T \in S^*$ is G representable.*

P. Vámos [24] has proved parts (a) and (c) by algebraic methods. Part (f) appears as Theorem 9 in Piff [23]. Piff's Theorem 5 (If S is a matroid and F is a finite field, then S is VS representable over F iff every $T \in S^*$ is VS representable over F.) is also easily proved using Model Theory.

Proof. To prove part (a), diagram the statement that S is a matroid imbedded in a vector space and apply the Compactness Theorem. Parts (b) and (c) are proved by adding appropriate statements about characteristic to the diagram of part (a). Parts (d)–(f) are proved similarly. We need only check that the n-place predicate "$\{x_1, \cdots, x_n\}$ is a dependent set" is in the language. For p-dependence this follows from the fact that any polynomial is equal to one with degree less than p in each variable. For the case of graph matroids, just note that "$x_1, \cdots, x_n$ is a 1-cycle" is in the language.

For more details of the proof, see [17]. The theorem can also be proved using ultraproducts. Here we quote two theorems of R. Rado [12] which we will use in deriving corollaries to Theorem 5.

Theorem D (Rado). *Let S be a finite matroid and K be a field. Then S is VS representable over K iff S is VS representable over a simple algebraic extenison of the prime field of K.*

Theorem E (Rado). *Suppose the finite matroid S is VS-0 representable. Then there is a positive number c and a positive integer s such that, for every prime $p > c$, S is VS representable over* $\mathrm{GF}(p^s)$. *Also, for infinitely many primes p, S is VS representable over* $\mathrm{GF}(p)$.

Corollary 1. *Let S be a finite matroid. Then the following are equivalent:*

(a) *S is VS-0 representable;*

(b) *S is VS representable over a simple algebraic extension of* $\mathbf{Q}$;

(c) *there is a positive number c and a positive integer s such that for all primes $p > c$, S is VS representable over* $\mathrm{GF}(p^s)$;

(d) *there is a positive number c such that S is VS-p representable for every prime $p > c$;*

(e) *S is VS-p representable for infinitely many primes p.*

Proof. By Theorem D, (a) $\Leftrightarrow$ (b). By Theorem E, (a) $\Rightarrow$ (c). Clearly (c) $\Rightarrow$ (d) $\Rightarrow$ (e). Finally, (e) $\Rightarrow$ (a) by Theorem 5 (c).

Corollary 2. *Let S be a matroid. Then S is VS representable iff every $T \in S^*$ is VS representable over some finite field (which may depend on T).*

Proof. Suppose S is VS-0 representable. Then every $T \in S^*$ is VS-0 representable. By Corollary 1, T is representable over some $\mathrm{GF}(p^s)$. If S is VS-p representable ($p \neq 0$), then every $T \in S^*$ is, too. By Theorem D, T is representable over a finite field. The converse is immediate from Theorem 5 (a).

The next corollary follows from Theorem 3 and Corollary 1 (d).

Corollary 3. *If a finite matroid S is VS-0 representable, then there is a positive number c such that S is IE-p and TR-p representable for all $p > c$.*

Logic is a powerful tool for studying matroid representation. We use a theorem of J. Ax to give a quick proof of another result of Vámos. For a finite matroid S, let $T(S, q)$ be the statement that S is *not* VS representable over $\mathrm{GF}(q)$. By Corollary 2, S is VS representable iff the statement

$$(*) \qquad (\forall q) T(S, q)$$

is false. Ax [20] has proved that the theory of statements true in all finite fields is decidable. Hence there is an algorithm for deciding the truth of $(*)$. Thus we obtain Vámos' theorem [24] that there is an algorithm for deciding whether or not a finite matroid S is VS representable.

We now prove the two results mentioned in §§3 and 4. The facts about derivations which are used in their proofs are in Chapter 2 of Zariski and Samuel [19].

Theorem 6. *An extension L of a field K of characteristic* 0 *(as a matroid with algebraic independence as the independence function) is VS representable over a field of characteristic* 0. *If* tr.d. $(L/K) < \infty$, *then L is VS representable over L.*

Proof. By Theorem 5 (a), it suffices to prove just the second statement. Let $\{x_1, \cdots, x_n\}$ be a transcendence basis for L over K. There are derivations D_i $(1 \leq i \leq n)$ of $K(x_1, \cdots, x_n)$ such that $D_i x_j = \delta_{ij}$ (Kronecker delta). These derivations extend uniquely to L and form a basis for the vector space $\mathfrak{D}_{L/K}$ of K-derivations of L. Let V be the dual space of $\mathfrak{D}_{L/K}$ and let $dx_1, \cdots, dx_n$ be a dual basis for $D_1, \cdots, D_n$. For $y \in L$, let

$$dy = \sum_{i=1}^{n} D_i y \, dx_i \in V.$$

We will show that $y \mapsto dy$ is a representation of L into V.

Suppose $dy = dz$. Then for all i, $D_i(y - z) = 0$, so $y - z \in K$, and hence y and z are algebraically dependent. In this situation it suffices to show that d preserves independent and dependent sets.

If $A \subset L$ is algebraically dependent, then A contains a circuit $\{y_1, \cdots, y_m\}$. Let $P \in K[Y_1, \cdots, Y_m]$ be of least degree in Y_1 such that $P(y_1, \cdots, y_m) = 0$. Let $P_i = \partial P / \partial Y_i$. Thus $P_1(y_1, \cdots, y_m) \neq 0$. Applying D_i to $P(y_1, \cdots, y_m)$ we get

$$0 = D_i P(y_1, \cdots, y_m) = \sum_{=1}^{m} P_j(y_1, \cdots, y_m) D_i y_j,$$

so

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{m} P_j(y_1, \cdots, y_m) D_i y_j dx_i = \sum_{j=1}^{m} P_j(y_1, \cdots, y_m) dy_j,$$

and $\{dy_1, \cdots, dy_m\}$ is linearly dependent.

If $A \subset L$ is algebraically independent, extend A to a transcendence basis $\{y_1, \cdots, y_n\}$ for L over K. Then there are K-derivations D_i' $(1 \leq i \leq n)$ on L such that $D_i' y_j = \delta_{ij}$ and $\{D_1', \cdots, D_n'\}$ is a basis for $\mathfrak{D}_{L/K}$. It is easy to see that $dy_1, \cdots, dy_n$ is the dual basis of $D_1', \cdots, D_n'$, so that dA is linearly independent.

Corollary 1. *The statement*

(f) *S is TR-0 representable*

is equivalent to statements (a)–(e) *of Corollary* 1 *of Theorem* 5.

Corollary 2. *VS-0, DM-0, and TR-0 representabilities are equivalent.*

Corollary 3. *There is a function from* $\mathbf{R}$ *into a vector space which is one-to-one on the set of transcendental reals and which has the property that a set of real numbers is algebraically independent over* $\mathbf{Q}$ *iff its image is linearly independent.*

Theorem 7. *A purely inseparable algebraic extension L of exponent* 1 *of*

a field K of characteristic $p \neq 0$ (as a matroid with independence meaning p-independence) is VS representable over some field of characteristic p. If $[L:K] < \infty$, then L is VS representable over L.

Proof. The proof is just like that of Theorem 6 and so is omitted.

Corollary. *For every prime p, VS-p, DM-p, and IE-p representabilities are equivalent.*

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