TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 176, February 1973

PREHOMOGENEOUS VECTOR SPACES AND VARIETIES(1)

BY

FRANK J. SERVEDIO

ABSTRACT. An affine algebraic group G over an algebraically closed field k of characteristic 0 is said to act prehomogeneously on an affine variety W over k if G has a (unique) open orbit $o(G)$ in W. When W is the variety of points of a vector space V, $G \subseteq GL(V)$ and G acts prehomogeneously and irreducibly on V (We say an irreducibly prehomogeneous pair (G, V).), the following conditions are shown to be equivalent: 1. the existence of a nonconstant semi-invariant P in $k[V] \cong S(V^*)$, 2. (G', V) is not a prehomogeneous pair (G' is the commutator subgroup of G, a semisimple closed subgroup of G.), 3. if $X \in o(G)$, then $G_X^0 \subseteq G'$. (G_X^0 is the connected identity component of G_X, the stabilizer of X in G.) Further, if such a P exists, the criterion, due to Mikio Sato, "$o(G)$ is the principal open affine U_P if and only if G_X^0 is reductive" is stated.

Under the hypothesis G reductive, the condition "there exists a Borel subgroup $B \subseteq G$ acting prehomogeneously on W" is shown to be sufficient for $G \backslash W$, the set of G-orbits in the affine variety W to be finite.

These criteria are then applied to a class of irreducible prehomogeneous pairs (G, V) for which G' is simple and three further conjectures, one due to Mikio Sato, are stated.

Introduction. We shall investigate pairs (G, V) where V is a finite dimensional vector space over an algebraically closed field k of characteristic zero and where $G \subseteq GL(V)$ is a linear algebraic group with a dense orbit in V.

The principal results generalize the phenomena which can be witnessed in the following classical example. Let P be a nondegenerate quadratic form on V. Denote by $O(P)$ the corresponding orthogonal group and let $G = GO(P)$ be the group $(k^* \mathrm{Id}_V) \cdot O(P)$, the full group of similitudes for the form.

The orbits of G in V can easily be determined with the aid of Witt's Theorem. One finds that G has the following three orbits:

(1) $U_P = \{X \mid P(X) \neq 0\}$ = the set of nonisotropic vectors for P,

(2) $Z(P) - \{0\} = \{X \mid P(X) = 0, X \neq 0\}$, the set of nonzero isotropic vectors, and

(3) $\{0\}$.

Received by the editors September 13, 1971.

AMS (MOS) subject classifications (1970). Primary 20G05; Secondary 57E25.

Key words and phrases. Linear algebraic group, affine variety, Zariski open orbit, semi-invariant form, reductive linear algebraic group, Borel subgroup, simple algebraic group, irreducible representation.

(1) This work comprises part of the Ph.D. thesis at Columbia University, June 1970. The support of the National Science Foundation is gratefully acknowledged.

The properties of the example which are significant from our point of view are the following:

1. V is irreducible under $O(P)$ and $GO(P)$. $GO(P)$ has a Zariski open dense orbit, U_P, in V, whereas $O(P)$ does not have an open dense orbit in V.

2′. The open dense orbit of $GO(P)$, U_P, is an affine variety.

2″. If X belongs to the open dense orbit, then $GO(P)_X$ = (the full isotropy subgroup of X in $GO(P)$) = $\{g \in GO(P) \mid gX = X\}$ is the full group of isometries of $X\perp$ relative to P_X = the form P restricted to $X\perp$, which is nondegenerate. Hence $GO(P)_X$ is reductive.

3. V is a finite union of orbits under $GO(P)$.

It can be shown that these phenomena are characteristic of a larger class of vector spaces, groups and forms. Let (G, V) be a pair consisting of a finite dimensional vector space V over k, algebraically closed of zero characteristic, and of a connected algebraic group $G \subseteq GL(V)$. Call the pair *irreducible* if and only if G acts irreducibly on V; *prehomogeneous* if and only if G acts prehomogeneously on V (i.e. G has a unique Zariski open dense orbit $o(G)$ in V).

If (G, V) is irreducible then either G is isogenous to $k^* \times G'$ or $G = G'$ where G', the commutator subgroup of G, is a connected semisimple algebraic group acting irreducibly on V. Let (G, V) be irreducible and prehomogeneous.

1. There exists a nonconstant semi-invariant form P for G if and only if (G', V) is not a prehomogeneous pair.

2. The following conditions are equivalent.

2′. $o(G)$ is the affine variety $U_P = \{X \mid P(X) \neq 0\}$ where P is a semi-invariant in $k[V]$ for G.

2″. G_X, the isotropy subgroup of X in $o(G)$, is reductive.

3. If G has a Borel subgroup B which has an open dense orbit in V, or in $Z(P) = \{X \mid P(X) = 0\}$ in case one of the conditions in 2 holds, then G has only finitely many orbits in V.

The irreducible prehomogeneous pairs (G, V) above arise quite naturally, aside from their own intrinsic interest, in a theory of zeta functions expounded by by Professor Mikio Sato in a course of lectures given at Columbia University (1964–1965) and at Osaka University and at the Tokyo University of Education in 1963. The theorem realized in the statements labeled 2 above was communicated orally to me by Mikio Sato when I was a graduate student at the time. His references were Matsushima's article *Espaces homogènes de Stein des groupes de Lie complexes* in the Nagoya Math J. 16 (1960), pp. 205–218. We state 2 as Theorem (4.1) and use it in deriving applications of 1 which is Corollary (3.4) and of 3 which is Theorem (6.2). These applications, especially (6.5) 3, indicate that a

converse to (6.2) is impossible. A partial list of irreducible prehomogeneous pairs is found in §7. A good part of the list is due to Mikio Sato; much of it is "common knowledge". But indeed startling, for example, are Sato's results that the skew-symmetric tensor representations of order 3 of $GL(6)$, $GL(7)$ and $GL(8)$ give irreducible prehomogeneous pairs each with nondegenerate semi-invariant forms. As verification of prehomogeneity and of the existence of a form P, I have indicated points in the large orbits and their isotropy subgroups. Sato's proofs for these three examples are quite interesting, but very special, going much more deeply into properties particular to each of the pairs. The examples date back to at least 1965.

I have learned recently that Mikio Sato has published a lecture note in Japanese entitled *The theory of pre-homogeneous vector spaces* in the journal Sûgaku no Ayumi 15–1 (1970). There he has a list of examples which has a nonempty intersection with ours.

Some further work has been done on the classification of all irreducible prehomogeneous pairs, but the classification is far from complete.

For the work a threefold debt of gratitude is owing first to Professor Mikio Sato who introduced me to the subject, and second to Professor James Humphreys who has made many valuable suggestions and remarks, and third, but most, to Professor Hyman Bass for his careful scrutiny, lucid insights and many helpful suggestions.

0. **Preliminaries.** Throughout, *k denotes an algebraically closed field of characteristic zero*, k^* always denotes the multiplicative group of units in k.

(0.1) All varieties are algebraic varieties and these are understood to be defined over k. If W is an affine variety, we identify W with $W(k) = \mathrm{Hom}_{k\text{-alg}}(k[W], k)$ where $k[W]$ is the ring of regular functions on W. In general, if B is any k-algebra we write $W(B) = \mathrm{Hom}_{k\text{-alg}}(k[W], B)$ as the set of B valued points of the affine variety W. Thus if $f \in k[W]$ and $X \in W(B)$ we define $f(X) = X(f)$!

In this work we shall frequently make reference to standard results which may be found in Armand Borel, *Linear algebraic groups*, Benjamin, New York, 1969. MR 40 #4273. Such a reference will always be indicated by "Borel (.)".

G shall always denote a connected affine algebraic group defined over k. If H is an algebraic group defined over k, H^0 shall always denote the identity component of H, H' the connected identity component of the commutator subgroup of H. Hence by Borel (1.10), G is isomorphic to a closed subgroup of some $GL(V)$, the full automorphism group of a finite dimensional vector space V over k. As above $G = G(k)$ and $G(B) = \mathrm{Hom}_{k\text{-alg}}(k[G], B)$.

By an action of G on a variety W we mean a morphism

$$\alpha\colon G \times W \longrightarrow W, \qquad \alpha\colon (g, X) \mapsto gX = \alpha(g, X)$$

satisfying $eX = X$ and $g(hX) = (gh)X$ for e the identity element of G, g, h in G and for all X in W. For W an affine variety and B a k-algebra and α a morphism as above,

$$\alpha(B)\colon G(B) \times W(B) \longrightarrow W(B), \qquad \alpha(B)\colon (g_B, X_B) \mapsto g_B X_B$$

shall be the morphism given by the following diagram

$$\begin{tikzcd} k[W] \arrow[r, "\alpha^0"] \arrow[dr, "g_B X_B"'] & k[G] \otimes_k k[W] \arrow[d, "g_B \otimes X_B"] \\ & B \end{tikzcd}$$

for α^0 the comorphism of α above.

If $g \in G$ and W is affine we denote by λ_g the comorphism of

$$\begin{aligned} L_g\colon W &\longrightarrow W \\ : X &\mapsto gX. \end{aligned}$$

Then $\lambda_g\colon k[W] \longrightarrow k[W]$ with $(\lambda_g f)(X) = f(g^{-1}X)$ is a linear automorphism of $k[W]$ called left translation of functions by g. The map

$$\begin{aligned} \lambda : G &\longrightarrow \mathrm{Aut}_{k\text{-alg}}(k[W]), \\ : g &\mapsto \lambda_g \end{aligned}$$

is easily seen to be a group homomorphism.

(0.2) The *Lie algebra* $L(G)$ *of* G is the tangent space at the identity e of G, the k-vector space of point derivations $k[G] \longrightarrow k$ at e. By Borel (3.3), it is naturally isomorphic to the left invariant derivations of $k[G]$. $T(G) = G(k[\delta])$ is the tangent bundle of G where $k[\delta]$ is the algebra of dual numbers, $\delta^2 = 0$.

$G(k[\delta]) = \mathrm{Hom}_{k\text{-alg}}(k[G], k[\delta])$ with a typical member $g^{\delta A} = g + \delta A\colon f \longrightarrow f(g) + \delta A(f)$.

$$L(G) = \{A \mid e + \delta A \text{ is in the tangent space at the identity}\}.$$

If α is an action of G on an affine variety W and if

$$T_X(W) = \{Y \mid X + \delta Y \in W(K[\delta])\}$$

is the tangent space to W at X, then there is a natural map

$$\begin{aligned} (d\alpha)_X\colon L(G) &\longrightarrow T_X(W), \\ : A &\mapsto (d\alpha)_X(A) \end{aligned}$$

defined as follows: $(d\alpha)_X(A)$ is the unique element of $T_X(W)$ for which the diagram below commutes.

$$\begin{tikzcd} k[W] \arrow[r, "\alpha^0"] \arrow[dr, "X + \delta(d\alpha)_X(A)"'] & k[G] \otimes_k k[W] \arrow[d, "(e + \delta A) \otimes X"] \\ & k[\delta] \end{tikzcd}$$

If $X \in W(k)$, we let

$$\pi\colon G \longrightarrow GX$$
$$: g \mapsto gX$$

be the orbit morphism. This is a surjective k-morphism. In fact, from Borel (1.8) and (6.7) we know that GX is a smooth variety defined over k, locally closed in W and that its boundary is a union of orbits of lower dimension. If $G_X = \{g \in G \mid gX = X\}$ is the stabilizer of X in G then π is an orbit map for the action of G_X on G by right translation. We know also that π is a quotient of G by G_X over k and that $(d\pi)_e\colon L(G) \longrightarrow T_X(GX)$ is surjective. When W is affine, we may identify the images of $(d\pi)_e$ and $(d\alpha)_X$ in $T_X(W)$.

(0.3) The symmetric algebra S is a functor from k-vector spaces to graded k-algebras. It induces a functor S from the category of finite dimensional k-vector spaces to the category of graded k-algebras of finite type. Linear transformations of k-vector spaces induce k-algebra homomorphisms of the associated symmetric algebras. The linear transformations are natural for the functor S.

Let W be a finite dimensional k-vector space, W^* its k-dual. If $h\colon V \longrightarrow W$ is a homomorphism of vector spaces, define $h^*\colon W^* \longrightarrow V^*$ by requiring $(h^*Y)(X) = Y(hX)$ for all Y in W^* and all X in V. h^* is k-linear. h in $GL(W)$ implies h^* in $GL(W^*)$. Let $\bar{h} = h^{*-1}$ for h in $GL(W)$. The map

$$^{-}\colon GL(W) \longrightarrow GL(W^*),$$
$$: h \mapsto \bar{h}$$

is an isomorphism of algebraic groups over k. If H is any subgroup of $GL(W)$ denote the image of H under $^{-}$ by $\bar{H}$ or H^{-}. Let H be any group and let

$$\rho\colon H \longrightarrow GL(W),$$
$$: h \mapsto \rho(h)$$

be a linear representation of H. Then associated with ρ we have the *contragredient* representation

$$\bar{\rho}\colon H \longrightarrow GL(W^*),$$
$$: h \mapsto \rho(h)^{-}.$$

If V is of finite dimension n over k, V may be identified with $\mathbf{A}^n$, affine n-space, $k[V]$ with $S(V^*)$. Via this identification, the action $\lambda_g\colon k[V] \longrightarrow k[V]$

for g in $GL(V)$ becomes $S(\bar{g})\colon S(V^*) \to S(V^*)$ induced by $\bar{g}\colon V^* \to V^*$.

The Lie algebra $L(GL(V))$ of $GL(V)$ can be identified canonically with $\mathrm{End}_k(V)$ by considering $\mathrm{Id}_V + \delta A$ in $GL_{k[\delta]}(V \otimes k[\delta])$ for every A in $\mathrm{End}_k(V)$. The isomorphism

$$^{-}\colon GL(V) \to GL(V^*),$$

$$^{-}\colon g \mapsto \bar{g} = g^{*-1}$$

has differential

$$^{-\prime}\colon \mathrm{End}_k(V) \to \mathrm{End}_k(V^*),$$

$$^{-\prime}\colon A \mapsto \bar{A}' = -A^*.$$

$GL(V)$ acts on $S(V)$ via

$$S\colon GL(V) \to GL(S(V)),$$

$$S\colon g \mapsto S(g).$$

When $S(g)$ is restricted to $S(V)^1$, this is just g. Therefore the differential of this action is given by

$$D\colon \mathrm{End}_k(V) \to \mathrm{Der}_{k\text{-alg}}(S(V)),$$

$$D\colon A \mapsto D(A),$$

where $D(A)$ is the unique k-derivation which extends $A\colon V \to V$, by Proposition 1 of [1]. $GL(V)$ also acts on $S(V^*)$ via

$$\bar{S}\colon GL(V) \to GL(S(V^*)),$$

$$\bar{S}\colon g \mapsto \bar{S}(g) = S(\bar{g}).$$

$\bar{S}(g)^1 = \bar{g}$, and in this case, as above, the differential is given by

$$\bar{D}\colon \mathrm{End}_k(V) \to \mathrm{Der}_{k\text{-alg}}(S(V^*)),$$

$$\bar{D}\colon A \mapsto \bar{D}(A) = D(\bar{A}'),$$

$D(\bar{A}')\colon S(V^*) \to S(V^*)$ is the unique k-derivation which extends $\bar{A}'\colon V^* \to V^* = S(V^*)^1$. The derivations $D(A)$ and $\bar{D}(A)$ are of degree 0, i.e., they preserve degrees.

The examples and reasonings which follow use these simple basic facts.

CHAPTER I. PREHOMOGENEOUS PAIRS WITH SEMI-INVARIANTS

1. **On semi-invariant forms.** Let V be a k-vector space and let H be a subgroup of $GL(V)$. $v \in V$ is said to be *semi-invariant* if H stabilizes kv, the subspace generated by v. If $v \neq 0$ (i.e. if $\dim kv = 1$) then there exists a unique character $\chi_v\colon H \to k^*$ such that $h(v) = \chi_v(h) \cdot v$ for all h in H. We say that v is of weight χ_v for H. Note that if $H = H'$, the commutator subgroup of H, then $\chi_v(h) = 1$ for all h.

Let V now be a finite dimensional k-vector space and let H be a subgroup of $GL(V)$. H acts naturally on $k[V]$ which we identify with $S(V^*)$ the symmetric algebra on V^*, the dual to V. Recall that the action of H on $k[V]$ is by left translation of functions:

$$\lambda: H \longrightarrow \mathrm{Aut}_{k\text{-alg}}(k[V])$$
$$: h \mapsto \lambda_h \quad \text{where } (\lambda_h f)(X) = f(h^{-1}X).$$

H has a contragredient representation on V^* and this induces its action on $S(V^*)$. The action is precisely that given by $\bar{S}$ via the identification $k[V] = S(V^*)$.

$k[V]$ is a graded algebra and H preserves the grading. Hence we have

(1.1) **Lemma.** *If P in $k[V]$ is a semi-invariant of weight χ for H, then so is each homogeneous component of P. Therefore, if H has a nonconstant semi-invariant in $k[V]$, it has one of the same weight which is a form in $k[V]$.*

(1.2) **Proposition.** *Let S denote the set of nonzero semi-invariants for H in $k[V]$. Then*

(a) *$k^* \subseteq S$, and if $P, Q \in S$ then $PQ \in S$, i.e. S is a commutative monoid in $k[V]$ which contains the units k^* of $k[V]$.*

(b) *If H is a connected algebraic subgroup of $GL(V)$ then S is generated as a monoid by k^* and the irreducible polynomials in S. In particular, $PQ \in S$ if and only if $P \in S$ and $Q \in S$.*

Proof. (a) is clear.

(b) We now assume H to be a connected algebraic subgroup of $GL(V)$. $k[V]$ which is isomorphic to a polynomial ring in dim V indeterminates over k is a factorial ring or a unique factorization domain. Suppose $P \in S$, then $P = \prod_{i \in I} P_i^{e_i}$ where I is finite and the P_i are irreducible, uniquely determined up to scalar multiples.

$$\lambda_g P = \chi_P(g) \cdot P \quad \text{implies} \quad \prod_{i \in I} (\lambda_g P_i)^{e_i} = \chi_P(g) \prod_{i \in I} P_i^{e_i}.$$

Therefore $\lambda_g P_i = \chi_i(g) P_{\sigma(g)(i)}$ for some map $\sigma: H \longrightarrow$ Sym group (I).

$$\lambda_{hg} P_i = \lambda_h \lambda_g P_i = \chi_i(g) \lambda_h P_{\sigma(g)(i)} = \chi_i(g) \chi_{\sigma(g)(i)}(h) P_{\sigma(h)(\sigma(g)(i))}$$

entails that $\sigma: H \longrightarrow$ Sym group (I) is a homomorphism since $\lambda_{hg} P_i = \chi_i(hg) P_{\sigma(hg)(i)}$. With H connected, we must have $\ker \sigma = H$ or, equivalently, $\sigma(g)i = i$ for all $i \in I$ and all $g \in G$. See [3].

2. Prehomogeneous pairs.

(2.1) *Terminology.* By a *pair* we mean an ordered pair (G, V) where V is a finite dimensional vector space over k and where G is a connected algebraic subgroup of $GL(V)$.

We call the pair (G, V) *prehomogeneous* if G has a (Zariski) dense orbit in V. By the proposition in Borel (1.8), each such orbit is open in V and is clearly unique. Therefore we denote the orbit by $o(G)$.

We call the pair *irreducible* if the representation of G on V is irreducible.

(2.2) **Lemma.** *If the pair (G, V) is irreducible then the commutator subgroup G' of G is semisimple and G is contained in $(k^*\mathrm{Id}_V)\cdot G'$, an almost direct product of algebraic groups, i.e. $G' \subset G \subset (k^*\cdot \mathrm{Id}_V)\cdot G'$.*

Proof. From Theorem 5.2 of Borel's paper [3], $G = S\cdot G'$, almost direct, where S is a central algebraic torus and G' is semisimple. Let $\mathrm{X}(S)$ be the character group of S into k; by Borel (8.5), $\mathrm{X}(S)$ is free abelian of rank $=\dim S$ and forms a k-basis of $k[S]$. Let $\mathrm{X}_V(S)$ be the subset of eigenfunctions of S in V, then $V = \bigoplus_{\omega\in \mathrm{X}_V(S)} V_\omega$ where $V_\omega \neq (0)$ is the eigenspace for ω and the sum is G-direct. Since V is irreducible, $\mathrm{X}_V(S)$ is singleton and the conclusion follows since $\mathrm{X}_V(S)$ is a set of generators of $\mathrm{X}(S)$.

The following examples show that the conditions of irreducibility and of prehomogeneity for pairs are quite independent.

(2.3) **Example.** Prehomogeneity does not imply irreducibility. Let $V =$ vector space of $n\times 1$ matrices over k, $G = T(n)$ the lower triangular group in $GL_n(k)$. (G, V) is a prehomogeneous pair which is not irreducible if $n > 1$. Let $X =$ column $(1, 0, \cdots 0)$,

$$GX = \{\mathrm{col}\,(x_1, \cdots, x_n) \mid x_1 \neq 0\} \text{ is } o(G).$$

(2.4) **Example.** Irreducibility does not imply prehomogeneity. Since $(d\pi)_e\colon L(G) \to T_X(GX)$ is surjective, prehomogeneity requires $\dim V \leq \dim G$, whereas $SL_2(k)$, for example, has irreducible representations of arbitarily large dimensions.

(2.5) **Example.** Prehomogeneity of (G, V) does not imply the prehomogeneity of $(\overline{G}, V^*)$. Let $V = M_{2\times 1}(k) = \{\mathrm{col}\,(x_1, x_2) \mid x_i \in k\}$. We may identify V^* with $M_{1\times 2}(k) = \{(y_1, y_2) \mid y_i \in k\}$ by setting $Y(X) = (y_1, y_2)\mathrm{col}\,(x_1, x_2) = y_1x_1 + y_2x_2$, the matrix product.

Let $G = \{[\begin{smallmatrix} a & 0\\ b & 1\end{smallmatrix}] \mid a \in k^*, b \in k\} \subset GL(V)$. Then

$$\begin{bmatrix} a & 0\\ b & 1\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix} = \begin{bmatrix} ax_1\\ bx_1 + x_2\end{bmatrix}.$$

$$\begin{bmatrix} a & 0\\ b & 1\end{bmatrix}^{-}(y_1, y_2) = (y_1, y_2)\begin{bmatrix} a^{-1} & 0\\ -a^{-1}b & 1\end{bmatrix} = (a^{-1}y_1 - a^{-1}by_2, y_2).$$

One easily sees that (G, V) is prehomogeneous with $o(G) = G[\begin{smallmatrix} 1\\ 0\end{smallmatrix}] = \{\mathrm{col}\,(a, b) \mid a \in k^*, b \in k\}$, but that $(\overline{G}, V^*)$ is not prehomogeneous since the coordinate y_2 is fixed.

3. Criteria for the existence of semi-invariant forms for prehomogeneous pairs.

(3.1) **Proposition.** *Let (G, V) be any pair. Suppose that there exists a nonconstant semi-invariant P for G in $k[V]$. Then P is invariant for G' and every orbit of G' in V is contained in a level hypersurface, $P(X) = c$, of P. In particular (G', V) is not prehomogeneous.*

Proof. The first statement has been shown. For all $g \in G'$, $P(gX) = \lambda_{g^{-1}}P(x) = P(X)$. The closure of the orbit $G'X$ is contained in the set of zeros of $P - P(X)$, for any $X \in V$.

(3.2) **Lemma.** *Let (G, V) be an irreducible pair, so that $G' \subset G \subset k^*\mathrm{Id}_V \cdot G'$. For any X in V,* codim $(G'X$ *in* $GX) = 0$ *or* 1. *Equivalently, for any X in V,* $\dim G'X = \dim GX$ *or* $\dim G'X = \dim GX - 1$.

Proof. Consider the surjective orbit morphism $\pi\colon G \to GX$. By Theorem 3, of Mumford's *Introduction to algebraic geometry,* [5, p. 93], there exists a nonempty open $U \subset GX$ such that for all irreducible closed subsets $W \subset GX$ such that $W \cap U \neq \varnothing$ and for all components Z of $\pi^{-1}(W)$ such that $Z \cap \pi^{-1}(U) \neq \varnothing$, codim $(Z$ in $G) =$ codim $(W$ in $GX)$. Since GX is homogeneous, we may assume $U = GX$. Consider $W = \overline{G'X}$. Then codim $(G'X$ in $GX) =$ codim $(\overline{G'X}$ in $GX) =$ codim $(G'$ in $G) = 0$ or 1 since $\dim G' \leq \dim G \leq \dim k^* + \dim G'$.

(3.3) **Theorem.** *Let (G, V) be any irreducible prehomogeneous pair. Let S denote the set of semi-invariants for G in $k[V]$, then either*

(a) *$S = k$ and (G', V) is prehomogeneous or*

(b) *$S = \{aP^n \mid a \in k, n \in \mathbf{Z}, n \geq 0\}$ for some irreducible homogeneous polynomial P determined uniquely up to a nonzero scalar in k and (G', V) is not prehomogeneous.*

Proof. Suppose $S \neq k$. Then there exists a nonconstant semi-invariant P for G in $k[V]$. By Proposition (3.1) above, (G', V) is not prehomogeneous. Since $G \subset GL(V)$, every homogeneous component of P is semi-invariant. By Proposition (1.2) it suffices to show that there is only one irreducible generator for S. Suppose that P_1, P_2 are nonassociate (noncollinear) irreducible semi-invariants for G. Since (G, V) is irreducible, Lemma (3.2) entails that $G'X$ is of dimension $n - 1$ for $X \in o(G)$. But $G'X$ is contained in $Z(P_1 - P_1(X)) \cap Z(P_2 - P_2(X))$, an algebraic set of codimension ≥ 2 in V. This is impossible. All the claims of (b) are shown. On the other hand, if $S = k$ we show that (G', V) is prehomogeneous. Suppose (G', V) is not prehomogeneous. Then for $X \in o(G)$, $G'X$ is of dimension $n - 1$, of pure codimension 1 in V. Since $k[V]$ is factorial, $G'X = Z((f))$ for some

nonconstant $f \in k[V]$ by Proposition 4, p. 87 of Mumford [4] or by the remark on p. 238 of Zariski and Samuel [5]. f is G'-semi-invariant. Let $f = f_0 + \cdots + f_l$, $l > 0$, be its decomposition into homogeneous components. Then f_l is G'-semi-invariant and f_l is $k^*\mathrm{Id}_V \cdot G'$ semi-invariant and a fortiori G-semi-invariant. This contradicts $S = k$.

(3.4) **Corollary.** *Let (G, V) be an irreducible prehomogeneous pair. Assume that $G = (k^*\mathrm{Id}_V) \cdot G'$. Then the following conditions are equivalent.*

(a) *G has a nonconstant semi-invariant in $k[V]$.*

(b) *(G', V) is not a prehomogeneous pair.*

(c) *If $X \in o(G)$ then G_X^0 is contained in G'.*

Proof. The equivalence of (a) and (b) is immediate from the theorem.

(a) implies (c): If there is a P and if $X \in o(G)$ then $P(gX) \neq 0$ and $\chi_P(g^{-1})P(X) = (\lambda_g - 1P)(X) = P(gX) = P(X)$ for all g in G_X^0. Therefore, $G_X^0 \subset (\ker \chi_P)^0$ where P is of weight χ_P. If G is irreducible, then G' is semisimple and of codimension ≤ 1 in G. Therefore $(\ker \chi_P)^0 = G'$ and $G_X^0 \subset G'$.

(c) implies (b): If $G_X^0 \subset G'$, then $\pi_X\colon G' \to G'X$ is an orbit map for the action of G_X^0 on G'. Therefore $G'X$ is of dimension $n - 1$ for every X in $o(G)$ and thus (G', V) is not prehomogeneous.

(3.5) **Example.** $(GL(V), V)$ is irreducible prehomogeneous. $GL(V)' = SL(V)$ and $(SL(V), V)$ is prehomogeneous. There exist no nonconstant semi-invariants in $k[V]$ for $GL(V)$.

(3.6) **Example.** (a) Let V be the k-vector space of skew-symmetric matrices of size $2n + 1$ over k, $G = \{T_g \mid g \in GL_{2n+1}(k)\}$ where

$$\begin{aligned} T_g &: V \to V, \\ &: X \mapsto gXg^t, \qquad G \cong GL_{2n+1}(k)/\text{finite group}. \end{aligned}$$

(G', V) is irreducible prehomogeneous and (G', V) is prehomogeneous since $L(G)X_0 = (d\pi)_{X_0}(L(G)) = V$ and $L(G')X_0 = (d\pi)_{X_0}(L(G')) = V$ where

$$X_0 = \begin{bmatrix} \mathbf{0_n} & 0 & \mathbf{I_n} \\ 0 & 0 & 0 \\ -\mathbf{I_n} & 0 & \mathbf{0_n} \end{bmatrix} \quad (\mathbf{0_n}, \mathbf{I_n} \text{ are the zero and identity square matrices of size } n)$$

and $L(G) = \{D_A \mid A \in gl_{2n+1}(k)\}$,

$$\begin{aligned} D_A &: V \to V, \\ &: X \mapsto AX + XA^t. \end{aligned}$$

(b) If V is the k-vector space of skew-symmetric matrices of size $2n$ over k, $G = \{T_g \mid g \in GL_{2n}(k)\}$, as above, then (G, V) is irreducible prehomogeneous and (G', V) is not prehomogeneous.

$$S = \{aP^n \mid a \in k,\ P = (\text{Pfaffian form on } V) = \sqrt{\det X},\ n \in \mathbf{Z},\ n \geq 0\}.$$

For the Pfaffian form, see Hermann Weyl, *The classical groups*, Princeton Univ. Press, Chapter VI 1, or N. Bourbaki, *Algèbre*, Chapter IX, §5, No. 2.

4. Characterizations of irreducible prehomogeneous pairs with open affine orbit.

(4.1) **Theorem.** *For an irreducible prehomogeneous* (G, V) *the following conditions are equivalent* [16, *p.* 134].

(i) $o(G) = U_P$ *the principal open affine set with* P *a nonconstant semi-invariant for* G.

(ii) $o(G)$ *is an affine variety.*

(iii) G_X *the isotropy subgroup of* X *in* $o(G)$ *is reductive.*

Proof. (i) $\Rightarrow$(ii): (i) $\Rightarrow$ (ii) is obvious since U_P is an affine variety.

(ii) $\Rightarrow$ (i): $o(G)$ is open in V since it is an orbit. Hence $o(G)$ is an open affine variety in V. In his thesis Jacob Eli Goodman states as a general proposition which is "well known but which does not appear in the literature", the following "Proposition 1, Chapter I. Suppose U an open affine subset of a variety V. Then the complement W of U in V is purely of codimension one" [6]. Now since $k[V]$ is a factorial ring $V - o(G) = Z(P)$ for some $P \in k[V]$ by Proposition 4, p. 87 in Mumford [4]. G leaves $Z(P)$ invariant and hence in its action on $k[V]$, G preserves the line $k \cdot P$. Thus $o(G) = V - Z(P) = U_P$ and P is semi-invariant for G.

(ii) $\Leftrightarrow$(iii): The following statement is most convenient: Theorem 3.5 in the paper of Borel and Harish-Chandra [7]. "Let G be a connected reductive algebraic group and H an algebraic subgroup. Then the quotient variety G/H is an affine algebraic variety if and only if H is reductive." Form Borel (6.7), we know that the orbit map $\pi: G \to GX$ is an orbit map for the action of $G_X = \{g \in G \mid gX = X\}$ by right translation, and that π is a quotient of G by G_X. $o(G)$ is isomorphic to G/G_X for any $X \in o(G)$. By the theorem above, $o(G)$ is affine if and only if G_X is reductive for $X \in o(G)$.

(4.2) **Examples.** 1. We have seen that theorem illustrated in the case $(GO(P), V)$ where P is a nondegenerate symmetric bilinear form.

2. Consider $(G(S), V)$ where $V = M_{n\times n}(k)$ and $G(S) \leq GL(V)$ is isogenous to $k^* \times SL_n(k) \times S$ and where S is a connected subgroup of $SL_n(k)$ acting irreducibly on $V(n)$. A typical element of $G(S)$ is

$$(c\,\mathrm{Id}_V, g, h): V \to V,$$
$$: X \mapsto cgXh^{-1}.$$

P in this case is det.

(i), (ii): $o(G) = U_P = \{X \mid \det X \neq 0\}$ is affine.

(iii): Take $X = I_n$ in $o(G(S))$; then $G_X \cong S$ which is reductive.

CHAPTER II. FINITENESS OF $G\backslash W$, THE SET OF G ORBITS IN W

5. **Examples where the cardinality of orbits is not finite.** Let (G, V) be the irreducible prehomogeneous pair consisting of the tensor product of the following irreducible representations: $SL_n(k)$ on $V(n)$ and ρ_n of $SL_2(k)$ on $V(n) = S(V(2))^{n-1}$ (see (4.2) 2). $V \cong N_{n\times n}(k)$ and G is isogenous to $k^* \times SL_n(k) \times SL_2(k)$. With

$$T(c, g, h): V \longrightarrow V, \qquad c \in k^*,\ g \in SL_n(k),\ h \in SL_2(k),$$

$$: X \otimes Y \mapsto cgX \otimes \rho_n(h)Y.$$

Let $I = X_1 \otimes Y_1 + \cdots + X_n \otimes Y_n$, then $G_I \cong SL_2(k)$ and $U_P = GL_n(k)$. The algebraic set of matrices of rank 1 in V is G-invariant. By the action of $SL_n(k)$, any such rank 1 matrix X may be made equivalent to some

$$\begin{bmatrix} a_{11}, & a_{12}, & \cdots, & a_{1n} \\ 0 & 0 & & 0 \\ 0 & 0 & & 0 \end{bmatrix}.$$

If $n \geq 5$, $k^*\mathrm{Id}_v \cdot SL_2(k)$ does not have a finite number of orbits in $V(n)$, since $\dim k^*\mathrm{Id}_v \cdot SL_2(k) = 4$. Now the action of $k^*\mathrm{Id}_v \cdot SL_2(k)$ on the matrices above is just its action on $V(n)$. Hence the orbits of rank 1 are not finite for G. $P = \det$ is a semi-invariant for G in V. However, if we take the full group $G_P \subset GL(V)$ leaving $P = \det$ semi-invariant; G_P is isogenous to $k^* \times SL_n(k) \times SL_n(k)$. The G_P orbits are just the matrices of a given rank.

Let ST_L be the lower triangular group in $SL_n(k)$. It is easy to verify that the Borel subgroup $k^* \times ST_L \times ST_L$ acts with an open dense orbit on $Z(P) = \{X \mid \det X = 0\}$.

6. **A condition guaranteeing the finiteness of $G\backslash W$ for reductive G.** In this section fix G a reductive connected linear algebraic group and assume that $G \times W \longrightarrow W$ is an action of G on an affine variety W; $G \times W \longrightarrow W$ is a morphism and W is irreducible.

One can easily verify the following phenomenon in the example of §5. No Borel subgroup of $G = k^*\mathrm{Id}_v \cdot SL_n(k) \times SL_2(k)$ (almost-direct) can act so as to have an open dense orbit in $W = Z(\det)$. A dimension comparison confirms this. The dimension of a Borel subgroup of G is $n(n+1)/2 + 2$ whereas $\dim Z(\det) = n^2 - 1$ and $n^2 - 1 > n(n+1)/2 + 2$ if $n \geq 5$. The main statement of this section guarantees that the orbits are finite in number if there is a Borel subgroup which has an open dense orbit in W.

(6.1) **Definitions.** Let H be an algebraic group, W an affine variety, and $H \times W \longrightarrow W$ an action of H on W, each defined over k. Say that H acts *prehomogeneously* on W if H has a unique open dense orbit $o(H)$ in W.

Let G be as above and let R be a G-module. Call R *multiplicity free* if distinct irreducible G-submodules of R are nonisomorphic.

(6.2) **Theorem.** *Consider the following conditions.*

(a) *There exists a Borel subgroup B of G such that B acts prehomogeneously on W.*

(b) *The G-module $k[W]$ is multiplicity free.*

(c) *G has only finitely many orbits in W.*

Then (a) *implies* (b), *and* (b) *implies* (c).

Proof. (6.3) (a) $\Rightarrow$ (b).

Lemma. *If an affine group H acts prehomogeneously on an affine variety W by the morphism $H \times W \to W$, $(g, X) \mapsto gX$, then there exist no nonconstant rational functions on W fixed under H. Thus, if f is a semi-invariant for H in $k(W)$ of weight χ_f, the character induced by χ_f on H is nontrivial.*

Proof: Let $o(H)$ be the open dense orbit, let $f \in k(W)$ be an invariant for H, i.e. $\lambda_g f = f$ for all $g \in H$. Let O_f be the open dense set on which f is defined. $o(H) \cap O_f$ is open dense since W is irreducible. But for all $X \in o(H) \cap O_f$, $f(X) = f(g^{-1}X_0) = (\lambda_g f)(X_0) = f(X_0)$ for some fixed $X_0 \in o(H)$, where $g \in H$ depends on X. Hence f is constant on an open dense set. f is a constant function on W.

The action $G \times W \to W$ induces an action of G on $k[W]$ via λ, left translation of functions. By Borel(1.9), $k[W]$ is a sum of finite dimensional irreducible G-modules. Suppose W_1 and W_2 are distinct G-isomorphic irreducible G-submodules of $k[W]$. Since B is a Borel subgroup, a minimal parabolic subgroup of G, there exist unique lines $kf_1 \subset W_1$ and $kf_2 \subset W_2$ which are B-invariant and which induce the same character on B; see [3, p. 17, §7]. Since $W_1 \neq W_2$, f_1 and f_2 are not k-collinear. But then f_1/f_2 is a rational function on W which is B-fixed and nonconstant. This contradicts the Lemma.

(6.4) **Proof of** (b) $\Rightarrow$ (c). We use a Lefschetz principle to reduce to the case when $k = \mathbf{C}$, the complex numbers. G, W and $G \times W \to W$ are defined over k. We take a field of definition k_0 algebraically closed and of finite type over the rationals, $\mathbf{Q}$, and an embedding of k_0 into $\mathbf{C}$. The $G(k_0)$ irreducible submodules of $k_0[W]$ are absolutely irreducible [8]. Since $k_0[W]$ is multiplicity free as a $G(k_0)$-module, $\mathbf{C}[W] = \mathbf{C} \otimes k_0[W]$ is multiplicity free as a $G(\mathbf{C})$-module. Therefore assume $G = G(\mathbf{C})$ and $W = W(\mathbf{C})$.

It suffices to show that the following statement is true. If a reductive complex group G acts on a complex variety W with multiplicity free action on $\mathbf{C}[W]$, then G has only finitely many orbits in W.

We do this by induction on the dimension of the Noetherian space W. We

claim that G has an open orbit U in W. Once this is shown the proof is completed by induction since $W - U$ is closed, G acts on $W - U$, and $\dim (W - U) < \dim W$. Since G is connected, each irreducible component of $W - U$ is G-stable, and we may assume for the induction step that $W - U$ is irreducible. The hypothesis is satisfied—G has multiplicity free action on $\mathbf{C}[W - U]$, for $\mathbf{C}[W - U]$ is isomorphic to a G-direct summand of $\mathbf{C}[W]$.

The G-orbits in W are open in their closures in W and hence there is a bijection between G-orbits in W and their closures in W;

$$GX \leftrightarrow \text{closure of } GX \text{ in } W = \text{closure of } GX.$$

Since these closures are G-invariant algebraic sets in W, the cardinality of the closures is the cardinality of the G-invariant radical ideals in $\mathbf{C}[W]$ by the Nullstellensatz. Since $\mathbf{C}[W]$ is Noetherian, this is the cardinality of the finitely generated G-submodules of $\mathbf{C}[W]$ which must be countable by the hypothesis that $\mathbf{C}[W]$ is multiplicity free. The Zariski topology is contained in the topology on W as a complex analytic variety and in this topoloty W is locally compact. We state the category theorem, "If a locally compact space M is a countable union $M = \bigcup M_i$, where each M_i is closed, then at least one M_i contains an open subset of M." [9] It now follows that one of the G-invariant closed subsets contains a strong open set U, and hence an open orbit. Q.E.D.

(6.5) **Examples.** 1. Let $G = GL(V)$ and $W = V$.

(a) Any Borel subgroup B of G acts prehomogeneously on V; see (2.3).

(b) The decomposition of $k[V] = S(V^*)$ into irreducible $GL(V)$ modules is $\bigoplus_{i \geq 0} S(V^*)^i$.

(c) The $GL(V)$ orbits in V are $V - \{0\}$ and $\{0\}$.

2. Let V be a k-vector space of dimension $2l\,(l \geq 1)$, let P be a nondegenerate quadratic form and let $G = GO(P) = k^* \mathrm{Id}_V \cdot O(P)$ be the full group of similitudes of P. We may assume dual bases $\mathbf{B} = \{X_i \mid i = 1, \cdots, 2l\}$ in V and $\mathbf{B}^* = \{Y_j \mid j = 1, \cdots, 2l\}$ in V^* chosen so that $P = Y_1 Y_{l+1} + \cdots + Y_l Y_{2l}$; the Lie algebra of G is

$$L(GO(P)) = k\,\mathrm{Id}_{2l} \oplus \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11}^{\mathrm{tr}} \end{bmatrix} \middle| \begin{array}{l} A_{12} \text{ and } A_{21} \text{ skew-symmetric} \\ \text{and } A_{ij} \in M_{l \times l}(k) \end{array} \right\}. \quad [10]$$

A Borel subalgebra of $L(GO(P))$ is

$$N = k\mathrm{Id}_{2l} \oplus \left\{ \begin{bmatrix} T_L & 0 \\ sk & -T_L^{\mathrm{tr}} \end{bmatrix} \middle| \begin{array}{l} sk \text{ skew-symmetric in } M_{l\times l}(k) \\ \text{and } T_L \text{ lower triangular} \end{array} \right\}.$$

(a) If a Borel subgroup B has $L(B) = N$, then $B(X_1 + X_{l+1})$ is open in V since $N(X_1 + X_{l+1}) = V$.

(b) $O(P)$ has rank l. The irreducible $O(P)$-modules are indexed by their highest weights relative to an ordering in the weight space for a Cartan subgroup $H \subset O(P)$. If a module M is finite dimensional, the highest weight is a linear combination with nonnegative integer coefficients of simple roots relative to that ordering. Let $(s_1, \cdots, s_l) = (1, 0, \cdots, 0)$ correspond to the representation of $O(P)$ in V. Let s_0 index the character on $k^*\mathrm{Id}_v$ in $GO(P)$. As an $O(P)$-module,

$$V \cong V^* \leftrightarrow (-1, 1, 0, \cdots, 0) = (s_0, s_1, \cdots, s_l).$$

Then $k[V] = S(V^*) = \bigoplus_{i\geq 0} S(V^*)^i$.

$$S(V^*)^0 = k = (0, 0, 0, \cdots, 0),$$

$$S(V^*)^1 = V^* = (-1, 1, 0, \cdots, 0),$$

$$S(V^*)^2 = (-2, 2, 0, \cdots, 0) \oplus (-2, 0, 0, -0) = (-2, 2, 0, \cdots, 0) \oplus kP,$$

$$\vdots$$

$$S(V^*)^{2m} = \bigoplus_{j=0}^{m} (-2m, 2(m-j), 0, \cdots, 0),$$

$$= \bigoplus_{j=0}^{m} P^j \cdot (-2(m-j), 2(m-j), 0, \cdots, 0),$$

$$S(V^*)^{2m+1} = \bigoplus_{j=0}^{m} (-(2m+1), 2(m-j)+1, 0, \cdots, 0),$$

$$= \bigoplus_{j=0}^{m} P^j(-(2(m-j)+1), (2m-j)+1, 0, \cdots, 0).$$

$k[V]$ is multiplicity free.

(c) The G orbits in V are $U_P = \{X \mid P(X) \neq 0\}$, $Z(P) - \{0\}$ and $\{0\}$; this follows easily from Witt's Theorem.

3. Let $W = V = \{X = a_3u^3 + a_1u^2v + a_{-1}uv^2 + a_{-3}v^3 \mid a_i \in k\}$ = vector space

of binary cubic forms. Choose a basis $\mathbf{B} = \{X_3 = u^3, X_1 = u^2v, X_{-1} = uv^2, X_{-3} = v^3\}$ for V and dual basis $\mathbf{B}^* = \{Y_3, Y_1, Y_{-1}, Y_{-3}\}$ for V^*. Let G be the full group of substitutions in these forms; $G \cong k^*\mathrm{Id}_v \cdot SL_2(k)$ where V is the irreducible $SL_2(k)$-module of highest weight 3. (G, V) is an irreducible prehomogeneous pair with nonconstant semi-invariant form

$$P = \text{discriminant} = -6Y_3Y_1Y_{-1}Y_{-3} + 4Y_3Y_{-1}^3 + 4Y_1^3Y_{-3} + Y_3^2Y_{-3}^2 - 3Y_1^2Y_{-1}^2.$$

With bases chosen above,

$$L(G) = k\mathrm{Id}_V \oplus \left\{ \begin{bmatrix} 3\alpha & \gamma & 0 & 0 \\ 3\beta & \alpha & 2\gamma & 0 \\ 0 & 2\beta & -\alpha & 3\gamma \\ 0 & 0 & \beta & -3\alpha \end{bmatrix} \middle| \alpha, \beta, \gamma \in k \right\}.$$

A Borel subalgebra of $L(G)$ is

$$N = \left\{ \begin{bmatrix} 3\alpha+\delta & 0 & 0 & 0 \\ 3\beta & \alpha+\delta & 0 & 0 \\ 0 & 2\beta & -\alpha+\delta & 0 \\ 0 & 0 & \beta & -3\alpha+\delta \end{bmatrix} \middle| \alpha, \beta, \delta \in k \right\}.$$

Let $W = Z(P) =$ zeros of P in V; $\dim W = 3$.

(a) If $X = u^3 - u^2v - uv^2 + v^3 = (u - v)^2(u + v) \in Z(P)$ and if B is a Borel subgroup with $L(B) = N$, then

$$NX = (3\alpha + \delta)u^3 + (3\beta - \alpha - \delta)u^2v + (-2\beta + \alpha - \delta)uv^2 + (-\beta - 3\alpha + \delta)v^3$$

is a vector space of dimension 3. Hence B acts prehomogeneously on $Z(P)$ and, by (6.2), G has finitely many orbits in $Z(P)$.

Note. If $X_0 = u^3 + v^3 \in U_P$, $L(G)X = V$. Therefore, G_{X_0} is finite and hence reductive. By (4.1), $o(G) = U_P$ and $V = U_P \cup Z(P)$ is a finite union of G orbits; i.e. (c) holds for (G, V). However (a) does not hold for (G, V) since the dimension of any Borel subgroup B of G is 3 whereas $\dim V = 4$. We see (c) $\not\Rightarrow$ (a) in general.

7. Determination of some irreducible prehomogeneous pairs for G' simple. In this section we consider irreducible representations of a simple group G', ρ: $G' \to GL(V)$ and take $G = k^*\mathrm{Id}_v \cdot \rho(G')$ (a semidirect product). $\rho(G')$ is in fact a Chevalley group.

A necessary condition for irreducible (G, V) to be prehomogeneous is that $\dim G = \dim G' + 1 \geq \dim V$. A table of pairs $(\rho(G'), V)$ for which $\dim G' > \dim V$ is found in [11]. A Cartan subgroup is a connected isotropy subgroup of a point

whose orbit is of maximal dimension in the adjoint representation. From this and from $\dim G = \dim G' + 1$ we infer that the pair (G, V) for the adjoint representation can only be prehomogeneous in the rank 1 case. (The authors of [11] mistakenly include this case in their list.) The table needs to be augmented only by the pair $(k^* \mathrm{Id}_V \cdot \rho(A_1), V)$ where V is the representation space of highest weight 3λ, λ a fundamental weight (see (6.5)3).

We subdivide according to the classes of simple groups or simple Lie algebras, and list only the representations for which (G, V) prehomogeneous is possible. From left to right the format includes for each V a brief description, its highest weight λ_ρ (a linear combination of fundamental weights λ_i, $1 \le i \le l$) described when possible as a linear combination of ω_i, $1 \le i \le l$, an orthonormal basis for the Killing form on a Cartan subalgebra of $L(G')$, and $\dim V$. We give a description of a semi-invariant if it exists (see §3). We then include information on $o(G)$, namely G_X and a description of $o(G)$ (§4). We further consider B, a Borel subgroup of G with a description, a statement of its prehomogeneity on V or on $Z(P)$. The last column contains some remarks about G and G_P = (the connected stabilizer of the semi-invariant P in $k[V]$) $= \{g \mid g \in GL(V),\ \lambda_g P = c_g \cdot P$ for some $c_g \in k^*\}$.

If a column is empty, information has not been obtained.

Those examples which were revealed to me by Mikio Sato have been accredited in one of two ways. If the example is found in the Sûgaku no Ayumi lecture note, it is referred to by [16] and its page number. If it is not found there, "(Sato)" appears next to its description in the table.

For the basic representations of F_4 and E_6 we refer to a paper of Chevalley and Schafer [13].

"The exceptional simple Jordan algebra $\mathfrak{J}$ over k is the non-associative algebra of dimension 27 whose elements are 3×3 Hermitian matrices

$$X = \begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix}$$

with elements in the Cayley Algebra $\mathfrak{C}$ of dimension 8 over k, multiplication being defined by $X \circ Y = \frac{1}{2}(XY + YX)$ where XY is the ordinary matric product." The main theorem states: "The exceptional simple Lie algebra F_4 of dimension 52 and rank 4 over k is the derivation algebra $\mathfrak{D}$ of the exceptional Jordan algebra $\mathfrak{J}$ of dimension 27 over k. The exceptional simple Lie algebra E_6 of dimension 78 and rank 6 over k is the Lie algebra $\mathfrak{D} + \{R_Y, \text{Trace } Y = 0\}$ spanned by the derivations of $\mathfrak{J}$ and the right multiplications of elements Y of trace 0." An additional remark is in order. F_4 is the full Lie algebra of endomorphisms of $\mathfrak{J}$

A_l $(l \geq 1)$ $\qquad G' = \rho(A_l)$; $G = k^*\mathrm{Id}_V \cdot G'$; $\dim G = (l+1)^2$

V			X	P	
Description	highest weight λ_ρ	$\dim V$		Description	λ_ρ (cont'd.)
For all l [16, p. 139]: V = usual representation; $\rho(A_l) \cong SL_{l+1}(k)$ with basis elements X_i, $1 \leq i \leq l+1$, $kX_i = V_{\omega_i}$, $kX_{l+1} = V_{-(\omega_1+\cdots+\omega_l)}$.	$\lambda_1 = \omega_1$	$l+1$	X_i	Does not exist. $\rho(A_l)X_1$ is open dense.	λ_1
For all l [16, p. 146]. $V = S(V^{\lambda_1})^2$; $\rho(A_l) = S(A_l)^2$. $V \cong$ vector space of quadratic forms on V^{λ_1*}. $\cong$ vector space of $(l+1)\times(l+1)$ symmetric matrices M with $\rho(g)(M) = gMg^{tr}$ for $g \in SL_{l+1}(k)$.	$2\lambda_1 = 2\omega_1$	$(l+1)(l+2)/2$	$X_1^2 + X_2^2 + \cdots + X_n^2$ I_n	$P(X)$ = discriminant of X. $P(M)$ = determinant of M.	$2\lambda_1$
For all l (Sato): $V = \Lambda^2(V^{\lambda_1})$; $\rho(A_l) = \Lambda^2(A_l)$. $V \cong$ vector space of $(l+1)\times(l+1)$ skew-symmetric matrices M with $\rho(g)(M) = gMg^{tr}$, $g \in SL_{l+1}(k)$.	$\lambda_2 = \omega_1 + \omega_2$	$l(l+1)/2$	l odd: $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$	$P(M)$ = Pfaff (M) (3.5).	λ_2 (l odd)
			l even: $\begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	Does not exist.	λ_2 (l even)
$l = 1$; $V = S(V^{\lambda_1})^3$ = vector space of binary cubic forms on V^{λ_1*}, $\dim G = 4$ (Sato).	$3\lambda_1$	4	$X_1^3 + X_2^3$	P = discriminant	$l = 1$, $3\lambda_1$.
$l = 5$; $V = \Lambda^3(V^{\lambda_1})$; V = vector space elements of degree 3 in the Grassmann algebra on V^{λ_1}; $\dim G = 36$ (Sato).	$\lambda_3 = \omega_1 + \omega_2 + \omega_3$	20	$X_1 \wedge X_2 \wedge X_3 + X_4 \wedge X_5 \wedge X_6$	P exists deg P is even	$l = 5$, $\omega_1 + \omega_2 + \omega_3 = \lambda_3$.
$l = 6$; $V = \Lambda^3(V^{\lambda_1})$; V = elements of degree 3 in the Grassmann algebra on V^{λ_1}; $\dim G = 49$ (Sato).	$\omega_1 + \omega_2 + \omega_3 = \lambda_3$	35	$X_1 \wedge X_2 \wedge X_3 + X_4 \wedge X_5 \wedge X_6 + X_7 \wedge (X_1 \wedge X_4 + X_2 \wedge X_5 + X_6 \wedge X_7)$	P exists	$l = 6$, $\omega_1 + \omega_2 + \omega_3 = \lambda_3$.
$l = 7$; $V = \Lambda^3(V^{\lambda_1})$; $\dim G = 64$ (Sato).	$\omega_1 + \omega_2 + \omega_3 = \lambda_3$	56	$X_1 \wedge X_2 \wedge X_3 + X_4 \wedge X_5 \wedge X_6 + X_7 \wedge (X_1 \wedge X_4 - X_2 \wedge X_5) + X_8 \wedge (X_1 \wedge X_4 - X_3 \wedge X_6)$	P exists	$l = 7$, λ_3.

Table continues on facing page

C_l $(l \geq 3)$ $\qquad G = k^*\mathrm{Id}_V \cdot \rho(C_l)$ $\qquad \dim G = 2l^2 + l + 1$

V			X	P
Description	highest weight λ_ρ	$\dim V$		description
$l \geq 3$; V = natural representation. Basis: X_i = eigenvector for ω_i, $1 \leq i \leq l$. X_{l+i} = eignevector for $-\omega_i$, $1 \leq i \leq l$.	$\lambda_1 = \omega_1$	$2l$	X_1	Does not exist; $\rho(C_l)X_1$ is open dense (3.3).
$l \geq 3$; V = representation given by the vector space of skew-symmetric matrices ($= \Lambda^2(V^{\lambda_1})$) modulo the invariant line $k \cdot \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. $X = \begin{bmatrix} X_1 & X_2 \\ -X_2^t & X_3 \end{bmatrix}$, X_1, X_3 skew $l \times l$ matrices; Trace $(X_2) = 0$ in $M_{l\times l}(k)$.	$\lambda_2 = \omega_1 + \omega_2$	$2l^2 - l - 1$	V is not prehomogeneous, P_1 = Trace $(X_1X_3 - X_2^2)$ is G semi-invariant and P_2 = Pfaff(X) is also G semi-invariant. These are nonassociate (not collinear) (3.3).	
$l = 3$ ($\dim G = 22$); V is a subspace of the exterior algebra $\Lambda^3(V^{\lambda_1})$. $V = \{\sum x_{abc}X_aX_bX_c \mid x_{i14} + x_{i25} + x_{i36} = 0,\ 1 \leq i \leq 6\}$. See [12, Prop. 7, p. 1026].	$\lambda_3 = \omega_1 + \omega_2 + \omega_3$	14	$X_1X_2X_3 - 2[-I]^{1/2} \cdot X_4X_5X_6$	$P(X)$ is of degree 4 $f(X)$ of [12, Prop. 7].

Table continues on facing page

$o(G)$		B a Borel subgroup of G $B = k^*\mathrm{Id}_v \cdot \rho(T_L)$; T_L = lower triangular subgroup of $SL_{l+1}(k)$; $\dim B = (l+2)(l+1)/2$		Finiteness of orbits, G orbit decomposition.	Remarks
G_X	Description of $o(G)$	on V	on $Z(P)$		
$\begin{bmatrix} 1 & k^l \\ 0 & SL_l(k) \end{bmatrix}$	$V - \{0\}$	$BX_1 = U_{Y_1}$ open dense	———	$V = (V - \{0\}) \cup \{0\}$	———
$G_X \cong O(Y_1^2 + \cdots + Y_n^2)$ $\cong B_l$, $l \geq 2$ even; D_l, l odd.	$o(G) = U_P$ = nondegenerate forms	BX is open dense	B acts prehomogeneously on $Z(P)$	Yes $V = U_P \cup \bigcup_{i=0}^{l} \{X \mid \mathrm{rank}\, X = i\}$	$G = G_P$
$G_X \cong SP(l+1)$ $\cong C_{(l+1)/2}$	U_P = skew-symmetric matrices of rank $l+1$	BX is open dense	B acts prehomogeneously on $Z(P)$	$V = U_P \cup \bigcup_{0 \leq i \leq l/2} \{X \mid \mathrm{rank}\, X = 2i\}$	$G = G_P$
$G_X \cong \left[\begin{array}{c\|c} SP(l, k) & k^l \\ \hline 0 & a \end{array}\right]$	$o(G)$ = skew-symmetric matrices of rank l	BX is open dense		$V = \bigcup_{0 \leq i \leq l/2} \{X \mid \mathrm{rank}\, X = 2i\}$	———
finite	$o(G) = U_P$ = forms with no multiple factors	$\dim B = 3$, B does not act prehomogeneously on V.	B acts prehomogeneously on $Z(P)$ (6.5)	Yes, $V = U_P \cup \{X \mid X$ has a double root$\} \cup \{X \mid X$ has a triple root$\} \cup \{0\}$.	$G = G_P$
$G_X \cong SL_3(k) \times SL_3(k)$ $\cong A_2 \times A_2$	$o(G) = U_P$ (4.1)	not known $\dim B = 21$ $\dim V = 20$	not known	not known	What is G_P?
$G_X \cong G_2$	$o(G) = U_P$ (4.1)	$\dim B = 28$ $\dim V = 35$ impossible	impossible	not known	What is G_P?
$G_X \cong SL_3(k)$	$o(G) = U_P$ (4.1)	$\dim B = 36$, $\dim V = 56$ impossible	impossible	not known	What is G_P? Is it $k^*\mathrm{Id}_V \cdot \rho(E_7)$ where $\lambda\rho = \lambda_1$? (See E_7.)

Table continued from facing page

λ_p	$o(G)$		B a Borel subgroup acts prehomogeneously		Finiteness of orbits	Remarks
(cont'd.)	G_X	Description of $o(G)$	on V	on $Z(P)$	G-decomposition	
λ_1	$L(G_X) = \begin{array}{c} \left[\begin{array}{c\|c} A & B \\ \hline 0 \begin{array}{c\|c} 0 & 0 \\ \hline C & -A^{tr} \end{array} \end{array}\right] \begin{array}{l} l \\ 1 \\ l-1 \end{array} \\ 1 \quad l-1 \quad l \end{array}$ B symmetric, $(l \times l)$; C symmetric, $(l-1) \times (l-1)$		Yes $\left[\begin{array}{c\|c} T_L & 0 \\ \hline S_{21} & -T_L^{tr} \end{array}\right] X_1$ -1	———	Yes (6.2).	
λ_2	———	———	———	———	———	———
λ_3	$G_X = SL_3(k)$	$o(G) = U_P$	$\dim B = 13$ not possible	B acts prehomogeneously on $Z(P)$	Yes, the orbits are described in [12, Prop. 7].	

Table continued from facing page

B_l $(l \geq 2)$	$G = k^* \mathrm{Id}_V \cdot \rho(B_l)$		$\dim G = 2l^2 + l + 1$	
V			X	P
Description	Highest weight λ_ρ	$\dim V$		Description
For all $l \geq 2$: V = natural representation with basis elements X_i = eigenvector for ω_i, $1 \leq i \leq l$; X_{l+1+i} = eigenvector for $-\omega_i$, $1 \leq i \leq l$; X_l = eigenvector for 0 weight [16, p. 142].	$\lambda_1 = \omega_1$	$2l + 1$	X_l	$\sum_{i=1}^{l} Y_i Y_{l+1+i} + Y_l^2$

$l = 3, 4, 5, 6$ Spin Representations—not complete

λ_ρ	$o(G)$		B a Borel subgroup of G		Finiteness of orbits	Remark
(cont'd.)	G_X	$o(G)$	on V	on $Z(P)$	G orbit decomposition	
λ_1	$G_X \cong D_l$ $(l \geq 4)$; A_3 $(l = 3)$; $A_1 \times A_1$ $(l = 2)$	U_P = nonisotropic vectors	BX is open dense in V	B acts prehomogeneously on $Z(P)$	$V = U_P \cup (Z(P) - \{0\}) \cup \{0\}$	$G = G_P$

For the Spin and Half-Spin representations we refer to the paper of Jun-Ichi Igusa [12]. In accommodating to Igusa's results, we change our format.

$G = k^* \mathrm{Id}_V \cdot \mathrm{Spin}_{2l+1}$, V = Spin representation of B_l.

l	dim of Spin rep.	dim G	$o(G)$ G_X, X in $o(G)$	$o(G)$	P
3	8	22	G_2	U_P	P is of deg 2 [12, Prop. 4].
4	16	37	Spin_7	U_P	P is of deg 2 [12, Prop. 5].
5	32	56	$SL_5(k)$	U_P	P is of deg 4 [12, Prop. 6].
6	64	79			

For $l = 3, 4, 5$, the dimension of a Borel subgroup B of G is 13, 21, 31, respectively. Hence it is possible that B acts prehomogeneously on $Z(P)$. The orbits are finite in these cases by the results in Propositions 4, 5, 6 of [12]. For $l = 6$, B has dimension 43. Hence neither V nor $Z(P)$ can have a dense B-orbit.

E_6	$G = k^* \mathrm{Id}_V \cdot \rho(E_6)$		$\dim G = 79$						
V			X	P	$o(G)$		B a Borel subgroup	Finiteness of orbits	Remarks
Description	highest weight	$\dim_V$			G_X	$o(G)$	on V on $Z(P)$		
$V = \mathfrak{J}$ the exceptional simple Jordan algebra of 3×3 Hermitian matrices with elements in the Cayley algebra of dimension 8, $\mathfrak{C}$.	λ_1	27	$\begin{bmatrix} l & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{bmatrix}$ $0 \in \mathfrak{C}$ $l \in \mathfrak{C}$	$P(X) = \mathrm{Tr}(X \circ X \circ X)$	$G_X \cong F_4$	$o(G) = U_P$	$\dim B = 43$		$G = G_P$

D_l $(l \geq 4)$ $\dim G = 2l^2 - l + 1$

V			X	P
Description	λ_ρ	$\dim V$		Description
For all $l \geq 4$: V = natural representation with basis elements X_i = eigenvector for ω_i, X_{i+l} = eigenvector for $-\omega_i$ [16, p. 142].	$\lambda_1 = \omega_1$	$2l$	$X_1 + X_{l+1}$	$P = Y_1 Y_{l+1} + \cdots + Y_l Y_{2l}$

For $l = 4, 5, 6, 7$ Half-spin representations—not complete

(cont'd.)	$o(G)$		B a Borel subgroup		Finiteness of orbits	Remark
	G_X	$o(G)$	on V	on $Z(P)$		
λ_1	$G_X \cong B_{l-1}$	$o(G) = U_P$	Yes	Yes	Yes $V = U_P \cup (Z(P) - \{0\}) \cup \{0\}$	$G = G_P$
			See (6.5)2.			

$G = k^* \mathrm{Id}_V \cdot \mathrm{Spin}_{2l}$

l	dim of Half-Spin rep.	$\dim G$	$o(G)$: G_X, X in $o(G)$	$o(G)$	if it exists.
4	8	29	Spin_7	U_P	P is of deg 2 [12, Prop. 1].
5	16	46	$\mathrm{Spin}_7 \cdot (G_a)^8$ semidirect prod.	not reductive [12, Prop. 2]	
6	32	67	$SL_6(k)$	U_P	P is of deg 4 [12, Prop. 3].
7	64	92			

For $l = 4, 5, 6$ the dimension of a Borel subgroup B of G is 17, 26, and 37 respectively. It is possible that B acts prehomogeneously on V in each case. The finiteness of orbits for these cases follows from Propositions 1, 2, and 3, respectively, of Igusa's paper. For $l = 7$, B has dimension 50 and prehomogeneity for B on V or on $Z(P)$, if P exists, is not possible.

G_2 $G = k^* \mathrm{Id}_V \cdot \rho(G_2)$ $\dim G = 15$

V			X	P	$o(G)$	
Description	highest weight	$\dim V$		description	G_X	$o(G)$
If R = 3 dimensional vector algebra with bilinear scalar product and skew-symmetric vector product, V = Trace 0 elements in 8-dim split Cayley algebra over k $\cong \left\{ \begin{bmatrix} \alpha & a \\ b & -\alpha \end{bmatrix} \middle\vert \begin{matrix} \alpha \in k, \\ a, b \in R \end{matrix} \right\}$, see [10, pp. 142–145].	λ_1	7	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = X_0$	Trace (XX)	$G_{X_0} \cong A_2 \cong SL_3(k)$	U_P

$\exists$ a Borel subgroup B such that $L(B) = k\mathrm{Id}_V + \{D_{e_1, a_{12}} \mid a \in R\} + L(B_0)$		Finiteness of orbits G-orbit decomposition	Remarks
on V	on $Z(P)$		
	If $(u, v) = 0$ in R and $L(B_0)u$ has dim 3 and $L(B_0)v$ has dim 2 then $L(B) \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}$ has codim 1.	$V = U_P \cup Z(P)$, $Z(P)$ is a finite union of G orbits.	$G_P = GO(P) \supsetneq G$

leaving the homogeneous polynomials $P_1(X) = \text{Trace } X \circ X$ and $P_2(X) = \text{Trace } X \circ X \circ X$ invariant; [13, p. 139]. $\mathfrak{J}_0$, the subspace of trace 0 elements in $\mathfrak{J}$ above, is the space V for F_4 with highest weight λ_1. From Theorem (3.2), we see that $(k^*\mathrm{Id}_V \cdot \rho(F_4), \mathfrak{J}_0)$ cannot be prehomogeneous.

However from the Chevalley-Schafer theorem we see that $V = \mathfrak{J}$ is the space of the representation ρ of E_6 with the property that if $X = I \in \mathfrak{J}$, then $L(G)I = \mathfrak{J}$ and hence GI is open dense in V. Since $L(E_6)I = \mathfrak{J}_0$, there exists a P by (3.3).

We fix notation as in the table for E_6. On $\mathfrak{J}$ one can define a nondegenerate symmetric bilinear form $(\ ,\ )$: $\mathfrak{J} \times \mathfrak{J} \to k$ by $(X, Y) = \text{Trace } X \circ Y = (D_Y P_1)(X)$. One can also define a bilinear mapping $\square$: $\mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ where $X \square Y$ is defined by the identity $(X \square Y, Z) = D_Y D_X P_2(Z)$ for all Z in $\mathfrak{J}$ [14]. We now consider E_7 (see the table for E_7).

E_7	$G = k^*\mathrm{Id}_V \cdot \rho(E_7)$							$\dim G = 134$
V			X in $o(G)$	P	$o(G)$		B a Borel subgroup	Remarks
Description [14]	Highest weight	$\dim V$		Description	G_X	$o(G)$		
As a vector space, V is isomorphic to $\mathfrak{J} \oplus \mathfrak{J} \oplus k \oplus k$. Let $M, N \in \mathfrak{J}$ and $\sigma, r \in k$. We represent X in V by $X = \begin{bmatrix} M \\ N \\ \sigma \\ r \end{bmatrix}$. Let $\nu \in k$ and let $\Phi \in L(E_6)$ in its natural representation with weight λ_1. The action of $L(E_7)$ on V may be represented as left multiplication by the matrix $\begin{bmatrix} \Phi + 1/3\nu\mathrm{Id}_3 & 2B\square & 0 & A \\ 2A\square & \Phi^* - 1/3\nu\mathrm{Id}_{3^*} & B & 0 \\ 0 & (A, \ ) & -\nu & 0 \\ (B, \ ) & 0 & 0 & \nu \end{bmatrix}$	λ_1	56	$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$	$P(X) = (M \square M, N \square N) - \sigma \det M - r \det N - \frac{1}{4}((M, N) - \sigma r)^2$ of degree 4	$L(G_X) \cong L(E_6)$	$o(G) = U_P$	$\dim B = 71$	$G = G_P$, G has finitely many orbits in V [15].

8. **Conjectures.** One can see that there is no irreducible prehomogeneous pair (G, V) in the list for which P exists but for which $o(G) \neq U_P$, i.e., for which the condition G_X is reductive fails (4.1). Thus it is tempting to conjecture that every irreducible prehomogeneous pair (G, V) which admits a nonconstant semi-invariant P has $o(G) = U_P$. This is false however. Mikio Sato has an example of an irreducible prehomogeneous pair (G, V) for which $o(G) \subsetneq U_P$ [16, p. 140]. It is the pair with $G \cong k^*\mathrm{Id}_V \cdot (\mathrm{Sp}(2n, k) \times O(3, k))$ and $V \cong$ the vector space over k of matrices of size $2n$ by 3. (G', V) is the tensor product of the pairs $(\mathrm{Sp}(2n, k), k^{2n})$ and $(O(3), k^3)$; the action is given by $gX = cAXB^{-1}$ for $c \in k^*$, $A \in \mathrm{Sp}(2n, k)$, $B \in O(3)$. In this case, there exists a P of degree 4. Let $X = (X_1, X_2, X_3)$ with X_i in k^{2n}. Then $P(X) = [X_1, X_2]^2 + [X_2, X_3]^2 + [X_3, X_1]^2$,

where $[X, Y]$ is the skew symmetric nondegenerate bilinear form which $\mathrm{Sp}(2n, k)$ leaves invariant. The point

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

lies in $U_P - o(G)$, $n = 2$. A reasonable question is whether $G = G_P$. Hence we conjecture the following.

1. If an irreducible prehomogeneous pair (G, V) admits a nonconstant semi-invariant P, then $o(G_P) = U_P$.

One can also notice from the list that whenever $G = G_P$ the condition "A Borel subgroup $B \subset G$ acts prehomogeneously on $Z(P)$" is not implausible from the point of view of dimensions. Whenever a B is known, this has been verified. Hence a second conjecture:

2. If (G_P, V) is irreducible prehomogeneous, then a Borel subgroup $B \subset G_P$ acts prehomogeneously on the hypersurface $Z(P)$.

A consequence of 1 and 2 is a conjecture of Mikio Sato.

3. If (G, V) is irreducible prehomogeneous with nonconstant semi-invariant P, then G_P has finitely many orbits in V.

BIBLIOGRAPHY

1. *Séminaire H. Cartan et C. Chevalley*, 8e année: 1955/56. Géométrie algébrique, Secrétariat mathématique, Paris, 1956; P. Cartier, Exposé 13, Prop. 1, pp. 13–30. MR 20 #3871.

2. H. Weyl, *The classical groups*, 2nd ed., Princeton Univ. Press, Princeton, N. J., 1953, p. 250.

3. Armand Borel, "Linear algebraic groups," *Algebraic groups and discontinuous subgroups*, Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, R. I., 1966, pp. 3–19. MR 34 #4371.

4. David Mumford, *Introduction to algebraic geometry*, Preliminary version of first three chapters; Department of Mathematics, Harvard University, Cambridge, Mass.

5. Oscar Zariski and Pierre Samuel, *Commutative algebra*. Vol. I, University Series in Higher Math., Van Nostrand, Princeton N. J., 1958. MR 19, 833.

6. Jacob Eli Goodman, Thesis, Columbia University, New York, 1967.

7. Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) 75 (1962), 485–535. MR 26 #5081.

8, C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. 11, Interscience, New York, 1962, p. 203. [29.15] MR 26 #2519.

9. Sigurdur Helgason, *Differential geometry and symmetric spaces*, Pure and Appl. Math., vol. 12, Academic Press, New York, 1962, p. 110. MR 26 #2986.

10. Nathan Jacobson, *Lie algebras*, Interscience Tracts in Pure and Appl. Math., no. 10, Interscience, New York, 1962, p. 141. MR 26 #1345.

11. E. M. Andreev, È. B. Vinberg and A. G. Èlašvili, *Orbits of greatest dimension in semi-simple linear Lie groups*, Funkcional. Anal. i Priložen. 1 (1967), no. 4, 3–7 = Functional Anal. Appl. 1 (1967), 257–261. MR 42 #1942.

12. Jun-ichi Igusa, *A classification of spinors up to dimension twelve*, Amer. J. Math. 92 (1970), 997–1028. MR 43 #3291.

13. Claude Chevalley and R. D. Schafer, *The exceptional simple Lie algebras F_4 and E_6*, Proc. Nat. Acad. Sci. U.S.A. 36 (1950), 137–141. MR 11, 577.

14. Hans Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene*. I, Nederl. Akad. Wetensch. Proc. Ser. A 57 = Indag. Math. 16 (1954), 218–230. MR 16, 108.

15. Stephen J. Haris, *Some irreducible representations of exceptional algebraic groups*, Amer. J. Math. 93 (1971), 75–106. MR 43 #4829.

16. Mikio Sato, *The theory of pre-homogeneous vector spaces*, Sûgaku 15–1 (1970), 85–157; notes by T. Aratami, published by Association for Sûgaku no Ayumi, c/o S.S.S., Dr. Y. Morita, Department of Mathematics, Faculty of Science, Univ. of Tokyo, Tokyo, Japan

DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA