TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 176, February 1973

ON THE ASYMPTOTIC REPRESENTATION OF ANALYTIC SOLUTIONS OF FIRST-ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS IN SECTORS

BY

STEVEN BANK

ABSTRACT. In this paper, we treat first-order algebraic differential equations whose coefficients belong to a certain type of function field. (Our results include as a special case, the case when the coefficients are rational functions.) In our main result, we obtain precise asymptotic representations for a broad class of solutions of such equations.

1. **Introduction.** We treat first-order differential equations, $\Omega(z, y, dy/dz) = 0$, where Ω is a polynomial in y and dy/dz, whose coefficients belong to a certain type of field of meromorphic functions which was introduced and investigated by W. Strodt in [8]. Such a field consists of functions, each defined and analytic in a sectorial region approximately of the form,

$$a < \arg(z - te^{i(a+b)/2}) < b \tag{1}$$

(for fixed a and b in $(-\pi, \pi)$ and some $t \geq 0$), and has the property that there is a fixed nonnegative integer p (called the *rank* of the field) such that the field contains all logarithmic monomials of rank $\leq p$ (i.e. all functions of the form,

$$M(z) = Kz^{r_0}(\log z)^{r_1}(\log\log z)^{r_2}\cdots(\log_p z)^{r_p} \tag{2}$$

for real r_j and complex $K \neq 0$), and in addition, for every element f in the field except zero, there is a logarithmic monomial M of rank $\leq p$ which is asymptotically equivalent to f as $z \to \infty$ over a filter base (denoted $F(a, b)$) which consists essentially of the sectors (1) as $t \to +\infty$. (We are using here the slightly stronger concepts of "asymptotically equivalent" ($\sim$), and "smaller rate of growth" ($<<$), which were introduced by Strodt in his paper [6, §§13 and 17], where existence theorems for solutions of such equations were proved. For the reader's convenience, these concepts (as well as the precise definition of $F(a, b)$) are reviewed in §2 below.) The set of all rational combinations of logarithmic monomials of rank $\leq p$ is the simplest example of such a field, and every such field (even in the case $p = 0$) contains the field of rational functions.

Presented to the Society, April 10, 1972; received by the editors February 1, 1972.
AMS (MOS) subject classifications (1970). Primary 34A20, 34D05, 34E05.
Key words and phrases. Algebraic differential equations, analytic solutions, representation of solutions, asymptotic behavior of solutions.

(Hence our results include, as a special case, the case where Ω has polynomial coefficients.)

In [3], [6], [9], first-order equations, $\Omega(z, y, y') = 0$, whose coefficients belong to such fields of arbitrary rank p, were treated and existence theorems were proved for solutions which are themselves asymptotically equivalent to logarithmic monomials over $F(a, b)$. (We mention here, in particular, the very powerful existence theorems in [9, pp. 221–225].) In [2, p. 132], existence theorems were proved for solutions of $\Omega = 0$ (again for arbitrary rank p), which are of larger rate of growth than all logarithmic monomials over $F(a, b)$, and for solutions which are of smaller rate of growth than all logarithmic monomials. (Each of these solutions was of the form $\exp \int W$, for some function W which is asymptotically equivalent to a logarithmic monomial of rank $\leq p$.)

The converse problem of determining the form of arbitrary solutions of such first-order equations was begun in [1], where it was shown (again for arbitrary rank p) that any meromorphic solution which is of larger rate of growth than a predetermined power of z must be of the form $\exp \int M(z)(1 + \epsilon(z))$, where M is a logarithmic monomial of rank $\leq p$, and where the analytic function $\epsilon(z) \to 0$ over $F(a, b)$. (As indicated in [1, p. 303], there is an algorithm which can be applied at the outset to produce a finite set of monomials, among which are the possible monomials M mentioned above (see also §4 below).) A corresponding result for nonidentically zero analytic solutions which are of smaller rate of growth than a predetermined power of z was also proved in [1] and it was shown that such solutions are also of the form $\exp \int M(z)(1 + \epsilon(z))$, where M is a monomial of rank $\leq p$, and $\epsilon(z) \to 0$ over $F(a, b)$. Thus the results of [1] provide an asymptotic representation for "large" and "small" solutions. In this paper, we treat solutions of "intermediate" growth, in the case when the coefficients of Ω belong to a field of rank zero. (This restriction will later be shown to be necessary, but as indicated previously, our results still include as a special case, the case of polynomial coefficients.) More specifically, we treat all solutions $y_0(z)$ which are defined and meromorphic in a sector of the form (1), and which are "comparable" with all logarithmic monomials of the form, $M(z) = z^{\alpha_0}(\log z)^{\alpha_1}$ (in the sense that for any such M, one of the relations $y_0 \ll M$, $y_0 \gg M$ or $y_0 \sim cM$ for some constant $c \neq 0$ is valid over $F(a, b)$). Our main result states that any such solution, which is not of larger rate of growth than all powers of z, and is not of smaller rate of growth than all powers of z, must be asymptotically equivalent over $F(a, b)$ to a logarithmic monomial of the form $Kz^{\alpha}(\log z)^{\beta}$. (As above, an algorithm can be applied to Ω at the outset to produce the finite set of possible pairs of exponents (α, β), and this algorithm shows that β must be rational (see §4 below).) Thus the combination of this result with those in [1] mentioned above provides a representation theorem for *all* solutions which are comparable with all monomials $z^{\alpha_0}(\log z)^{\alpha_1}$ over $F(a, b)$, in the case of first-order equations with coefficients in fields of rank zero. This result is stated in §3 below.

We then consider the case of first-order equations whose coefficients belong to fields of rank higher than zero. In this case, we show (§5) that our main result no longer holds by constructing a very simple equation (for rank one) which has a solution y_0 such that over $F(-\pi, \pi)$, we have $y_0 << z^\epsilon$ for all $\epsilon > 0$, while $y_0 >> (\log z)^N$ for all real N. Such a solution is comparable with all monomials (of arbitrary rank), but it is clearly not of larger rate of growth than all powers of z, nor is it of smaller rate of growth than all powers of z, and it is easy to see that the solution is not asymptotically equivalent to any logarithmic monomial. In a future paper, the author hopes to investigate the appearance of such solutions (as well as solutions in the case of rank zero which violate the comparability condition in every $F(a, b)$, such as the principal branch of z^i (where $i^2 = -1$), which is not comparable with $M(z) \equiv 1$).

In §6 is an appendix containing four lemmas which are needed several times in the proof of the main result. They are put at the end of the paper to avoid unduly interrupting the main line of thought.

2. **Preliminaries.** (a) [6, §94]. Let $-\pi \leq a < b \leq \pi$. For each nonnegative real-valued function ψ on $(0, (b-a)/2)$, let $T(\psi)$ be the union (over $\delta \in (0, (b-a)/2)$) of all sectors

$$a + \delta < \arg(z - \psi(\delta)\exp(i(a+b)/2)) < b - \delta. \tag{3}$$

The set of all $T(\psi)$ (for all choices of ψ) is denoted $F(a, b)$ and is a filter base which converges to ∞ by [6, §95]. Each $T(\psi)$ is simply-connected by [6, §93]. If $W(z)$ is analytic in $T(\psi)$, then the symbol $\int W$ will stand for a primitive of W in $T(\psi)$.

(b) Unless otherwise indicated, $\log z$ will denote the principal branch of the logarithm in S: $|\arg z| < \pi$. It is then easy to see that $\log(\log z)$ is defined and analytic for those points in S where $|z| > 1$. We denote this function by $\log_2 z$. By induction, the function $\log_{q+1} z = \log(\log_q z)$ is defined and analytic for those points in S where $|z| > e_q(0)$ (where $e_q(z)$ is the qth iterate of the exponential function). A logarithmic monomial of rank $\leq p$ is a function of the form (2) (which, of course, is defined to be $K(\exp \sum_{j=0}^{p} r_j \log_{j+1} z)$). Clearly, for any logarithmic monomial, there is an element of $F(a, b)$ on which it is defined and analytic.

(c) [6, §11]. If f is meromorphic in an element of $F(a, b)$ and λ is a complex number, then $f \to \lambda$ over $F(a, b)$ means that for any $\epsilon > 0$, there is an element of $F(a, b)$ on which $|f(z) - \lambda| < \epsilon$. Similarly, $f \to \infty$ over $F(a, b)$ means that for any $N > 0$, there is an element of $F(a, b)$ on which $|f(z)| > N$. If f

and g are meromorphic in an element of $F(a, b)$, with $g \not\equiv 0$, we will occasionally use the notation $f = o(g)$ over $F(a, b)$ to mean $f/g \to 0$. Similarly, $f = O(g)$ over $F(a, b)$ will mean that f/g is bounded on some element of $F(a, b)$. From Cauchy's formula for derivatives, it follows [5, p. 309] that if $f \to 0$ over $F(a, b)$, then $zf'(z) \to 0$ over $F(a, b)$.

(d) [6, §§13 and 17]. If f is analytic in an element of $F(a, b)$ then $f << 1$ over $F(a, b)$ means that, in addition to $f \to 0$, for all positive integers j and k we have $\theta_j^k f \to 0$ where $\theta_j f = (z \log z \cdots \log_{j-1} z)f'$ and where θ_j^k if the kth iterate of the operator θ_j. Then $f << g$, $f >> g$, $f \sim g$ and $f \approx g$ mean respectively, $f/g << 1$, $g/f << 1$, $f - g << g$ and finally $f \sim cg$ for some constant $c \neq 0$. The crucial property [6, §28] of the relation "<<" is that if $f << 1$ over $F(a, b)$, then $\theta_j f << 1$ over $F(a, b)$ for all $j > 0$. From this it follows easily that if $f << M$, where M is a nonconstant logarithmic monomial of any rank, then $f' << M'$. A function f is said to be *comparable* with a function g over $F(a, b)$ if one of the relations $f << g$, $f \approx g$, $f >> g$ is valid over $F(a, b)$. Finally, if $f \sim M$ where M is the monomial (2), then $\delta_0(f)$ will denote r_0.

(e) [8, p. 244]. A *logarithmic field* of rank p over $F(a, b)$ is a set L of functions, each defined and meromorphic in an element of $F(a, b)$, with the following properties: (i) L is a field (where, as usual, we identify two elements of L if they agree on an element of $F(a, b)$; (ii) L contains all logarithmic monomials of rank $\leq p$; and (iii) for every element f in L except zero, there exists a logarithmic monomial M of rank $\leq p$ such that $f \sim M$ over $F(a, b)$.

3. We now state our main result which deals with logarithmic fields of rank *zero.*

Theorem. *Let* $\Omega(z, y, y') = \sum_{i,j \geq 0} f_{ij}(z) y^i (y')^j$ *be a polynomial in* y *and* y', *whose coefficients* f_{ij} *belong to a logarithmic field of rank zero over* $F(a, b)$, *and let some coefficient be not identically zero. Let* $y_0(z)$ *be a function which is defined and meromorphic in an element of* $F(a, b)$, *and which satisfies* $\Omega(z, y_0(z), y_0'(z)) \equiv 0$. *Then if* y_0 *is comparable with all logarithmic monomials of the form* $z^{r0}(\log z)^{r1}$ *over* $F(a, b)$, *then one of the following three conclusions must hold.*

(A) *For every real* α, *we have* $y_0 >> z^\alpha$ *over* $F(a, b)$, *and there exist in an element of* $F(a, b)$, *an analytic function* $W(z)$ *and a logarithmic monomial* cz^λ, *such that* $W(z) = cz^\lambda(1 + o(1))$ *over* $F(a, b)$ *and* $y_0 = \exp \int W$.

(B) *For every real* α, *we have* $y_0 << z^\alpha$ *over* $F(a, b)$, *and if* $y_0 \not\equiv 0$, *then there exist in an element of* $F(a, b)$, *an analytic function* $V(z)$ *amd a logarithmic monomial* dz^σ *such that* $V(z) = dz^\sigma(1 + o(1))$ *over* $F(a, b)$ *and* $y_0 = \exp \int V$.

(C) *There exist real numbers* α_0 *and* β, *and a complex number* $c \neq 0$ *such* *that* $y_0 \sim cz^{\alpha_0}(\log z)^{\beta}$ *over* $F(a, b)$.

Proof. (*Note.* In this proof, all asymptotic relations defined in §2 (c), (d) will take place over the given $F(a, b)$, unless otherwise indicated.) We begin by letting A be the set of all real α for which $y_0 << z^{\alpha}$. We distinguish three possibilities. If A is empty, then by comparability, we must have $y_0 >> z^{\alpha}$ for all α. (Note that if $y_0 \approx z^{\alpha}$, then $y_0 << z^{\alpha+1}$ so that $\alpha + 1$ would belong to A.) In this case, the remainder of conclusion (A) follows from [1, §3(a)]. The second possibility for the set A is that it is nonempty but unbounded from below. In this case it follows that $y_0 << z^{\alpha}$ for all real α, and the remainder of conclusion (B) follows from [1, §3(b)]. The last possibility is that A is nonempty and bounded from below. (In this case, we will prove conclusion (C) holds.) Let α_0 be the infimum of A. Then clearly for any $\epsilon > 0$, $y_0 << z^{\alpha_0+\epsilon}$. Furthermore, $\alpha_0 - \epsilon$ cannot belong to A, so by comparability, either $y_0 \approx z^{\alpha_0-\epsilon}$ or $y_0 >> z^{\alpha_0-\epsilon}$. However the first possibility would lead to $\alpha_0 - \epsilon/2$ belonging to A contradicting the definition of α_0. Thus for any $\epsilon > 0$, $z^{\alpha_0-\epsilon} << y_0 << z^{\alpha_0+\epsilon}$. Without loss of generality, we can assume $\alpha_0 = 0$ in this inequality, because we can replace y_0 by $w_0 = z^{-\alpha_0}y_0$ (which clearly also satisfies the comparability assumption), and replace $\Omega = 0$ by the equation

$$\sum f_{ij}(z)(z^{\alpha_0}w)^i(z^{\alpha_0}w' + \alpha_0 z^{\alpha_0-1}w)^j = 0,$$

which clearly has w_0 for a solution and has coefficients in the same field as Ω does. (Obviously, if conclusion (C) holds for w_0, say $w_0 \sim M$, then $y_0 \sim z^{\alpha_0}M$ so conclusion (C) holds for y_0.) Hence we may assume that

$$z^{-\epsilon} << y_0 << z^{\epsilon} \quad \text{for all } \epsilon > 0. \tag{4}$$

In view of (4) and the fact that $F(a, b)$ is a filter base which converges to ∞, clearly there exists an element S_0 in $F(a, b)$ with the following properties:

$$y_0 \text{ is analytic and nowhere zero on } S_0, \text{ and} \tag{5}$$

$$|z| > e \quad \text{for all } z \text{ in } S_0. \tag{6}$$

In view of (5) and the fact that S_0 is simply-connected (by [6, §93]), there is an analytic branch $L(z)$ of $\log y_0$ in S_0. We will fix this branch, and for any real α, we will denote by y_0^{α}, the branch of y_0^{α} determined by $L(z)$. That is,

$$y_0^{\alpha} = \exp(\alpha L) \quad \text{on } S_0. \tag{7}$$

(From this point on, all elements of $F(a, b)$ which appear in the proof will be assumed to be contained in S_0.)

Now since y_0 solves $\Omega = 0$, we have

$$\sum f_{ij}(z)(y_0(z))^i(y_0'(z))^j \equiv 0 \quad \text{on } S_0. \tag{8}$$

Let J_1 be the set of (i, j) for which $f_{ij} \not\equiv 0$. Since the f_{ij} belong to a logarithmic field of rank zero, it easily follows that for (i, j) in J_1, we have

$$f_{ij} = c_{ij} z^{a_{ij}}(1 + E_{ij}), \tag{9}$$

where c_{ij} is a nonzero constant, a_{ij} is real, and where E_{ij} also belongs to the field and is $\ll 1$. Thus clearly, either $E_{ij} \equiv 0$ or $\delta_0(E_{ij}) < 0$. Let d be the maximum of the numbers $a_{ij} - j$ for (i, j) in J_1, and let J be the subset of J_1 consisting of those (i, j) for which $a_{ij} - j = d$.

Now for any $\epsilon > 0$, $y_0 \ll z^\epsilon$ by (4). Hence for any nonnegative integer i, $y_0^i = O(z^{i\epsilon})$. By §2 (d), $y_0' \ll z^{\epsilon - 1}$, so $(y_0')_j = O(z^{\epsilon j - j})$. Hence with (9) we obtain

$$f_{ij} y_0^i (y_0')^j = O(z^{a_{ij} - j + (i+j)\epsilon}) \quad \text{for any } \epsilon > 0. \tag{10}$$

If $(i, j) \notin J$, then $\epsilon_{ij} = d - (a_{ij} - j) > 0$, and so

$$f_{ij} y_0^i (y_0')^j = O(z^{d - \epsilon_{ij} + (i+j)\epsilon}) \quad \text{for any } \epsilon > 0. \tag{11}$$

If $(i, j) \in J$, then either $E_{ij} \equiv 0$ or $r_{ij} = -\delta_0(E_{ij}) > 0$. Thus since $E_{ij} = O(z^{-r_{ij}})$, we have

$$c_{ij} z^{a_{ij}} E_{ij} y_0^i (y_0')^j = O(z^{d - r_{ij} + (i+j)\epsilon}) \quad \text{for any } \epsilon > 0. \tag{12}$$

Letting ϵ_1 be a positive number which is less than all the numbers $\epsilon_{ij}/2$ (in (11)) and $r_{ij}/2$ (in (12)), we can choose the arbitrary number $\epsilon > 0$ so small that the right sides of (11) and (12) are all $O(z^{d - \epsilon_1})$. In view of (8), (9), (11) and (12), we clearly have

$$\sum_{(i,j)\in J} c_{ij} z^{a_{ij}} y_0^i (y_0')^j = O(z^{d - \epsilon_1}). \tag{13}$$

Since $a_{ij} = j + d$ for (i, j) in J, we thus have

$$\sum_{(i,j)\in J} c_{ij} y_0^i (z y_0')^j = O(z^{-\epsilon_1}). \tag{14}$$

At this point we pause to prove a lemma.

Lemma A. *Let $\Omega(z, y, y')$ be a polynomial in y and y' whose coefficients belong to a logarithmic field of rank zero. Let y_0 be a solution of $\Omega = 0$ which satisfies $z^{-\epsilon} \ll y_0 \ll z^\epsilon$ over $F(a, b)$ for all $\epsilon > 0$. Then it is impossible that $y_0 \gg (\log z)^N$ for all $N > 0$.*

Proof. Let $\Omega(z, y, y') = \Sigma f_{ij} y^i (y')^j$, so that (8) holds. Let J_1, J and ϵ_1 be as in (9)–(14) so that 14 holds. (We point out here that (14) was derived only under the assumption $z^{-\epsilon} \ll y_0 \ll z^{\epsilon}$, and did not require the comparability hypothesis of the main result. This lemma does not require the comparability hypothesis.) Let $n = \max\{i + j: (i, j) \in J\}$, and let $J' = \{(i, j) | (i, j) \in J$ and $i + j = n\}$. Let $m = \max\{j: (n - j, j) \in J'\}$. We now isolate the terms of degree n in (14) and divide through by y_0^n. We note that $O(z^{-\epsilon_1})/y_0^n = O(z^{-\epsilon_1/2})$ since $1/y_0^n = O(z^{\epsilon n})$ for all $\epsilon > 0$, and we can choose ϵ so that $\epsilon n < \epsilon_1/2$. Thus we obtain

$$(15) \qquad \sum_{(n-j,j)\in J'} \left(\frac{zy_0'}{y_0}\right)^j = - \sum_{i+j<n} c_{ij} \frac{(zy_0')^j}{y_0^{n-i}} + O(z^{-\epsilon_1/2}).$$

We now assume the lemma is false, i.e. that

$$(16) \qquad y_0 \gg (\log z)^N \quad \text{for all } N > 0.$$

Thus for any $\epsilon > 0$, the function $h_{\epsilon N} = (\log z)^{\epsilon N}/y_0^{\epsilon}$ (where y_0^{ϵ} is as in (7)) tends to 0. Thus $zh'_{\epsilon N} \to 0$ over $F(a, b)$ (see §2 (c)). But $zh'_{\epsilon N}$ is equal to $\epsilon N(\log z)^{-1} h_{\epsilon N} - \epsilon(\log z)^{\epsilon N}(zy_0'/y_0^{1+\epsilon})$, and hence since ϵ and N are arbitrary > 0, we see that for any $\epsilon > 0$ and any $M > 0$, we have

$$(17) \qquad zy_0'/y_0^{1+\epsilon} = o((\log z)^{-M}).$$

Let $\theta = 2m + 1$. We now examine each term in the sum on the right of (15). If $j = 0$, then since $n - i > 0$, it follows from (16) that the term is $o((\log z)^{-\theta})$. If $j > 0$ then since the term is $c_{ij}(zy_0'/y_0^{1+\epsilon})^j$ where $\epsilon = (n - (i + j))/j > 0$, we have from (17) that it is $o((\log z)^{-\theta})$. Since $z^{-\epsilon_1/2}$ is also $o((\log z)^{-\theta})$, we may write (15) as

$$(18) \qquad G(w_0(z)) = o((\log z)^{-\theta}),$$

where

$$(19) \qquad G(w) = \sum_{(n-j,j)\in J'} w_j \quad \text{and} \quad w_0 = \frac{zy_0'}{y_0}.$$

Since the $c_{n-j,j}$ are nonzero constants, clearly (18) shows that $m > 0$. Letting $\lambda_1, \dots, \lambda_q$ be the distinct roots of G with λ_j of multiplicity m_j, it follows from (18) and Appendix Lemma 1 that for some t, $1 \le t \le q$, we have

$$(20) \qquad w_0(z) - \lambda_t = O((\log z)^{-\theta/m_t}) \quad \text{over } F(a, b).$$

Letting $V(z) = L(z) - \lambda_t \log z$ (where L is the fixed branch of $\log y_0$ in (7)), and noting that $\theta/m_t > 2$, we have from (20) that $V' = O(z^{-1}(\log z)^{-2})$. By Appendix Lemma 2, there is an element S in $F(a, b)$ and a constant $K_1 > 0$, such that if

z_0 is an element of S with $\arg z_0 = (a + b)/2$, then

(21) $$|V(Rz_0) - V(z_0)| \leq K_1(\log|z_0|)^{-1} \quad \text{for all } R \geq 1.$$

We fix a point z_0, and hence from (21), $|V(Rz_0)|$ is bounded as a function of R on $[1, +\infty)$. Thus $\exp V(Rz_0)$ is also bounded, so there is a constant $K_2 > 0$ (independent of R) such that

(22) $$|\exp V(Rz_0)| \leq K_2 \quad \text{for all } R \geq 1.$$

But clearly, $\exp V = y_0 \exp(-\lambda_t \log z)$. Letting $\lambda_t = c + id$ (c, d real), and noting that $\arg Rz_0 = (a + b)/2$, it follows from (22) that for all $R \geq 1$,

(23) $$|y_0(Rz_0)| \leq K_2 R^c |z_0|^c \exp(-d(a + b)/2).$$

But by (16), clearly $y_0 >> 1$ and it easily follows that $\lim_{R \to +\infty} |y_0(Rz_0)| = +\infty$. Hence from (23), clearly

(24) $$c > 0.$$

But if we set $U(z) = -V(z)$, then from (21), we see that $U(Rz_0)$ is also bounded for all $R \geq 1$ and hence so is $\exp U(Rz_0)$. If $K_3 > 0$ is such a bound, then noting that $\exp U = (1/y_0) \exp(\lambda_t \log z)$, we see that for all $R \geq 1$,

(25) $$|y_0(Rz_0)| \geq R^c(|z_0|^c / K_3)\exp(-d(a + b)/2).$$

But by (24), $c > 0$ and hence by hypothesis, $y_0 << z^{c/2}$. Thus there is an element T in $F(a, b)$ on which $|y_0(z)| < |z|^{c/2}$. From the definition of $F(a, b)$, such an element T contains the points Rz_0 for all sufficiently large R and hence $|y_0(Rz_0)| < R^{c/2}|z_0|^{c/2}$ for all sufficiently large R. This clearly contradicts (25) since $c > 0$. This contradiction establishes Lemma A.

Lemma B. *Under the same hypothesis on Ω and y_0 as in Lemma A, it is impossible that $y_0 << (\log z)^{-N}$ for all $N > 0$.*

Proof. Since $z^{-\epsilon} << y_0 << z^{\epsilon}$ for all $\epsilon > 0$, the same property holds for $u_0 = 1/y_0$. Clearly u_0 is a solution of

(26) $$\sum f_{ij}(z)(-1)^j u^{\sigma - (i+2j)}(u')^j = 0,$$

where $\sigma = \max\{i + 2j \mid f_{ij} \not\equiv 0\}$, and this equation has coefficients in the same field as Ω does. By Lemma A, it is impossible that $u_0 >> (\log z)^N$ for all $N > 0$ and this proves Lemma B.

Continuing with the proof of our main result, we let B be the set of all real α such that $y_0 << (\log z)^{\alpha}$. It follows from Lemma A and the comparability of y_0 with all $(\log z)^{\alpha}$, that B cannot be empty. Similarly, from Lemma B it follows that B must be bounded from below. If we let β be the infimum of B, then from comparability, it follows that

$$(27) \qquad (\log z)^{\beta-\epsilon} << y_0 << (\log z)^{\beta+\epsilon} \quad \text{for all } \epsilon > 0.$$

Our aim is to prove conclusion (C) for y_0. For reasons that will become evident as the proof proceeds (e.g. see the proof of (30) below), the proof will be by contradiction. Thus we assume the contrary, that is, for every logarithmic monomial $M = cz^{r_0}(\log z)^{r_1}$,

$$(28) \qquad \text{the relation } y_0 \sim M \text{ is false.}$$

It easily follows from this assumption that for every M of the above form,

$$(29) \qquad y_0/M \to 0 \quad \text{or} \quad \infty \text{ over } F(a, b).$$

(For if y_0/M fails to $\to 0$ or ∞, the relations $y_0 << M$ and $y_0 >> M$ are both false, and hence by comparability, $y_0 \approx M$ which would contradict (28) for some multiple of M.)

We now proceed under assumption (28) to eventually obtain a contradiction.

We first prove that there is an element S_1 in $F(a, b)$, contained in S_0 (see (5)) such that

$$(30) \qquad y_0' \text{ is nowhere zero on } S_1.$$

To prove (30), let $t = \min\{j: f_{ij} \not\equiv 0 \text{ for some } i\}$. Then equation (8) takes the form

$$(31) \qquad (y_0')^t \sum_{j \geq t} f_{ij}(z) y_0^i (y_0')^{j-t} \equiv 0 \quad \text{on } S_0.$$

It follows that we must have

$$(32) \qquad \sum_{j \geq t} f_{ij}(z) y_0^i (y_0')^{j-t} \equiv 0 \quad \text{on } S_0.$$

(If $t = 0$, this is clear. If $t > 0$, then by the analyticity of both factors in (31), failure of (32) would lead to $(y_0')^t \equiv 0$. Thus y_0 would be a constant say c. If $c = 0$, this violates (4), while if $c \not\equiv 0$, it violates (28).) Let θ and σ be respectively the minimum and maximum of the set $\{i \mid f_{it} \not\equiv 0\}$, and consider the algebraic polynomial,

$$(33) \qquad H(z, v) = \sum_{i=\theta}^{\sigma} f_{it}(z) v^i.$$

Since $f_{\sigma t} \not\equiv 0$ and belongs to a logarithmic field, clearly there is an element T_1 of $F(a, b)$, $T_1 \subset S_0$, such that

$$(34) \qquad f_{\sigma t} \text{ is nowhere zero on } T_1.$$

If $\sigma = 0$, then y_0' cannot have any zeros on T_1, for if z_0 was such a zero, then by (32) we would have $f_{\sigma t}(z_0) = 0$ contradicting (34). Thus if $\sigma = 0$, (30) holds.

If $\sigma > 0$, then H is an algebraic polynomial of positive degree σ with coefficients in a logarithmic field of rank zero. It follows from [8, Theorem II, p. 244] (by applying this result to, in the terminology of [8, p. 246], the logarithmic quadruple $(F, E_0^*(F), R, S_0)$, where $F = F(a, b)$ and R is the set of real numbers), that there exists a logarithmic field of rank zero over $F(a, b)$ in which $H(z, v)$ factors completely. Hence there exist distinct functions $B_1, \cdots, B_q$, each defined and analytic in an element T_2 of $F(a, b)$, such that each B_j is $\sim$ to a logarithmic monomial of rank zero, and there exist positive integers $m_1, \cdots, m_q$ with $\Sigma m_j = \sigma - \theta$ such that

$$H(z, v) = f_{\sigma t}(z) v^{\theta} (v - B_1(z))^{m_1} \cdots (v - B_q(z))^{m_q}, \tag{35}$$

for all meromorphic functions $v = v(z)$ defined on T_2. By assumption (29), for each j, $y_0/B_j \to 0$ or ∞ and hence $1 - (B_j/y_0) \to \infty$ or 1 over $F(a, b)$. Hence there exists $T_3 \subset T_2$ in $F(a, b)$ such that

$$|1 - (B_j(z)/y_0(z))| \geq 1/2 \quad \text{on } T_3, \text{ for } j = 1, \cdots, q. \tag{36}$$

Now let S_1 be an element of $F(a, b)$ contained in T_1 and T_3. This S_1 will satisfy (30), for if z_0 is a point in S_1 at which $y_0'(z_0) = 0$, then by (32) and (33), we would have $H(z_0, y_0(z_0)) = 0$. Since $y_0(z_0) \neq 0$ (by (5)), we would have $H(z_0, y_0(z_0))/(y_0(z_0))^{\sigma} = 0$. But in view of (35), the left side of this expression is

$$f_{\sigma t}(z_0)(1 - (B_1(z_0)/y_0(z_0)))^{m_1} \cdots (1 - (B_q(z_0)/y_0(z_0)))^{m_q},$$

which is not zero by (34) and (36). This contradiction shows that S_1 does satisfy (30).

Since S_1 is simply-connected and $|z| > e$ for all z in S_1 (by (6)), it follows from (30) that there is an analytic branch $L_1(z)$ of $\log(zy_0')$ on S_1. We will fix this branch, and for any real α, we will denote by $(zy_0')^{\alpha}$, the branch determined by $L_1(z)$, that is

$$(zy_0')^{\alpha} = \exp(\alpha L_1) \quad \text{on } S_1. \tag{37}$$

(As before, all elements of $F(a, b)$ which will arise as the proof proceeds will be assumed to be contained in S_1.)

We now return to the inequality (14). We set

$$\gamma = \max\{\beta(i + j) - j : (i, j) \in J\} \quad \text{(where } \beta \text{ is as in (27))}, \tag{38}$$

and

$$\Delta = \{(i, j) : (i, j) \in J \text{ and } \beta(i + j) - j = \gamma\}. \tag{39}$$

Now for any $\epsilon > 0$, we have from (27), that for $i \geq 0$, $y_0^i = O((\log z)^{i(\beta+\epsilon)})$. Since $y_0 << (\log z)^{\beta+\epsilon}$, we have $zy_0' << (\log z)^{\beta+\epsilon-1}$ (see §2 (d)). Thus for $j \geq 0$, we have $(zy_0')^j = O((\log z)^{j(\beta+\epsilon-1)})$. Hence,

$$y_0^i(zy_0')^j = O((\log z)^{\beta(i+j)-j+\epsilon(i+j)}) \quad \text{for any } \epsilon > 0. \tag{40}$$

Now if $(i, j) \in J - \Delta$, then $\epsilon_{ij} = \gamma - (\beta(i + j) - j) > 0$. Choosing a positive number ϵ_2 so that $\epsilon_2 < \epsilon_{ij}/2$ for all $(i, j) \notin \Delta$, we can clearly choose the arbitrary ϵ in (40) so small that the right side of (40) is $O((\log z)^{\gamma-\epsilon_2})$ if $(i, j) \notin \Delta$. In view of (14) (and the fact that $z^{-\epsilon_1} = O((\log z)^{\gamma-\epsilon_2})$ since $\epsilon_1 > 0$), we thus have

$$\sum_{(i,j)\in\Delta} c_{ij}y_0^i(zy_0')^j = O((\log z)^{\gamma-\epsilon_2}). \tag{41}$$

Now if (i, j) and (i_1, j_1) are distinct elements of Δ, then $i + j \neq i_1 + j_1$ (for otherwise by (39), $j = j_1$ and thus also $i = i_1$). Hence if we let (i_1, j_1), (i_2, j_2), $\cdots$, (i_n, j_n) be the distinct elements of Δ, then the numbers $q_k = i_k + j_k$ are all distinct, and we may assume (by renumbering, if necessary) that they are so arranged that

$$q_1 < q_2 < \cdots < q_n. \tag{42}$$

Letting c_k denote c_{i_k, j_k}, we may rewrite (41) as

$$\sum_{k=1}^{n} c_k y_0^{q_k - j_k}(zy_0')^{j_k} = O((\log z)^{\gamma-\epsilon_2}). \tag{43}$$

(Note that the c_k are nonzero constants by (9).)

At this point, we observe that

(44) If Δ has only one element (i.e. $n = 1$), then $j_1 > 0$.

To see this, we note that if $n = 1$ but $j_1 = 0$, then (43) shows $c_1 y_0^{q_1} = O((\log z)^{\gamma-\epsilon_2})$, and by (39), $\gamma = \beta q_1$. In view of (27), we would thus have $c_1 = O((\log z)^{\epsilon q_1 - \epsilon_2})$ for any $\epsilon > 0$. Choosing ϵ so small that $\epsilon q_1 - \epsilon_2 < 0$, this would contradict the fact that c_1 is a nonzero constant. Thus (44) holds.

Returning to (43), we have (by (39)) that $j_k = \beta q_k - \gamma$ for each k. Thus by our convention (7), we clearly have $y_0^{q_k - j_k} = y_0^\gamma y_0^{q_k(1-\beta)}$. Since q_k is a non-negative integer, we further have $y_0^{q_k(1-\beta)} = (y_0^{1-\beta})^{q_k}$. Similarly (using (37)), we have $(zy_0')^{j_k} = (zy_0')^{-\gamma}((zy_0')^\beta)^{q_k}$. Again observing that q_k is an integer, we can therefore write (43) as

$$y_0^\gamma(zy_0')^{-\gamma}E(v_0) = O((\log z)^{\gamma-\epsilon_2}), \tag{45}$$

where

$$v_0 = y_0^{1-\beta}(zy_0')^\beta \quad \text{and} \quad E(v) = \sum_{k=1}^{n} c_k v^{q_k}. \tag{46}$$

Now by (7) and (37), $v_0 = \exp((1-\beta)L(z) + \beta L_1(z))$ and hence v_0 is analytic and nowhere zero in S_1. Clearly $v_0 = y_0$ if $\beta = 0$. If $\beta \neq 0$, set

$$g(z) = ((1/\beta) - 1)L(z) + L_1(z), \tag{47}$$

so that

$$v_0 = \exp(\beta g(z)). \tag{48}$$

Now set

$$V = \beta y_0^{1/\beta}, \quad \text{so } V = \beta \exp(L(z)/\beta). \tag{49}$$

It follows easily from (7) and (37) that V is analytic in S_1 and we have

$$zV' = \exp(g(z)) = y_0^{(1/\beta)-1}(zy_0') \quad \text{on } S_1. \tag{50}$$

Now $E(v)$ in (46) is a polynomial, with constant coefficients, of degree q_n. (This degree is always > 0 for if $n > 1$, then $q_n > q_1 \geq 0$, while if $n = 1$, then by (44), $q_n \geq j_1 > 0$.) Letting $\mu_1, \cdots, \mu_t$ be the distinct nonzero roots of E (if any), with μ_j of multiplicity m_j, and setting $\mu_0 = 0$ and $m_0 = q_1$, we clearly have

$$E(v) = c_n(v-\mu_0)^{m_0}(v-\mu_1)^{m_1}\cdots(v-\mu_t)^{m_t}. \tag{51}$$

We will now distinguish the two cases: $\gamma \geq 0$ and $\gamma < 0$. In each case, we will distinguish subcases according to the sign of β. (We remind the reader that in view of assumption (28), the theorem will be proved if each case leads to a contradiction.)

Case I. $\gamma \geq 0$. Thus if $\epsilon > 0$, then in view of (27), $y_0^{-\gamma} = O((\log z)^{\epsilon\gamma-\beta\gamma})$, and by §2 (d), $zy_0' \ll (\log z)^{\beta+\epsilon-1}$, so $(zy_0')^\gamma = O((\log z)^{\beta\gamma+\epsilon\gamma-\gamma})$. Hence from (45), $E(v_0) = O((\log z)^{2\epsilon\gamma-\epsilon_2})$. Letting $\epsilon_3 = \epsilon_2/2$ (so $\epsilon_3 > 0$), we can choose the arbitrary ϵ so small that we have $E(v_0) = O((\log z)^{-\epsilon_3})$. In view of (51) and Appendix Lemma 1, there is a k, $0 \leq k \leq t$, such that

$$v_0 - \mu_k = O((\log z)^{-\epsilon_4}), \quad \text{where } \epsilon_4 = \epsilon_3/m_k > 0. \tag{52}$$

We distinguish two subcases.

Subcase A. $\beta = 0$. Then $v_0 = y_0$ (by (46)), so unless $\mu_k = 0$, (52) is a contradiction of (29) (for $M = 1$). But if $\mu_k = 0$, then (52) gives $y_0 = O((\log z)^{-\epsilon_4})$. However, this is also impossible, since by (27) and the fact that $\beta = 0$ and $\epsilon_4 > 0$, we have $y_0 \gg (\log z)^{-\epsilon_4}$.

Subcase B. $\beta \neq 0$. We assert that

$$\beta > 0. \tag{53}$$

To prove (53), we observe that if $n \geq 2$, then $q_n > q_1 \geq 0$, so since $\beta q_n - j_n = \gamma \geq 0$ and $j_n \geq 0$, we have $\beta \geq 0$. Since $\beta \neq 0$, we have (53). If $n = 1$, then by (44), $q_1 \geq j_1 > 0$, so again $\beta \geq j_1/q_1 > 0$.

If in (52), $\mu_k = 0$, then since $|v_0|^{1/\beta} = |zV'|$ (by (48) and (50)), we would have $zV' = O((\log z)^{-\epsilon_4/\beta})$, since $\beta > 0$. Thus by Appendix Lemma 3, we would have $V = O((\log z)^{1-\epsilon'})$ for some $\epsilon' > 0$. But then by (49) (and (53)), $y_0 = O((\log z)^{\beta - \beta\epsilon'})$. Since $\beta\epsilon' > 0$, this contradicts (27).

The other possibility in (52) is that $\mu_k \neq 0$. Let $\phi = v_0 - \mu_k$, so that

$$\phi = O((\log z)^{-\epsilon_4}). \tag{54}$$

Let $A(z)$ be a branch of $\log(\mu_k + z)$ for $|z| < |\mu_k|$, and let $W(z) = \exp((1/\beta)A(z))$. Then W is analytic for $|z| < |\mu_k|$ and if $\lambda_1 = W(0)$ then $\lambda_1 \neq 0$. From the power series expansion for W, we may write

$$W(z) = \lambda_1 + zh(z), \tag{55}$$

where h is analytic for $|z| < |\mu_k|$, so $|h(z)| \leq K_1$ on $|z| \leq |\mu_k|/2$, for some $K_1 > 0$. Now by (54), there exist S_2 in $F(a, b)$ and $K_2 > 0$ such that

$$|\phi(z)| \leq K_2 |\log z|^{-\epsilon_4} < |\mu_k|/2 \quad \text{on } S_2. \tag{56}$$

Thus on S_2, from (55),

$$W(\phi(z)) = \lambda_1 + \phi(z)h(\phi(z)) \quad \text{and} \quad |h(\phi(z))| \leq K_1. \tag{57}$$

But clearly $\exp A(\phi(z)) = v_0(z)$, so by (48) (and continuity), $A(\phi(z))$ and $\beta g(z)$ differ by a constant λ_2 on S_2. Thus clearly $W(\phi(z)) = \exp(g(z) + (\lambda_2/\beta))$, and so by (50) we have on S_2,

$$zV'(z) = \lambda_3 W(\phi(z)), \quad \text{where } \lambda_3 = e^{-\lambda_2/\beta}. \tag{58}$$

Thus if we let $U(z) = V(z) - \lambda_1\lambda_3 \log z$ on S_2, then by (58), (57) and (56), we see that on S_2,

$$|U'(z)| \leq |\lambda_3| K_1 K_2 |z|^{-1} |\log z|^{-\epsilon_4}. \tag{59}$$

Thus by Appendix Lemma 3, $U = O((\log z)^{1-\epsilon'})$ for some $\epsilon' > 0$. Thus clearly $V/(\lambda_1\lambda_3 \log z) \to 1$. In view of (49) (and the fact that $\beta \neq 0$), we have from Appendix Lemma 4, that $y_0/c_2(\log z)^{\beta} \to 1$ for some $c_2 \neq 0$. This contradicts our assumption (29), and ends Case I.

Case II. $\gamma < 0$. We let $\sigma = -\gamma$, so

$$\sigma > 0, \tag{60}$$

and from (45), we have

$$(zy_0')^{\sigma} y_0^{-\sigma} E(v_0) = O((\log z)^{\gamma - \epsilon_2}). \tag{61}$$

Furthermore, by definition of v_0 (i.e. (46)),

(62) $$|v_0/y_0| = |zy_0'/y_0|^{\beta}.$$

We now distinguish the three subcases $\beta > 0$, $\beta < 0$ and $\beta = 0$.

Subcase A. $\beta > 0$. Now from (61) and (62), we have

(63) $$|v_0/y_0|^{\sigma/\beta}|E(v_0)| \le K_1(|\log z|^{\gamma-\epsilon_2}) \quad \text{for some } K_1 > 0,$$

on some element of $F(a, b)$. Now by (27), for any $\epsilon > 0$, $y_0 \ll (\log z)^{\beta+\epsilon}$, so $y_0^{\sigma/\beta} = O((\log z)^{\sigma+(\epsilon\sigma/\beta)})$. Choosing $\epsilon = \beta\epsilon_2/2\sigma$ and noting that $\sigma = -\gamma$, it follows from (51) and (63) that on some element of $F(a, b)$, we have,

(64) $$|v_0 - \mu_0|^{q_1+(\sigma/\beta)}|v_0 - \mu_1|^{m_1} \cdots |v_0 - \mu_t|^{m_t} \le K_2(|\log z|^{-\epsilon_2/2}),$$

for some constant K_2. Since $q_1 + (\sigma/\beta) \ge (\sigma/\beta) > 0$, we can apply Appendix Lemma 1, to show that there is a k, $0 \le k \le t$, such that $v_0 - \mu_k = O((\log z)^{-\epsilon_4})$ for some $\epsilon_4 > 0$. This is exactly the same conclusion as (52), and since $\beta > 0$ here, as in Case I, Subcase B, the proof proceeds exactly as in that subcase to show that both $\mu_k = 0$ and $\mu_k \ne 0$ are impossible.

Subcase B. $\beta < 0$. We set $\eta = -\beta$, so from (62),

(65) $$|zy_0'/y_0|^{\sigma} = |y_0/v_0|^{\sigma/\eta}.$$

Now in view of (27) (and the fact that $\sigma/\eta > 0$), it follows that for any $\epsilon > 0$, $1/y_0^{\sigma/\eta} = O((\log z)^{(\sigma/\eta)(-\beta+\epsilon)})$, which is $O((\log z)^{\sigma+(\epsilon\sigma/\eta)})$ since $\eta = -\beta$. Choosing $\epsilon = \eta\epsilon_2/3\sigma$ (and noting that $\sigma = -\gamma$), it therefore follows from (65) and (61) that, on some element of $F(a, b)$,

(66) $$|1/v_0|^{\sigma/\eta}|E(v_0)| \le K_1|\log z|^{-2\epsilon_2/3} \quad \text{for some } K_1 > 0.$$

In view of (51), we see that there is an element S_2 of $F(a, b)$ such that

(67) $$|1/v_0(z)|^{\theta}|v_0(z) - \mu_1|^{m_1} \cdots |v_0(z) - \mu_t|^{m_t} < |\log z|^{-\epsilon_2/2} \quad \text{on } S_2,$$

where $\theta = (\sigma/\eta) - q_1$. We assert that

(68) $$\theta > 0.$$

To see (68), we observe that $\theta = (\gamma/\beta) - q_1 = -j_1/\beta$ by (38). If $n = 1$, then by (44), $j_1 > 0$ so since $\beta < 0$, we have $\theta > 0$. If $n > 1$, then $\beta q_n - j_n = \beta q_1 - j_1$ (by (38)), so if $j_1 = 0$, then $\beta = j_n/(q_n - q_1)$ so we would have $\beta \ge 0$ contradicting this subcase. Hence $j_1 \ne 0$ so $j_1 > 0$, and again $\theta > 0$, proving (68). (Since $\theta > 0$, Appendix Lemma 1 is not applicable to (67).)

Since $\mu_1, \cdots, \mu_t$ are distinct nonzero complex numbers, there exists $\delta > 0$ such that

(69) $$|\mu_i - \mu_j| \ge \delta \quad \text{if } i \ne j, \qquad \text{and} \quad |\mu_i| \ge \delta \quad \text{for each } i.$$

Choose $K_0 > 0$ so large that

(70) $$K_0 > 3/(2\delta) \quad \text{and} \quad 2K_0^2 - K^*K_0 - 1 > 0,$$

where

(71) $$K^* \text{ is a fixed number } \geq |\mu_j| \quad \text{for } j = 1, \cdots, t.$$

Since $\epsilon_2 > 0$ in (67), there exists S_3 in $F(a, b)$ such that

(72) $$|\log z|^{-\epsilon_2/2} < (1/(2K_0))^{\theta + m_1 + \cdots + m_t} \quad \text{on } S_3,$$

and $S_3 \subset S_2$. There are now two possibilities.

Possibility 1. There exists a point z_1 in S_3 and an index k, $1 \leq k \leq t$, such that

(73) $$|v_0(z_1) - \mu_k| < 1/(2K_0).$$

In this case, we assert that k works for every z in S_3 in the sense that

(74) $$|v_0(z) - \mu_k| \leq 1/K_0 \quad \text{for } all \ z \text{ in } S_3.$$

To prove (74), assume that (74) fails at a point z_2 in S_3, and let Γ be a curve in S_3 joining z_1 to z_2. Since the continuous function $|v_0(z) - \mu_k|$ is $< (1/2K_0)$ at z_1 and is $> (1/K_0)$ at z_2, clearly at some point z_3 on Γ, we would have

(75) $$|v_0(z_3) - \mu_k| = 1/K_0.$$

Now in view of (67) and (72) it is clearly impossible that each of the numbers $|1/v_0(z_3)|$, $|v_0(z_3) - \mu_1|, \cdots, |v_0(z_3) - \mu_t|$ be $> 1/(2K_0)$, so at least one is $\leq (1/(2K_0))$. If it is a $|v_0(z_3) - \mu_j|$, then by (75), $j \neq k$, and we have $|\mu_k - \mu_j| \leq (3/(2K_0))$ which is $< \delta$, contradicting (69). If it is $|1/v_0(z_3)|$ which is $\leq (1/(2K_0))$, then $|v_0(z_3)| \geq 2K_0$, which with (75) and (70) would imply $|\mu_k| > K^*$ contradicting (71). Thus (74) is established.

Hence for all $z \in S_3$ and $j \neq k$, we have, $|v_0(z) - \mu_j| \geq |\mu_j - \mu_k| - |v_0(z) - \mu_k| > (1/(2K_0))$ by (69), (70) and (74). Also $|v_0(z)| \leq K^* + (1/K_0)$ by (74) and (71). Using these estimates in (67), it follows that for some constant $K_2 > 0$, we have

(76) $$|v_0(z) - \mu_k|^{m_k} \leq K_2 |\log z|^{-\epsilon_2/2} \quad \text{on } S_3.$$

Hence if $\phi = v_0 - \mu_k$, then we have $\phi = O((\log z)^{-\epsilon_2/2m_k})$. Since $\mu_k \neq 0$, this is the same estimate as in (54), with $\epsilon_4 = \epsilon_2/2m_k$. Since $\beta \neq 0$ here, the proof now proceeds *exactly* as from (54) on, to the end of Case I, where we would obtain the same contradiction, $y_0/c_2(\log z)^\beta \to 1$. Thus Possibility 1 leads to a contradiction.

Possibility 2. If Possibility 1 fails, then for each z in S_3 and each j, $1 \leq j \leq t$, we would have $|v_0(z) - \mu_j| \geq (1/(2K_0))$. Thus by (67), for some $K_3 > 0$, we would have $|v_0(z)|^{-\theta} \leq K_3 |\log z|^{-\epsilon_2/2}$ on S_3. Since $\theta > 0$ and $\eta > 0$, we would have $|v_0(z)|^{1/\eta} \geq K_4 \, |\log z|^{\epsilon_2/2\theta\eta}$, on S_3, where $K_4 > 0$. Since $\eta = -\beta$, we would obtain

(77) $$|v_0(z)|^{1/\beta} \leq (1/K_4)|\log z|^{-\epsilon_2/2\theta\eta} \quad \text{on } S_3.$$

But by (48) and (50), $|zV'| = |v_0|^{1/\beta}$, so in view of (77) and Appendix Lemma 3, we obtain

(78) $$V = O((\log z)^{1-\epsilon'}) \quad \text{for some } \epsilon' > 0.$$

But by (49), $|V| = |\beta| \, |y_0|^{1/\beta} = |\eta| \, |y_0|^{-1/\eta}$, where $\eta = -\beta > 0$. Thus from (78), it follows that on some element S_4 in $F(a, b)$, we have $|y_0(z)| \geq K_5 |\log z|^{\beta+\epsilon'\eta}$ for some $K_5 > 0$. Since $\epsilon'\eta > 0$, this clearly contradicts (27) for $\epsilon = \epsilon'\eta$ and hence Possibility 2 has also led to a contradiction, thus ending Subcase B of Case II.

Subcase C. $\beta = 0$. For this final subcase, we proceed as follows: By hypothesis, one of the three relations $y_0 \approx 1$, $y_0 >> 1$, $y_0 << 1$ must be valid. The first immediately contradicts the assumption (28). We will show that in this subcase (i.e. $\gamma < 0$, $\beta = 0$), the other two possibilities will also lead to contradictions.

Case (a). $y_0 >> 1$. Since $\beta = 0$, we have $v_0 = y_0$. Since $y_0 >> 1$, we have (in (46)), $E(y_0) = c_n y_0^{q_n}(1 + E_1)$, where $E_1 << 1$. It follows from (61), that in some T_1 in $F(a, b)$,

(79) $$|(zy_0')^{\sigma} y_0^{q_n - \sigma}| \leq K_1 |\log z|^{\gamma - \epsilon_2} \quad \text{on } T_1,$$

for some constant $K_1 > 0$. Now let $\zeta = (q_n - \sigma)/\sigma$. Since $\beta = 0$, we have $\sigma = -\gamma = j_n$ by (38), so that $\zeta \geq 0$. It is easily verified (using (7) and (37)), that the left side of (79) is $|z|^{\sigma} |y_0^{\zeta} y_0'|^{\sigma}$. Hence if we set

(80) $$U = y_0^{\zeta+1}/(\zeta + 1),$$

then from (79) (and noting that $\sigma = -\gamma$), we obtain

(81) $$|U'(z)| \leq (K_1)^{1/\sigma} |z|^{-1} |\log z|^{-1-(\epsilon_2/\sigma)} \quad \text{on } T_1.$$

Let z_0 be a fixed point in T_1 with $\arg z_0 = (a + b)/2$. Then by Appendix Lemma 2, we have that for all $R \geq 1$,

(82) $$|U(Rz_0) - U(z_0)| \leq K_2 (\log |z_0|)^{-\epsilon_2/\sigma},$$

for some constant K_2. Thus $|U(Rz_0)|$ is a bounded function of R on $[1, +\infty)$. But this is clearly impossible, since $y_0 \to \infty$ by assumption, and hence (since $\zeta \geq 0$), we see from (80), that $U \to \infty$ over $F(a, b)$. Thus Case (a) leads to a contradiction.

Case (b). $y_0 << 1$. Since $\beta = 0$, we have $v_0 = y_0$. Since $y_0 << 1$, we have (in (46)), $E(y_0) = c_1 y_0^{q_1}(1 + E_1)$, where $E_1 << 1$. It follows from (61), that there exist T_2 in $F(a, b)$, and a constant $K_3 > 0$ such that

(83) $$|(zy_0')^\sigma y_0^{q_1-\sigma}| \leq K_3 |\log z|^{\gamma - \epsilon_2} \quad \text{on } T_2.$$

Now let $\zeta = (q_1 - \sigma)/\sigma$. Since $\beta = 0$, we have $\sigma = -\gamma = j_1$, so $\zeta \geq 0$. It is again easily verified that the left side of (83) is $|z|^\sigma |y_0^\zeta y_0'|^\sigma$, so if we again set

(84) $$U = y_0^{\zeta+1}/(\zeta + 1),$$

then from (83),

(85) $$|U'(z)| \leq (K_3)^{1/\sigma} |z|^{-1} |\log z|^{-1-(\epsilon_2/\sigma)} \quad \text{on } T_2.$$

By Appendix Lemma 2, for any point z_1 in T_2, with $\arg z_1 = (a+b)/2$, and any $R \geq 1$, we have

(86) $$|U(Rz_1) - U(z_1)| \leq K_4 (\log |z_1|)^{-\epsilon_2/\sigma},$$

where $K_4 = \sigma(K_3)^{1/\sigma}/\epsilon_2$. Thus,

(87) $$|U(z_1)| - K_4(\log |z_1|)^{-\epsilon_2/\sigma} \leq |U(Rz_1)| \quad \text{for all } R \geq 1.$$

But since $y_0 \to 0$ in this case, and since $\zeta + 1 \geq 1$, it follows from (84) that $U \to 0$ over $F(a, b)$. Hence clearly, for any z_1 with $\arg z_1 = (a+b)/2$, clearly $\lim_{R \to +\infty} |U(Rz_1)| = 0$. Hence from (87), $|U(z_1)| \leq K_4 (\log |z_1|)^{-\epsilon_2/\sigma}$, so from (84), we see that for any point z_1 in T_2, with $\arg z_1 = (a+b)/2$,

(88) $$|y_0(z_1)| \leq (K_4(\zeta + 1))^{1/(\zeta+1)} (\log |z_1|)^{-\epsilon_3},$$

where $\epsilon_3 = \epsilon_2/\sigma(\zeta + 1)$. But $\epsilon_3 > 0$ and $\beta = 0$, so by (27), we have $y_0 >> (\log z)^{-\epsilon_3/2}$. Thus on some element T_3 of $F(a, b)$ contained in T_2, we have

(89) $$|y_0(z)| > (K_4(\zeta + 1))^{1/(\zeta+1)} (|\log z|^{-\epsilon_3/2}).$$

Now by definition of $F(a, b)$, T_3 contains points z_1, with $\arg z_1 = (a+b)/2$, and $|z_1|$ arbitrarily large. Since $|\log z_1| \leq \log |z_1| + |a+b|/2$ for such points, (88) and (89) would lead to the inequality,

(90) $$(1 + |a+b|/2 \log |z_1|)^{\epsilon_3/2} > (\log |z_1|)^{\epsilon_3/2},$$

for points z_1 with arbitrarily large modulus. This is of course impossible since as $|z_1| \to \infty$, the left side of (90) tends to 1 while the right side tends to ∞. Thus Case (b) has also led to a contradiction.

Hence every case has led to a contradiction because of assumption (28), and thus the theorem is proved.

4. **Remark.** Returning to the statement of the main result, we point out that there are algorithms which will produce (in a finite number of steps which can be bounded in advance):

(i) A finite number of logarithmic monomials of rank 0, among which are all possible cz^{λ} and dz^{σ} which can satisfy parts (A) and (B) respectively for some solution y_0.

(ii) A finite number of logarithmic monomials, $z^{\alpha}(\log z)^{\beta}$, with β rational, such that any monomial satisfying part (C) for some solution y_0 is a multiple of one of them.

To see this, it is shown in [1, p. 302], that if $M = cz^{\lambda}$ satisfies part (A) for a solution y_0, then there is a function $B(z)$ such that $B/M \to 1$ and such that $v = B(z)$ is a solution of the equation $G(z, v) = \sum_{i+j=p} f_{ij} v^j = 0$, where $p = \max\{i + j: f_{ij} \not\equiv 0\}$. In view of [8, §36], the algorithm in [8, §28], when applied to G, will produce all possible such M. To find all the monomials $M = dz^{\sigma}$ which can satisfy part (B) for some solution y_0, we make the change of variable $y = 1/u$ in Ω (see (26)) and apply the above reasoning to $-M$. In this way, it is easy to see that each such dz^{σ} would be produced by applying [8, §28] to $H(z, v) = \sum_{i+j=q} f_{ij} v^j$, where $q = \min\{i + j \mid f_{ij} \not\equiv 0\}$.

For (ii), it is proved in [3, §5] that any monomial M which satisfies part (C) for a solution y_0 would be a *critical* monomial for Ω as defined in [3, §4] (or [9, p. 5]). Hence the algorithm in [3, §§21, 26] or [9, p. 28], which produces all critical monomials for nth order differential polynomials with coefficients in a logarithmic field, would produce all possible such M. (For the case at hand, the algorithm easily shows that β must be rational, and that the number of steps will be bounded in advance (see [3, §17]).) The Strodt-Wright result [9, p. 221], essentially states, in part, that for any critical monomial of Ω, there is an asymptotically equivalent solution in a suitable $F(a, b)$.

5. **The case of positive rank.** We show here, by means of a simple example, that the main result may fail to hold when the rank of the coefficient field is positive (in this case, one). We consider

$$z(\log z)^{\beta} y' - y = 0, \quad \text{where } 0 < \beta < 1. \tag{91}$$

We assert that if y_0 is any $\not\equiv 0$ solution, then

$$\text{for any } \epsilon > 0,\ y_0 << z^{\epsilon} \quad \text{in } F(-\pi, \pi), \tag{92}$$

and

$$\text{for any } N > 0,\ (\log z)^N << y_0 \quad \text{in } F(-\pi, \pi). \tag{93}$$

It easily follows that for any logarithmic monomial M, $y_0 << M$ if $\delta_0(M) > 0$ while $y_0 >> M$ if $\delta_0(M) \leq 0$. Thus y_0 is comparable with all monomials (of arbitrary rank), and the statement $y_0 \sim M$ is false for every M (i.e. part (C) fails for every M). Clearly part (A) does not hold by (92) and part (B) does not hold by (93).

To prove (92) and (93), we first note that these properties would be trivial to verify if "<<" were replaced by the weaker "o" relation, by simply using the representation, $y_0 = c \exp[(\log z)^{1-\beta}/(1-\beta)]$. For the stronger statements, we proceed as follows: If $\epsilon > 0$, make the change of dependent variable $w = z^{-\epsilon}y$ in (91). We obtain

$$w - w'/V = 0, \quad \text{where } V = -\epsilon z^{-1} + z^{-1}(\log z)^{-\beta}. \tag{94}$$

Since $\beta > 0$, $V \sim -\epsilon z^{-1}$. The "indicial function" [7, §61] for V is the constant function $\equiv -1$ on $(-\pi, \pi)$. Thus by [4, Lemma δ, p. 271], every solution w of (94) is $\ll 1$ in $F(-\pi, \pi)$. Hence $y_0 \ll z^{\epsilon}$ in $F(-\pi, \pi)$, proving (92).

For (93), we make the change of dependent variable $u = (\log z)^{-N}y$ in (91), where $N > 0$ is arbitrary. We obtain

$$u - u'/U = 0, \quad \text{where } U = z^{-1}(\log z)^{-\beta} - Nz^{-1}(\log z)^{-1}. \tag{95}$$

Since $\beta < 1$, $U \sim z^{-1}(\log z)^{-\beta}$. The indicial function for U is the constant function $+1$ on $(-\pi, \pi)$. Hence by [4, Lemma δ, p. 271], just one solution of (95) is $\ll 1$, while every other solution is $\gg 1$ on $F(-\pi, \pi)$. (Clearly $u \equiv 0$ is the solution $\ll 1$.) Thus if $u_0 = (\log z)^{-N}y_0$, then $y_0 \gg (\log z)^N$ since $u_0 \gg 1$ proving (93).

6. Appendix.

Lemma 1. *Let $\lambda_1, \cdots, \lambda_q$ be $q \geq 1$ distinct complex numbers, and let $m_1, \cdots, m_q$ be positive real numbers. Let $v_0(z)$ be an analytic function in an element S of $F(a, b)$ such that for some $\epsilon > 0$ and $K_1 > 0$, we have*

$$|v_0(z) - \lambda_1|^{m_1} \cdots |v_0(z) - \lambda_q|^{m_q} \leq K_1 |\log z|^{-\epsilon} \quad \text{on } S. \tag{A1}$$

Then for some t, $1 \leq t \leq q$, we have $v_0 - \lambda_t = O((\log z)^{-\epsilon/m_t})$ over $F(a, b)$.

Proof. Let δ be a positive number such that $|\lambda_i - \lambda_j| \geq 2\delta$ if $i \neq j$. Since $\epsilon > 0$, there exists T_1 in $F(a, b)$, with T_1 contained in S, such that

$$K_1 |\log z|^{-\epsilon} < (\delta/2)^{m_1 + \cdots + m_q} \quad \text{for all } z \text{ in } T_1. \tag{A2}$$

Let z_1 be a point in T_1. In view of (A2) and (A1) it is clearly impossible that each of the numbers $|v_0(z_1) - \lambda_1|, \cdots, |v_0(z_1) - \lambda_q|$ be $> \delta/2$. Thus there exists an index t (depending on z_1, of course) such that

$$|v_0(z_1) - \lambda_t| \leq \delta/2. \tag{A3}$$

We now show that t will work for *every* z in T_1, by showing that for every z in T_1,

$$|v_0(z) - \lambda_t| \leq \delta. \tag{A4}$$

If (A4) fails for some z_2, let Γ be a curve in T_1 joining z_1 to z_2. Since the function $|v_0(z) - \lambda_t|$ is $\leq \delta/2$ at $z = z_1$, and is $> \delta$ at $z = z_2$, there exists z_3 on Γ such that

(A5) $$|v_0(z_3) - \lambda_t| = \delta.$$

But as above, from (A1) and (A2), there must be an index j such that $|v_0(z_3) - \lambda_j| \leq \delta/2$. From (A5), $j \neq t$, but $|\lambda_j - \lambda_t| \leq |\lambda_j - v_0(z_3)| + |v_0(z_3) - \lambda_t| \leq 3\delta/2$ contradicting the definition of δ. Thus (A4) holds. Therefore, from (A4) and the definition of δ, it follows that for z in T_1 and $j \neq t$, we have

(A6) $$|v_0(z) - \lambda_j| \geq |\lambda_j - \lambda_t| - |v_0(z) - \lambda_t| \geq \delta.$$

The lemma now immediately follows from (A1) and (A6).

Lemma 2. *Let $V(z)$ be analytic in an element S of $F(a, b)$, and let $\epsilon > 0$ and $K > 0$ be such that $|V'(z)| \leq K|z|^{-1}|\log z|^{-1-\epsilon}$ on S. Then for any point z_1 in S, with $\arg z_1 = (a+b)/2$ and $|z_1| > 1$, we have*

(A7) $$|V(Rz_1) - V(z_1)| \leq (K/\epsilon)(\log |z_1|)^{-\epsilon} \quad \textit{for all } R \geq 1.$$

Proof. Clearly rz_1 belongs to S for $r \geq 1$, and $|\log rz_1| \geq \log |rz_1| > 0$. Thus,

(A8) $$|V'(rz_1)z_1| \leq Kr^{-1}(\log r|z_1|)^{-1-\epsilon} \quad \text{for } r \geq 1.$$

Integrating inequality (A8) with respect to r from 1 to R, and using the obvious estimate, (A7) now follows from the fundamental theorem of calculus and the fact that $\log R|z_1| > 0$ for $R \geq 1$.

Lemma 3. *Let $V(z)$ be analytic in an element S of $F(a, b)$, where S has the property that $|z| > 1$ for all z in S, and let $\epsilon > 0$ and $K > 0$ be such that on S we have $|V'(z)| \leq K|z|^{-1}|\log z|^{-\epsilon}$. Then there exists a number ϵ', with $0 < \epsilon' < 1$, such that $V = O((\log z)^{1-\epsilon'})$ over $F(a, b)$.*

Proof. Let $\epsilon_1 = \min(\epsilon, 1/2)$, and let $F(z) = Kz^{-1}(\log z)^{-\epsilon_1/2}$. Let $E = V'/F$. Since $\epsilon_1/2 < \epsilon$, it follows by hypothesis that $E \to 0$ over $F(a, b)$. Since $-\epsilon_1/2 > -1$, it is proved in [5, Lemma 3, p. 304] that if z_0 is any fixed element of S, then $\int_{z_0}^{z} EF / \int_{z_0}^{z} F \to 0$ over $F(a, b)$. Since $EF = V'$, it follows that there exists an element S_1 of $F(a, b)$, contained in S, such that on S_1,

(A9) $$|V(z) - V(z_0)| \leq K_1 + (K/(1 - (\epsilon_1/2)))|\log z|^{1-(\epsilon_1/2)},$$

where $K_1 = (K/(1 - (\epsilon_1/2)))|\log z_0|^{1-(\epsilon_1/2)}$. Since z_0 is fixed and $1 - (\epsilon_1/2) > 0$, there is an element S_2 in $F(a, b)$ on which $K_1 + |V(z_0)| \leq |\log z|^{1-(\epsilon_1/2)}$, and hence from (A9), $V = O((\log z)^{1-(\epsilon_1/2)})$ proving Lemma 3.

Lemma 4. *Let $y_0(z)$ be analytic and nowhere zero in an element of $F(a, b)$ and let $L(z)$ be a branch of $\log y_0$. Let $V(z) = \beta \exp(L(z)/\beta)$, where β is a nonzero real number. Then if for some nonzero constant c_1 we have $V(z)/c_1 \log z \to 1$ over $F(a, b)$, then $y_0(z)/c_2(\log z)^\beta \to 1$ over $F(a, b)$ for some constant $c_2 \neq 0$.*

Proof. Let $T(z) = \exp(\beta \log z)$ be the principal branch of z^β in $|z - 1| < 1$, and let $w(z) = V(z)/c_1 \log z$. Since $w \to 1$, it follows easily that

$$T(w(z)) \to 1 \quad \text{over } F(a, b). \tag{A10}$$

Now if d is a value of $\log(\beta/c_1)$, then clearly, $(L(z)/\beta) - \log(\log z) + d$ is a branch of $\log w(z)$ in an element of $F(a, b)$, and hence differs from the principal branch $A(z)$ of $\log w(z)$ by a constant, say λ_1, on a convenient element of $F(a, b)$. Thus, since $T(w(z)) = \exp(\beta A(z))$, we have, in view of (A10), that

$$\exp(L(z) - \beta \log(\log z) + \beta(d - \lambda_1)) \to 1 \quad \text{over } F(a, b). \tag{A11}$$

Thus $y_0/c_2(\log z)^\beta \to 1$ where $c_2 = \exp(\beta\lambda_1 - \beta d)$, proving the lemma.

BIBLIOGRAPHY

1. S. Bank, *A representation theorem for large and small analytic solutions of algebraic differential equations in sectors*, Trans. Amer. Math. Soc. **159** (1971), 293–305.

2. ———, *On solutions having large rate of growth for nonlinear differential equations in the complex domain*, J. Math. Anal. Appl. 22 (1968), 129–143. MR **40** #5945.

3. ———, *On the instability theory of differential polynomials*, Ann. Mat. Pura Appl. (4) 74 (1966), 83–111. MR 34 #4623.

4. E. W. Chamberlain, *Families of principal solutions of ordinary differential equations*, Trans. Amer. Math. Soc. **107** (1963), 261–272. MR 26 #6470.

5. ———, *The univalence of functions asymptotic to nonconstant logarithmic monomials*, Proc. Amer. Math. Soc. 17 (1966), 302–309. MR 32 #7724.

6. W. Strodt, *Contributions to the asymptotic theory of ordinary differential equations in the complex domain*, Mem. Amer. Math. Soc. No. 13 (1954). MR 16, 702.

7. ———, *Principal solutions of ordinary differential equations in the complex domain*, Mem. Amer. Math. Soc. No. 26 (1957). MR **19**, 1177.

8. ———, *On the algebraic closure of certain partially ordered fields*, Trans. Amer. Math. Soc. **105** (1962), 229–250. MR 25 #3934.

9. W. Strodt and R. K. Wright, *Asymptotic behavior of solutions and adjunction fields for nonlinear first order differential equations*, Mem. Amer. Math. Soc. No. 109 (1971).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801