ON THE MULTIPLICATIVE COMPLETION OF CERTAIN BASIC SEQUENCES IN $L^p$, $1 < p < \infty$(1)

BY

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ABSTRACT. Boas and Pollard proved that given any basis $\{f_n\}_{n=1}^{\infty}$ for $L^2(E)$ one can delete the first $k$ basis elements and then find a bounded measurable function $M$ such that $\{Mf_n\}_{n=k+1}^{\infty}$ is total in $L^2(E)$, that is, the closure of the linear span of the set $\{Mf_n : n \geq k + 1\}$ is $L^2(E)$. We improve this result by weakening the hypothesis to accept bases of $L^p(E)$, $1 < p < \infty$, and strengthening the conclusion to read serially total, that is, given any $f \in L^2(E)$ one can find a sequence of reals $\{a_n\}_{n=k+1}^{\infty}$ such that $\sum_{n=k+1}^{\infty} a_n Mf_n$ converges to $f$ in the norm. We also show that certain infinite deletions are possible.

1. In this paper we strengthen and generalize the following result by Boas and Pollard [1].

Theorem 1.1. If $\{f_n\}_{n=1}^{\infty}$ is an orthonormal set which is not complete, but can be completed by the addition of a finite number of functions to the set, then there is a bounded measurable function $M$ such that $\{Mf_n\}_{n=1}^{\infty}$ is complete.

The strengthened theorem can be viewed as a first step toward changing totality into serial totality in [2].

Theorem 1.2. Let $\{\phi_n\}_{n=1}^{\infty}$ be a system of functions defined on the measurable set $E \subset [0, 1]$, $|E| > 0$, and forming a normal basis for $L^p(E)$, $1 < p < \infty$. Then, for any integer $N_0$ there exists a measurable function $M$, $0 \leq M(x) \leq 1$, such that for any given function $f$ in $L^p(E)$, there is a series

$$
\sum_{k=N_0}^{\infty} \alpha_k (M\phi_k), \quad \alpha_k \text{ a real number},
$$

with the properties

(a) the series (1.1) converges in $L^p$ to $f$;
(b) $\alpha_k \to 0$ as $k \to \infty$.

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2. The main tool in the proof of Theorem 1.2 is the following

Lemma 2.1. Let \( \Phi = \{ \phi_n : n = 1, 2, \ldots \} \) be a normalized basis for \( L^p(E) \), \( 1 < p < \infty \), \( f \) a measurable function finite almost everywhere on \( E \), and

\[
M(x) = \sum_{i=1}^{j} c_i x_i \quad \text{with} \quad E = \bigcup_{i=1}^{j} E_i \quad \text{and} \quad 0 < c_i < 1.
\]

Then given \( \epsilon > 0 \) and a positive integer \( n \), there exists a measurable set \( e_0 \) and a \( \Phi \)-polynomial \( P = \sum_{k=n}^{m} b_k \phi_k \) such that

1. \( e_0 \subset E \) and \( |e_0| < \epsilon \), where \( |e_0| \) is the Lebesgue measure of \( e_0 \);
2. \( |b_k| < \epsilon \) for \( n \leq k \leq m \);
3. \( \|MP - f\|_{E \setminus e_0} = \|(MP - f)1_{E \setminus e_0}\| < \epsilon \);
4. \( \|M \sum_{k=n}^{m} b_k \phi_k\|_e \leq \epsilon + \|f\|_e \) for all \( n \leq s \leq m \), and every measurable subset \( e \) of \( E \setminus e_0 \).

Proof. The lemma follows immediately from Lemma 3 of Talalyan [3].

3. Proof of Theorem 1.2. The required function will be a certain infinite product. The individual factors of this product are inductively determined. Let \( \epsilon_n = 2^{-n-2} \) and, for each \( n \), choose a positive \( \delta_n < \epsilon_n \) so that

\[
\|\phi_i\|_G < \epsilon_n/3, \quad i = 1, 2, \ldots, n, \quad \text{whenever} \quad |G| < \delta_n.
\]

Applying Lemma 2.1 with \( M(x) = 1 \) to \( \phi_1 \), we choose a set \( D_1 \) whose complement \( E_1 \) has measure less than \( \delta_1 \), and a \( \Phi \)-polynomial

\[
P_{11} = \sum_{j=\nu(1,0)+1}^{\nu(1,1)} b_j \phi_j, \quad \text{where} \quad \nu(1,0) = N_0,
\]

satisfying the following conditions:

1. \( |b_j| < \epsilon_1 \), if \( \nu(1,0) < j \leq \nu(1,1) \);
2. \( \|\phi_1 - P_{11}\|_{D_1} < \epsilon_1/3 \);
3. \( \sup_{\nu(1,0) < s \leq \nu(1,1)} \left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|_e < \frac{\epsilon_1}{2} + \|\phi_1\|_e \),

for all measurable subsets \( e \) of \( D_1 \).

Define

\[
M_1(x) = \begin{cases} 1, & \text{if} \ x \in D_1, \\ c_1, & \text{if} \ x \in E_1, \end{cases}
\]

where

\[
c_1 = \frac{\epsilon_1}{3} \left( 1 + \sup_{\nu(1,0) < s \leq \nu(1,1)} \left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|_e \right)^{-1}.
\]
With the help of $M_1$ we will be able to extend approximations analogous to (3.3) and (3.4) to the whole set and obtain estimates appropriate to the first step. In fact, it follows from (3.3), (3.1) and the definition of $M_1$ that
\[
\|\phi_1 - M_1 P_{11}\| \leq \|\phi_1 - P_{11}\|_{E_1} + \|\phi_1 - M_1 P_{11}\|_{E_1}
\]
(3.6)
\[
\leq \epsilon_1/3 + \|\phi_1\|_{E_1} + c_1 \|P_{11}\|_{E_1} \leq \epsilon_1/3 + \epsilon_1/3 + \epsilon_1/3 = \epsilon_1,
\]
and by virtue of (3.4) and the definition of $M_1$, we obtain for $\nu(1, 0) < s \leq \nu(1, 1)$
\[
\left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|_p = \left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|_{D_1} + \left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|_{E_1}
\]
(3.7)
\[
\leq \frac{\epsilon_1}{2} + \|\phi_1\|_p + \left(\frac{\epsilon_1}{3}\right) \leq 1 + \epsilon_1.
\]
Again, Lemma 2.1 allows us to choose, for $i = 1, 2$, sets $D_{2i}$ with respective complements $E_{2i}$ and $\Phi$-polynomials $P_{2i} = \sum_{j=\nu(2,i)}^{\nu(2,i+1)} b_j \phi_j$, with $\nu(1, 1) < \nu(2, 0) < \nu(2, 1) < \nu(2, 2)$, satisfying the following conditions:
\[
|E_2| < \delta_2/2, \quad i = 1, 2;
\]
(3.8)
\[
|b_j| < \epsilon_2, \quad j < \nu(2, 0) < j \leq \nu(2, 2);
\]
(3.9)
\[
\|P_{21} - P_{11}\|_{D_2} < \epsilon_2/3; \quad i = 1, 2;
\]
(3.10)
\[
\|\phi_{21} - P_{21}\|_{E_2} < \epsilon_2/3;
\]
(3.11)
\[
\|\phi_{22} - P_{22}\|_{D_2} < \epsilon_2/3;
\]
(3.12)
\[
\sup_{\nu(2,0) < s \leq \nu(2,1)} \left\| \sum_{j=\nu(2,0)+1}^{s} b_j \phi_j \right\|_p < \frac{\epsilon_2}{2} + \|\phi_{11} - P_{11}\|_{E_1} \leq 1 + \epsilon_1,
\]
for all measurable subsets $e$ of $D_{21}$;
\[
\sup_{\nu(2,1) < s \leq \nu(2,2)} \left\| \sum_{j=\nu(2,1)+1}^{s} b_j \phi_j \right\|_p < \frac{\epsilon_2}{2} + \|\phi_{22}\|_p < 1 + \epsilon_2/2
\]
(3.13)
for all measurable subsets $e$ of $D_{22}$.
\[
\sup_{\nu(2,1) < s \leq \nu(2,2)} \left\| \sum_{j=\nu(2,1)+1}^{s} b_j \phi_j \right\|_p < \frac{\epsilon_2}{2} + \|\phi_{22}\|_p < 1 + \epsilon_2/2
\]
(3.14)
Let $D_2 = D_{21} \cap D_{22}$ and $E_2 = E_{21} \cup E_{22}$.

Define
\[
M_2(x) = \begin{cases} 1, & \text{if } x \in D_2, \\ c_2, & \text{if } x \in E_2, \end{cases}
\]
(3.15)
where
\[
c_2 = \frac{\epsilon_2}{3} \left( 1 + \sum_{m=1}^{2} \sum_{k=1}^{m} \sup_{\nu(m,k) < s < \nu(m,k-1)} \left\| \sum_{j=\nu(m,k-1)+1}^{s} b_j \phi_j \right\|_p \right)^{-1}.
\]
With the help of $M_2$ we will be able to extend approximations analogous to (3.10) through (3.13) to the whole set and obtain the second-step estimates. In
fact, it follows from (3.10), (3.14), (3.8) and (3.1) that
\[
\|\phi_1 - M_1M_2(P_{11} + P_{21})\| \leq \|\phi_1 - M_1(P_{11} + P_{21})\|_{D_2} + \|\phi_1 - M_1M_2(P_{11} + P_{21})\|_{E_2}
\]
\[
\leq \epsilon_2/3 + \|\phi_1\|_{E_2} + c_2(\|P_{11}\| + \|P_{21}\|)
\]
\[
\leq \epsilon_2/3 + \epsilon_2/3 + \epsilon_2/3 = \epsilon_2,
\]
and by virtue of (3.12) and (3.15) we obtain for \(\nu(2, 0) < s \leq \nu(2, 1)\)
\[
\left\| M_1M_2 \sum_{j=\nu(2,0)+1}^{s} b_j \phi_j \right\|_{p} \leq M_1 \left\| \sum_{j=\nu(2,0)+1}^{s} b_j \phi_j \right\|_{D_2} + M_1M_2 \left\| \sum_{j=\nu(2,0)+1}^{s} b_j \phi_j \right\|_{E_2}
\]
\[
\leq \epsilon_2/2 + \|\phi_1 - M_1P_{11}\|^p + (\epsilon_2/3)^p
\]
\[
\leq \epsilon_2 + (\epsilon_1)^p \leq \epsilon_1 + \epsilon_2.
\]
Similarly, \(\|\phi_2 - M_1M_2P_{22}\| < \epsilon_2\), and for \(\nu(2, 1) < s \leq \nu(2, 2)\),
\[
\left\| M_1M_2 \sum_{j=\nu(2,1)+1}^{s} b_j \phi_j \right\| < 1 + \epsilon_2.
\]
Now we show that the estimates of the first step are weakened only slightly by the introduction of \(M_2\). As a matter of fact, (3.6) and (3.7) are changed only by the introduction of an additional \(\epsilon_2\).
\[
\|\phi_1 - M_1M_2P_{11}\| < \|\phi_1 - M_1P_{11}\|_{D_2} + \|\phi_1 - M_1M_2P_{11}\|_{E_2}
\]
\[
\leq \epsilon_1 + \|\phi_1\|_{E_2} + c_2\|P_{11}\|_{E_2} \leq \epsilon_1 + \epsilon_2/3 + \epsilon_2/3 < \epsilon_1 + \epsilon_2,
\]
and for \(\nu(1, 0) < s \leq \nu(1, 1)\),
\[
\left\| M_1M_2 \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\| = M_1 \left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|_{D_2} + M_1M_2 \left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|_{E_2}
\]
\[
\leq 1 + \epsilon_1 + (\epsilon_2/3)^p < 1 + \epsilon_2 + \epsilon_1.
\]
Suppose we have completed the first \(n\) steps; that is, for each pair \((m, k)\) with \(m = 1, 2, \ldots, n\) and \(k = 1, 2, \ldots, m\), we have
\[
(3.16) \quad \left\| \phi_k - \left( \prod_{i=1}^{n} M_i \right) \sum_{j=k}^{m} P_{jk} \right\| < \sum_{j=m}^{n} \epsilon_j;
\]
\[\sup_{(m, k - 1) \leq s \leq \nu(m, k)} \left\| \left( \prod_{i=1}^{n} M_i \right) \sum_{j=\nu(m, k-1)+1}^{s} b_j \phi_j \right\|_p \]

\[\leq \begin{cases} 
\sum_{j=m-1}^{n} \epsilon_j, & \text{if } m > k; \\
1 + \sum_{j=m}^{n} \epsilon_j, & \text{if } m = k;
\end{cases}\]

and

\[|b_j| < \epsilon_m, \quad \text{for all } \nu(m, 0) < j \leq \nu(m, m).\]

Now successively apply Lemma 2.1, with \(M = \prod_{i=1}^{n} M_i\), to the functions

\[\Psi_k = \phi_k - \left( \prod_{i=1}^{n} M_i \right) \left( \sum_{j=k}^{n} P_{jk} \right), \quad k = 1, 2, \ldots, n,\]

\[\Psi_{n+1} = \phi_{n+1}.
\]

By virtue of Lemma 2.1, we may choose, for \(k = 1, 2, \ldots, n + 1\), sets \(D_{n+1, k}\) with respective complements \(E_{n+1, k}\), and \(\Phi\)-polynomials

\[P_{n+1, k} = \sum_{j=\nu(n+1, k-1)+1}^{\nu(n+1, k)} b_j \phi_j,\]

where \(\nu(n, n) < \nu(n + 1, 0) < \nu(n + 1, 1) < \cdots < \nu(n + 1, n + 1)\), satisfying the following conditions:

\[|E_{n+1, k}| < \delta_{n+1}/(n + 1);\]

\[|b_j| < \epsilon_{n+1}, \quad \nu(n + 1, 0) < j \leq \nu(n + 1, n + 1);\]

\[\left\| \Psi_k - \left( \prod_{i=1}^{n} M_i \right) P_{n+1, k} \right\|_{D_{n+1, k}} \leq \frac{\epsilon_{n+1}}{3};\]

\[\sup_{\nu(n+1, k-1) \leq s \leq \nu(n+1, k)} \left\| \left( \prod_{i=1}^{n} M_i \right) \sum_{j=\nu(n+1, k-1)+1}^{s} b_j \phi_j \right\|_e \leq \frac{\epsilon_{n+1}}{2} + \left\| \Psi_k \right\|_e,
\]

for all measurable subsets \(e\) of \(D_{n+1, k}\).

(3.24) Let \(D_{n+1} = \bigcap_{k=1}^{n+1} D_{n+1, k}\) and \(E_{n+1} = \bigcup_{k=1}^{n+1} E_{n+1, k}\). Next, define

\[M_{n+1}(x) = \begin{cases} 
1, & \text{if } x \in D_{n+1}; \\
\epsilon_{n+1}, & \text{if } x \in E_{n+1};
\end{cases}\]

where
With the help of $M_{n+1}$ we will be able to extend approximations analogous to (3.22) and (3.23) to the whole set. In fact, it follows from (3.22), (3.19), (3.24), (3.1) and (3.25) that

$$
\left\| \phi_k - \left( \prod_{i=1}^{n+1} M_i \right) \sum_{j=k}^{n+1} P_{jk} \right\| \leq \left\| \Psi_k - \left( \prod_{i=1}^{n} M_i \right) P_{n+1,k} \right\| + \left\| \phi_k - \left( \prod_{i=1}^{n+1} M_i \right) \sum_{j=k}^{n+1} P_{jk} \right\|_{E_{n+1}} 
$$

$$
\leq \epsilon_{n+1} + \epsilon_{n+1} + \epsilon_{n+1} = \epsilon_{n+1}.
$$

Similarly,

$$
\sup_{\nu(n+1,k-1) < s \leq \nu(n+1,k)} \left\| \left( \prod_{i=1}^{n+1} M_i \right) \sum_{j=v(n+1,k-1)+1}^{s} b_j \phi_j \right\|^p \leq \epsilon_{n+1} + \frac{\epsilon_{n+1}}{3} \leq \epsilon_{n+1}.
$$

Now we show that the estimates of the first $n$ steps are weakened only slightly by the introduction of $M_{n+1}$. Indeed, (3.16) and (3.17) are changed by no more than an additional $\epsilon_{n+1}$. Let $m \leq n + 1$ and $k \leq m$; then by (3.20), (3.1), and (3.25)

$$
\left\| \phi_k - \left( \prod_{i=1}^{n+1} M_i \right) \sum_{j=k}^{m} P_{jk} \right\| \leq \left\| \phi_k - \left( \prod_{i=1}^{n} M_i \right) \sum_{j=k}^{m} P_{jk} \right\|_{D_{n+1}} + \left\| \phi_k \right\|_{\nu(n+1)} + \left\| \left( \prod_{i=1}^{n+1} M_i \right) \sum_{j=k}^{m} P_{jk} \right\|_{E_{n+1}} 
$$

$$
\leq \sum_{j=m}^{n} \epsilon_j + \frac{\epsilon_{n+1}}{3} + \frac{\epsilon_{n+1}}{3}.
$$

Similarly,
Completing certain sequences in $L^p$, $1 < p < \infty$

\[ \sup_{\nu(m, k-1) < s \leq \nu(m, k)} \left\| \left( \prod_{i=1}^{n+1} M_i \right) \sum_{j=\nu(m, k-1)+1}^{s} b_j \phi_j \right\|_p \]

\[ \leq \begin{cases} 
\sum_{j=m-1}^{n+1} \epsilon_j, & \text{if } m > k; \\
1 + \sum_{j=m}^{n+1} \epsilon_j, & \text{if } m = k.
\end{cases} \]

Therefore inequalities (3.16) and (3.17) hold for $n + 1$ and hence for all natural numbers.

We have constructed a sequence of measurable functions \( \{M_i\}_{i=1}^{\infty} \) with

\[ 0 < M_i(x) < 1 \quad \text{and} \quad \sum_{i=1}^{\infty} ||x: M_i(x) \neq 1|| < \sum_{i=1}^{\infty} ||E_i|| < \sum_{i=1}^{\infty} \epsilon_i = \sum_{i=1}^{\infty} 2^{-i-2} = \frac{1}{4}. \]

The sequence of partial products \( \prod_{i=1}^{n} M_i \) forms a nonincreasing sequence of positive functions. Hence

\[ M(x) = \lim_{n} \prod_{i=1}^{n} M_i(x) \]

exists, is measurable and satisfies $0 \leq M(x) \leq 1$.

Fix a pair \((m, k)\) with $m \geq k$. Since the norm is an absolutely continuous set function, we can find a $\delta > 0$ such that, whenever the measure of a set $G$ is less than $\delta$, we have

\[ \| \phi_k \|_G + \left\| M \sum_{j=k}^{m} P_{j,k} \right\|_G + \left( \sum_{\nu(m, k-1)+1}^{\nu(m, k)} \|Mb_j \phi_j \|_G \right)^p < 2^{-m-1}. \]

Now choose $n$ large enough so that

\[ B = \left\{ x: \prod_{i=n}^{\infty} M_i(x) \neq 1 \right\} \]

has measure less than $\delta$.

We obtain

\[ \left\| \phi_k - M \sum_{j=k}^{m} P_{j,k} \right\| \]

\[ \leq \left\| \phi_k - \left( \prod_{i=1}^{n} M_i \right) \sum_{j=k}^{m} P_{j,k} \right\|_E \setminus B + \left\| \phi_k - M \sum_{j=k}^{m} P_{j,k} \right\|_B \]

\[ \leq 2^{-m-1} + 2^{-m-1} = 2^{-m}. \]
Similarly,

\[
\begin{align*}
\sup_{\nu(m, k-1) < s \leq \nu(m, k)} \left\| \sum_{j=\nu(m, k-1)+1}^{s} b_j \phi_j \right\|_p = \begin{cases} 
2^{-m+2}, & \text{if } m > k; \\
2, & \text{if } m = k.
\end{cases}
\end{align*}
\]

Now we are ready to show that given any function \( f \) in \( L^p(E) \), we can find a series \( \sum a_j m\phi_j \) with \( \nu(m, 0) \leq j \leq \nu(m, m) \), \( m = 1, 2, \ldots \), which will converge to \( f \) in the norm.

In fact, if \( \sum_{k=1}^{\infty} a_k \phi_k \) is the Schauder basis expansion of \( f \), then \( \sum_{k=1}^{\infty} a_k m\phi_k \) converges to \( f \) in the norm.

Let \( \epsilon > 0 \) be given. Choose \( N_1 \) so that

\[
\left\| \sum_{k=1}^{n} a_k \phi_k - f \right\| < \frac{\epsilon}{3}, \quad \text{for all } n > N_1.
\]

Setting \( a = \sup_k |a_k| \), choose \( N_2 > N_1 \) so that, \( a \cdot n \cdot 2^{-n} < \epsilon/3 \), for all \( n > N_2 \).

By virtue of (3.28) we obtain

\[
\left\| \sum_{k=1}^{n} a_k \phi_k - f \right\| < \frac{\epsilon}{3}.
\]

Last, choose \( N_3 > N_2 \) so that

\[
\left\| \sum_{k=1}^{n} a_k \phi_k - f \right\| < \frac{\epsilon}{3}, \quad \text{whenever } n > N_3.
\]

By virtue of (3.30) and (3.31) we obtain

\[
\left\| \sum_{k=1}^{n} a_k \phi_k - f \right\| < \frac{2\epsilon}{3}, \quad \text{for all } n > N_3.
\]

Obviously,

\[
\left\| \sum_{k=1}^{n} a_k \phi_k - f \right\| < \frac{2\epsilon}{3}, \quad \text{for all } n > N_3.
\]

If we add in only part of the second sum, that is, \( \sum_{k=1}^{m} a_k m\phi_k \) with \( m < n + 1 \), then it is easy to see from (3.28) that the basis elements \( \phi_i, i = 1, 2, \ldots, m \), will be approximated better than before, by \( 2^{-n} \) instead of by \( 2^{-n} \). Hence via the calculations in (3.31), we find that

\[
\left\| \sum_{k=1}^{n} a_k \phi_k - f \right\| < \frac{2\epsilon}{3}.
\]

Lastly, if we add to the summations in (3.34) only part of the \( \Phi \)-polynomial

\[
a_{m+1} m\phi_{m+1}, \text{ let us say } \sum_{j=\nu(n+1, m)}^{\nu(n+1, m)+1} a_{m+1} b_j \phi_j,
\]

where \( \nu(n+1, m) < s < \nu(n+1, m+1) \), then (3.29) and (3.32) in addition to (3.34) give us
whenever $n > N_{1/2}$, $m < n + 1$, and $\nu(n + 1, m) < s < \nu(n + 1, m + 1)$. Thus, as a consequence of (3.33), (3.34) and (3.35), we obtain the desired series convergence. Furthermore, the coefficients of $M\phi_j$ go to zero, since the $a_n$ are bounded by $a$ and the $b_j$ go to zero.

The following remark is an immediate consequence of the above proof.

**Remark 3.1.** One can always delete certain infinite collections of the basis elements without affecting the conclusion of Theorem 1.2.

**Proposition 3.2.** The function $M$ defined in Theorem 1.2 is positive almost everywhere.

**Proof.** By virtue of (3.26), (3.27) and the definitions of $M_n$, $n = 1, 2, \ldots$, we have $M(x) = \Pi_{n=1}^\infty M_n(x)$, where

(a) $M_n(x)$ is measurable and $0 < M_n(x) \leq 1$;

(b) if $E_n = \{x: M_n(x) \neq 1\}$, then $\sum_{n=1}^\infty |E_n| < \infty$.

$M(x)$ is positive almost everywhere since the product defining $M(x)$ has infinitely many factors different from unity only on the set $\lim \sup E_n$. But (b) insures that $|\lim \sup E_n| = 0$.

**Definition 3.3.** $\{\phi_n\}_{n=1}^\infty$ is serially total in $L^p$ if and only if for any function $f \in L^p$ we can find a series $\sum_{k=1}^\infty a_k\phi_k$ which converges to $f$ in the norm.

**Theorem 3.4.** Let $\{\phi_n\}_{n=1}^\infty$ be a normalized basis for $L^p(E)$, $E \subset [0, 1]$, $1 < p < \infty$. Then, given any natural number $m$ and $\epsilon > 0$, there exists a set $G = G(m, \epsilon)$, $G$ contained in $E$ and satisfying $|G| > |E| - \epsilon$, such that $\{\phi_n\}_{n=m}^\infty$ is serially total in $L^p(G)$.

**Proof.** By virtue of Theorem 1.2 we can find a bounded measurable function $M$ such that $\{|M\phi_n\}_{n=m}^\infty$ is serially total in $L^p(E)$.

Choose $\alpha > 0$ so that $|[x: M(x) > \alpha]| > |E| - \epsilon$; denote this set by $G$. We assert that $\{|\phi_n\}_{n=m}^\infty$ is serially total in $L^p(G)$. To see this, take any $f$ in $L^p(G)$ and any positive $\delta$. If $\phi$ is the element of $L^p(E)$ that agrees with $f$ on $G$ and vanishes outside of $G$, then $M\phi \in L^p(E)$. Hence, by Theorem 1.2, there is a series $\sum_{k=m}^\infty a_k M\phi_k$, with $a_k \to 0$, with converges to $M\phi$ in the norm; that is,

$$\left| \sum_{k=m}^n a_k M\phi_k \right| < \alpha \delta,$$

for all $n > N(\alpha, \delta)$. 

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Now
\[ \alpha^p \left\| f - \sum_{k=m}^{n} a_k \phi_k \right\|_G^p = \int_G \alpha^p \left( f(x) - \sum_{k=m}^{n} a_k \phi_k(x) \right)^p \, dx \]
\[ \leq \int_G M^p(x) \left( f(x) - \sum_{k=m}^{n} a_k \phi_k(x) \right)^p \, dx \]
\[ \leq \int_E \left( M(x) \phi(x) - \sum_{k=m}^{n} a_k M(x) \phi_k(x) \right)^p \, dx \]
\[ = \left\| M\phi - \sum_{k=m}^{n} a_k M\phi_k \right\|_p^p < \alpha^p \delta^p. \]

Hence,
\[ \left\| f - \sum_{k=m}^{n} a_k \phi_k \right\|_G < \delta, \text{ for all } n > N(\alpha, \delta). \]

Since \( \delta \) may be taken to be arbitrarily small we have
\[ \{ \phi_k \}_{k=m}^{\infty} \text{ is serially total in } L^p(G). \]

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