GROUPS WHOSE HOMOMORPHIC IMAGES HAVE A TRANSITIVE NORMALITY RELATION

BY

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ABSTRACT. A group $G$ is a T-group if $H \triangleleft K \triangleleft G$ implies that $H \triangleleft G$, i.e., normality is transitive. A just non-T-group (JNT-group) is a group which is not a T-group but all of whose proper homomorphic images are T-groups. In this paper all soluble JNT-groups are classified; it turns out that these fall into nine distinct classes. In addition all soluble JNT-groups and all finite JNT-groups are determined; here a group $G$ is a T-group if $H \triangleleft K \triangleleft L \triangleleft G$ implies that $H \triangleleft L$. It is also shown that a finitely generated soluble group which is not a T-group has a finite homomorphic image which is not a T-group.

1. Introduction and statement of results.

(1.1) Definitions. If $P$ is a group theoretical property, a just non-P-group is a group which is not a P-group but all of whose proper homomorphic images are P-groups; for brevity we shall call these JNP-groups. For example, when $P$ is commutativity, soluble JNP-groups have been studied by Newman ([16] and [17]) and also by Rosati [21].

Denote by $T$ the property that normality is transitive; thus a group $G$ has $T$ if $H \triangleleft K \triangleleft G$ always implies that $H \triangleleft G$. Here we are concerned with JNT-groups; these can alternatively be defined as the groups whose nonnormal subnormal subgroups are core-free and form a nonempty set.

The principal object of this paper is to classify the soluble JNT-groups. This will be done by dividing the soluble JNT-groups into nine types (and eleven subtypes). While the descriptions of the different types vary in both complexity and precision, a rather clear picture of the structure of a soluble JNT-group emerges.

Notation.

$\langle X : \lambda \in \Lambda \rangle$: subgroup generated by the (subsets) $X_\lambda$, $\lambda \in \Lambda$.

$X^Y$: normal closure of $X$ in $Y$, i.e., the subgroup generated by all conjugates $x^y = y^{-1}xy$, $x \in X$, $y \in Y$.

$[X, Y] = [X, Y]$: commutator subgroup generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ ($x \in X$, $y \in Y$).

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\[X, Y = \{X; Y, Y\}\]

\(C_H(X):\) centraliser of \(X\) in \(H\).

\(\xi(G):\) centre of \(G\).

\(R^*:\) multiplicative group of units of \(R\), a ring with unity.

\(FG:\) group algebra of a group \(G\) over a field \(F\).

\(\phi:\) isomorphism of \(\Omega\)-operator groups.

\(X^+:\) additive subgroup of a ring generated by \(X\).

(1.2) The classification of soluble \(INT\)-groups. We shall now describe the nine types of groups which can occur.

I. The nonabelian nilpotent groups all of whose proper homomorphic images are abelian\(^{(2)}\), with the exception of the quaternion group of order 8.

II. Let \(L\) be an abelian group of type \(2^\infty\) with generators \(a_1, a_2, \cdots\) and relations \(a_1^2 = 1\) and \(a_{i+1}^2 = a_i\). Let \(D = \langle d \rangle\) be a group of order 2. Define \(G\) to be the group generated by the direct product \(L \times D\) and a cyclic group \(\langle t \rangle\) of order 2 or 8 where \(a^t = a^{-1}\), \(d^t = a_d\) and \(t^2 = 1\) or \(a_2d\) for all \(a \in L\).

III. Let \(L\) and \(D\) be as in II except that \(D\) may now have order 1. Let \(X\) be an extra-special 2-group generated by elements of order 2 and let \(C\) be the direct product of \(L \times D\) and \(X\) in which \(\langle a_1 \rangle\) and the centre of \(X\) are amalgamated. Choose an element \(\sigma\) from \(\text{Hom}(X/\langle a_1 \rangle, \langle a_1 \rangle)\) and let \(\langle t \rangle\) be a cyclic group of order 2 or 8. Define \(G\) to be the group generated by \(C\) and \(\langle t \rangle\) where for all \(a \in L\) and \(x \in X\),

\[
a^t = a^{-1}, \quad d^t = a_d, \quad (\text{if } d \neq 1), \quad x^t = x(x(a_1)^\sigma),
\]

and

\[
t^2 = 1 \quad \text{or} \quad a_2d, \quad (\text{if } d \neq 1).
\]

Moreover, if \(d \neq 1\) we can take \(\sigma = 0\).

IV. Let \(A\) be an elementary abelian \(p\)-group of order \(p^2\) where \(p\) is an odd prime, and let \(X\) be the group of automorphisms of \(A\) determined by a diagonal but nonscalar subgroup of \(\text{GL}(2, p)\). Define \(G\) to be the holomorph of \(A\) by \(X\).

V. Let \(P\) be an extra-special \(p\)-group of exponent \(p\), an odd prime, and let \(n\) be an integer lying strictly between 1 and \(p\). Let \(\{x_\lambda: \lambda \in \Lambda\}\) be a basis for \(P\) modulo its centre and define an automorphism \(t\) of \(G\) by the rule \(x_\lambda^t = x_\lambda^n, (\lambda \in \Lambda)\). Define \(G\) to be the holomorph of \(P\) by \(\langle t \rangle\).

VI. Let \(A\) be a group of type \(p^\infty\) with generators \(a_1, a_2, \cdots\) and relations \(a_{i+1}^p = a_i\) and \(a_1^p = 1\). Let \(\Gamma\) be a nonperiodic group of \(p\)-adic integers all of which are congruent to 1 modulo \(p\).

(a) Form an extension \(W\) of \(A\) by \(\Gamma\) using the natural coupling of \(\Gamma\) to \(A\). Let \(D\) be a nontrivial elementary abelian \(p\)-group of automorphisms of \(W\) which

\(^{(2)}\) Newman [17] calls these \(IN2\)-groups, however this conflicts with our present terminology.
acts trivially on \( \langle a_1 \rangle \) and \( W/\langle a_1 \rangle \). Define \( G \) to be the holomorph of \( W \) by \( D \). If \( p = 2 \) and \( -1 \in \Gamma \), then in addition require that \( t^2 = 1 \) or \( a_1 \) where \( -1 \to t^A \) in the isomorphism of \( \Gamma \) with \( W/A \). If \([D, t] \neq 1\), the possibility \( t^2 = a_1 \) can be dispensed with.

(b) Let \( p = 2 \) and \( -1 \in \Gamma \), and choose a group \( \langle u \rangle \) of order \( 2 \). Form an extension \( W \) of \( F = A \times \langle u \rangle \) by \( \Gamma \) using the natural coupling of \( \Gamma \) to \( A \), supplemented by \( a \to (u \to u) \) if \( a \equiv 1 \mod 4 \) and \( a \to (u \to a_1 u) \) if \( a \not\equiv 1 \mod 4 \); moreover, require that \( W' = A \) and \( t^2 = a_2 u \) where \( t \) is as in (a). Finally, form \( G \) as in (a) except that \( D \) is allowed to be \( 1 \) and the centre of \( G \) should not contain any element of order \( 2 \) except \( a_1 \). [This last condition is automatically satisfied in VI(a).]

VII. Let \( A \) and \( \Gamma \) be as in VI and let \( X \) be an extra-special \( p \)-group with generators of order \( p \). Write \( E \) for the direct product of \( A \) with \( X \) in which \( \langle a_1 \rangle \) and the centre of \( X \) are amalgamated.

(a) Form an extension \( W \) of \( E \) by \( \Gamma \) in which \( W' = A \), using the natural coupling of \( \Gamma \) to \( A \) supplemented by causing each \( a \) in \( \Gamma \) to correspond to an (outer) automorphism of \( X \) which acts trivially on \( \langle a_1 \rangle \) and \( X/\langle a_1 \rangle \). Let \( D \) be an elementary abelian \( p \)-group of automorphisms of \( W \) which acts trivially on \( \langle a_1 \rangle \) and \( W/\langle a_1 \rangle \). Define \( G \) to be the holomorph of \( W \) by \( D \) and suppose \( D \) is chosen so that the centre of \( G \) contains no elements of order \( p \) outside \( \langle a_1 \rangle \). If \( p = 2 \) and \( -1 \in \Gamma \), require also that \( t^2 = 1 \) where \( -1 \to t^E \) in the isomorphism of \( \Gamma \) with \( W/E \).

(b) Let \( p = 2 \) and \( -1 \in \Gamma \), and choose a group \( \langle u \rangle \) of order \( 2 \). Form an extension \( W \) of \( F = E \times \langle u \rangle \) by \( \Gamma \) using the coupling of \( \Gamma \) to \( E \) described in (a), supplemented by \( a \to (u \to u) \) if \( a \equiv 1 \mod 4 \) and \( a \to (u \to a_1 u) \) if \( a \not\equiv 1 \mod 4 \); moreover require that \( W' = A \) and \( t^2 = a_2 u \) where \( -1 \to t^F \) in the isomorphism of \( \Gamma \) with \( W/F \). Finally form \( G \) as in (a).

VIII. Let \( X \) be a soluble \( T \)-group and let \( A \) be a noncyclic abelian group which is faithful and irreducible as an \( X \)-module (so that \( A \) contains no proper nontrivial submodules). Define \( G \) to be the natural semidirect product of \( A \) by \( X \).

IX. Let \( p \) be any prime and let \( F \) be a subfield of the field of \( p \)-adic numbers; denote by \( Q \) the field of rational numbers. Choose \( X \) to be a group of \( p \)-adic integer units in \( F \) such that \( X \neq \langle -1 \rangle \) and \( X^+ < Q + X^+ = F \). Define \( G \) to be the natural semidirect product of \( F \) (as an additive group) by \( X \).

Our principal conclusion is, then,

**Theorem 1.** A group is a soluble JNT-group if and only if it is isomorphic with a group of type I to IX.

(1.21) Power automorphisms and soluble \( T \)-groups. The proof of Theorem 1 occupies \( \S \S 3 \) to 7. Not surprisingly, considerable use will be made of the theory of soluble \( T \)-groups. We present next a summary of the relevant facts from this theory.
A fundamental concept is that of a power automorphism of a group; this is an automorphism which leaves every subgroup of the group invariant, and so maps each element to a power of itself. A crucial result is that in an abelian group a power automorphism maps elements of the same order to the same power; moreover if an element of infinite order is present in the group, the only nontrivial power automorphism is the involution \( a \rightarrow a^{-1} \) ([7], [18, §4.1]). An extensive study of power automorphisms of nonabelian groups has been made by Cooper [2].

If \( G \) is a soluble \( T \)-group, then it is metabelian [18, Theorem 2.3.1]. Also \( L = [G', G] \) is the last term of the lower central series, \( G/L \) is a Dedekind group (i.e. every subgroup is normal) and \( C_G(G') = C_G(L) \) is the Fitting subgroup of \( G \).

Noncommutative soluble \( T \)-groups are divided into three classes:

(a) periodic groups,

(b) nonperiodic groups of type I, i.e. groups in which the centraliser \( C \) of the derived subgroup is nonperiodic,

(c) nonperiodic groups of type II, i.e. groups in which \( C \) is periodic.

If \( G \) is a periodic soluble \( T \)-group and \( L = [G', G] \), then \( L \) and \( G/L \) do not contain elements with the same odd prime order ([7], [18, Theorem 4.2.2]); also the 2-component of \( L \) is radicable.

If \( G \) is a soluble \( T \)-group of type I and \( C = C_G(G') \), then \( C \) is abelian and \( G = \langle t, C \rangle \) where \( |G:C| = 2 \), \( c^t = c^{-1} \) for all \( c \in C \), and \( \langle t^2, C^2 \rangle = \langle t^2, C^4 \rangle \) [18, Theorem 3.1.1].

If \( G \) is a soluble \( T \)-group of type II, somewhat less is known of its structure. However \( C = C_G(G') \) is abelian, \( G' \) is radicable and \( C = G' \times B \) where \( B \) lies in the centre of \( G \). If \( G' \) contains an element of prime order \( p \), the \( p \)-component of \( B \) has finite exponent, say \( p^{n(p)} \); if \( x \in G \), then \( x \) induces in the \( p \)-component of \( C \) the power automorphism \( a \rightarrow a^\alpha \) where \( \alpha \) is a \( p \)-adic integer unit satisfying \( \alpha \equiv 1 \ mod \ p^{n(p)} \); here, of course, \( a^\alpha \) is understood to mean \( a^{\alpha_1} \) where \( \alpha_1 \) is an integer congruent to \( \alpha \) modulo the order of \( a \) [18, Theorem 4.3.1].

(1.3) Remarks on the classification.

(1.3.1) Nilpotent just nonabelian groups. These groups—which include the extraspecial groups of Hall and Higman [10]—occur in our classification and deserve comment.

Let \( G \) be a nilpotent just nonabelian group and let \( Z \) be its centre. Then, as Newman has shown in [17], \( G \) is a \( p \)-group, \( Z \) is either cyclic or quasi-cyclic and \( G/Z \) is elementary abelian; moreover, \( G' \) lies in \( Z \) and has order \( p \). Let \( \{x_\lambda Z: \lambda \in A\} \) be a basis for \( G/Z \) and let \( G' = \langle a \rangle \). Then

\[
[x_\lambda, x_\mu] = a^f(\lambda, \mu)
\]

If we regard \( G/Z \) as a vector space over \( GF(p) \), then \( f \) is a nondegenerate alternating bilinear form. Also \( x^p_\lambda \in Z \) and it may be shown that \( x_\lambda \) can be chosen
so that either $x_i^p = 1$ or $x_i^p$ generates $Z$ [17, Lemma 3]. Indeed one can write
down generators and relations for $G$ by utilising (1), the position of the $x_i^p$ and
relations sufficient to make $[G', G] = 1$. Newman also proves that if $G$ is count-
able, it is a central product involving three rather simple kinds of groups [17,
Theorem 5].

It should be mentioned that soluble just nonabelian groups which are not
nilpotent have also been discussed by Newman [16]; these occur under our type
VIII heading unless they are metacyclic.

(1,32) Faithful irreducible representations. In connection with groups of type
VIII it is desirable to know which soluble $T$-groups $X$ can act faithfully and irre-
ducibly on an abelian group $A$. Of course in a situation of this kind $A$ is neces-
sarily isomorphic with the additive group of a vector space, i.e., it is either an
elementary abelian $p$-group or a direct product of copies of the additive group of
rational numbers. In the former event the problem reduced to the case where $X$ is
abelian. This is because of

Lemma 1. Let $F$ be a field and let $X$ be a soluble $T$-group.

(i) If $X$ is abelian, it has a faithful irreducible representation over $F$ if
and only if there is an extension field $E$ of $F$ such that $X \cong Y < E^*$ and $E = FY$.
(ii) In general, $X$ has a faithful irreducible representation over $F$ if and
only if the centre of its Fitting subgroup has such a representation.

Proof. (i) The proof is well known and we omit it.
(ii) We recall first that the Fitting subgroup $N$ of any $T$-group $X$ (whether
soluble or not) is nilpotent and coincides with $C_X(X')$ [18, Lemma 2.2.2]. Write
$C$ for the centre of $N$.

Suppose first that there exists a faithful irreducible (right) $FX$-module $M$. Let
$0 \neq a \in M$ and $D = C_N(a)$. Since $N$ is nilpotent, $D$ is subnormal in $X$ and hence
$D \triangleleft X$. Therefore $D$ fixes $ax$ for all $x \in X$, which shows that $D$ acts trivially
on $M$ and $D = 1$ since $M$ is irreducible and faithful. Consequently $M$ is fixed-
point-free with respect to each nonunit element of $N$.

Assume now that $N$ is nonperiodic. If $X$ is abelian, $C = N = X$ and $M$ is
already a faithful irreducible $FC$-module. Let $X$ be nonabelian—and thus a soluble
$T$-group of type I. It follows from the structure theory of these groups (§1.21) that
$N = C$ is abelian and $X = < x, C >$ where $a^{-1} = a'$ if $a \in C$, and $x^2 \in C$. Suppose
that $M_1$ is a proper nonzero $FC$-submodule of $M$; then $M_1x$ is also an $FC$-sub-
module because $C \triangleleft X$. Since $x^2 \in C$, we see that $M_1 + M_1x$ and $M_1 \cap M_1x$ are
$FX$-submodules of $M$; hence $M_1 \cap M_1x = 0$ and $M = M_1 \oplus M_1x$ by the irreducibility
of $M$. If $M_2$ is a proper nonzero $FC$-submodule of $M_1$, then $M = M_2 \oplus M_2x$ by the
same argument, with the result that $M_1 = M_1 \cap (M_2 \oplus M_2x) = M_2$. Therefore $M_1$
is an irreducible FC-module; it is faithful because $M$ is fixed-point-free.

Now suppose that $N$ is periodic. Let $Y$ be a finitely generated—and therefore finite-subgroup of $C$. Choose $a \neq 0$ from $M$; then $(a)FY$ has finite dimension over $F$, so it contains an irreducible FY-submodule, say $L$. Since the action of $N$ is fixed-point-free, $L$ is faithful. Consequently, by the first part of this lemma, $Y$ is isomorphic with a finite subgroup of an extension of the field $F$, which implies that $Y$ is cyclic and $C$ locally cyclic. Let $F$ be the algebraic closure of $F$.

Then the torsion-subgroup of $F^*$ is a direct product of $p^\infty$-groups, one for each $p$ not equal to the characteristic of $F$. We can identify $C$ with a subgroup of $F^*$. Having done this, define $E$ to be the subfield of $F$ generated by $C$. Since $E$ is algebraic over $F$, an FC-submodule of $E$ is an ideal. Thus $E$ is an irreducible FC-module, and it is obviously faithful.

Conversely, suppose $M_1$ is a right FC-module which is faithful and irreducible; the problem is to construct an FX-module that is faithful and irreducible. First we form the induced FX-module

$$l = M_1 \otimes_{FC} (FX)$$

and then we choose an FX-composition series for $l$; here the term composition series (or system) is used in the general sense of Kuroš [14, vol. 2, §56]. The composition factors are irreducible FX-modules, so we can assume that none is faithful. Now refine the series to an FC-composition series. If $1 \neq S \triangleleft X$, then $S \cap C \neq 1$; for $S \cap X' \leq S \cap C$ (since $X$ is metabelian) and $S \cap X' = 1$ implies $S \leq C$. It follows that none of the FC-composition factors can be faithful as FC-modules. It is straightforward to show that an irreducible FC-submodule of $l$ is FC-isomorphic with one of the factors of the FC-composition series, and hence is not faithful.

However, if $\{g_\lambda : \lambda \in \Lambda\}$ is a transversal to $C$ in $X$, with say $g_{\lambda_0} = 1$, then

$$l \cong \bigoplus_{\lambda \in \Lambda} (M_1 \otimes_{FC} (FX)g_\lambda)$$

and $M_1 \otimes_{FC} (FC)g_{\lambda_0} \cong M_1$, which gives the contradiction that $M_1$ is not faithful. This completes the proof.

Returning to the discussion of groups of type VIII, we see that Lemma 1 (with $F = GF(p)$) gives a complete description of the possible groups $X$ when $A$ is an elementary abelian $p$-group, although not in group theoretical terms if $A$ is infinite; when $A$ is finite the only restriction on the soluble $T$-group $X$ is that the centre of $C_X(X')$ be cyclic of order prime to $p$.

However, when $A$ is a direct product of copies of the additive group of rational numbers $(\mathbb{Q})$, the situation is less clear. Here the following theorem of Baer [1, Proposition] is relevant: a locally finite group cannot act as an irreducible group of automorphisms of a torsion-free abelian group $A \neq 1$. This tells us, at least, that $X$ cannot be periodic.
(1.33) Groups of type IX. While it would probably be difficult to give a purely
group theoretic characterisation of $G$ in this case, some possibilities can easily
be obtained.

Let $F$ be a subfield of $F_p$, the field of $p$-adic numbers, let $R_p$ be the ring of
$p$-adic integers and define $R = F \cap R_p$; now define $X = R^*$. Clearly $X^+ \leq R$. Let
$r \in R$; if $r \neq 0 \mod p$, then $r \in R^* = X$; if $r \equiv 0 \mod p$, then $1 + r \in R^* = X$ and
$r \in X^+$. Therefore $X^+ = R$. Also $R$ is not a field because $1/p \notin R$, so $X^+ \neq F$. Finally,
let $f \in F$; we can write $f = q + u$ where $q \in Q$ (the field of rational numbers) and
$u \in R_p$. Since $Q < F$, we have $u = f - q \in F$, so $u \in F \cap R_p = R$. Consequently,
$F = Q + X^+$. The natural semidirect product of $F$ with $X$ is of type IX. For example,
we could take $F = F_p$ and $X = R^*_p$.

On the other hand, in a group of type IX the subgroup $X$ cannot be periodic.
For suppose that $X$ is periodic; then $\langle -1 \rangle \neq X \leq R^*_p$ and the structure of $R^*_p$
show that $X$ is cyclic of order $m > 2$ dividing $p - 1$ where $p$ is odd. Thus $F = Q + X^+$ is a finite extension of $Q$. If $X = \langle x \rangle$, the irreducible polynomial of $x$ is
cyclotomic. Hence each element of $F$ can be written in the form $r + \sum_{i=1}^{n-1} \alpha_i x^i$
where $r$ is rational, $\alpha_i$ integral and $n = \phi(m) > 1$. But $x/p$ has no such represen-
tation since $1, x, \ldots, x^{n-1}$ are linearly independent.

2. Preliminary results. In this section we shall collect results of a general
nature about JNT-groups as well as some technical lemmas necessary for the
classification.

(2.1) Some properties of JNT-groups. Our first result indicates the complex
subnormal structure of JNT-groups in general.

Lemma 2. To each group $G$ there corresponds a JNT-group $G^*$ with sub-
groups $H$ and $K$ such that $K \triangleleft H$ and $H$ is subnormal in $G^*$ with subnormal
index $\leq 4$ while $G \cong H/K$.

Proof. The first step is to embed $G$ in a nonunit perfect group $G_1$ such that
$G$ is subnormal in $G_1$ with subnormal index $\leq 2$; that this is possible is a theorem
of Dark [3]. Write $Z$ for the set of integers in their natural order and form the
standard wreath power $G_2 = \mathfrak{W}_r G_1^Z$. The order-automorphism $n \mapsto n + 1$ of $Z$
gives rise to an automorphism $t$ of $G_2$ permuting the copies of $G_1$ in the same
way. Finally, let $G^*$ be the holomorph of $G_2$ by $\langle t \rangle$.

Let $Z_1$ and $Z_2$ be the sets of negative and nonnegative integers in natural
order. Then

$$G_2 \cong (\mathfrak{W}_r G_1^{Z_1}) \cup (\mathfrak{W}_r G_1^{Z_2})$$

where now not all the wreath products are standard; for these and other results
about wreath products see P. Hall [9]. The base group $L$ of the wreath product
(2) has $G_1$ as a homomorphic image, so that $G_1 \cong L/K$ for some $K \triangleleft L$. Hence
there exists a subgroup $H$ of $L$ such that $G \simeq H/K$ and $H$ is subnormal in $L$ with subnormal index $\leq 2$. Clearly $H$ is subnormal in $G^*$ with subnormal index $\leq 4$.

To show that $G^*$ is a JNT-group observe first that $G_2$—and hence $G^*$—is not a T-group. Also it follows by arguments of P. Hall [9, p. 183] that $G^*$ is monolithic with $G_2$ as its monolith; here $G_2$ is perfect since $G_1$ is, so $G_2 = G_2^l$. Hence every proper homomorphic image of $G^*$ is abelian.

Corollary. There exist JNT-groups with unbounded subnormal indices.

In contrast to this there is

Lemma 3. A JNT-group $G$ has all its subnormal indices $\leq 2$ if one of the following conditions is satisfied.

(i) $G$ contains a nontrivial normal abelian subgroup $A$.

(ii) $G$ contains a minimal normal subgroup $N$ which itself contains a minimal normal subgroup $N_1$.

Proof. Let $H$ be a nonnormal subnormal subgroup of $G$. Suppose that (i) is valid. Since $AH \triangleleft G$, we have

$$[A, H]^g \leq [A, AH] = [A, H]$$

for all $g \in G$. Hence $[A, H] \triangleleft G$ and $[A, iH] \triangleleft G$ for each $i \geq 1$. Therefore $\langle [A, iH] \rangle \triangleleft G$ provided $[A, iH] \neq 1$. Now if $s$ is the subnormal index of $H$ in $G$, then $[A, sH] \leq H$, so that $[A, sH] = 1$. Let $i$ be the least integer for which $[A, iH] = 1$. Then $i > 0$ and $H \triangleleft H[A, iH] \triangleleft G$; but also $H \triangleleft H[A, iH]$, so the result follows.

Now suppose that (ii) is valid. Clearly $N$ is the direct product of $N_1$ and certain of its conjugates. Therefore $N_1$ is simple. If $N_1$ is abelian, so is $N$ and the result follows from (i). If $N_1$ is nonabelian, a theorem of Wielandt [22] shows that $N$ normalises $H$, so $H \triangleleft HN \triangleleft G$.

These may be compared with the (more elementary) fact that a non-T-group whose proper subgroups are all T-groups has its subnormal indices $\leq 2$ [20].

We shall now consider the possibilities for minimal normal subgroups in JNT-groups.

Lemma 4. Let $G$ be a JNT-group:

(i) the number of minimal normal subgroups of $G$ equals 0, 1 or 2;

(ii) if this number equals 2, both minimal normal subgroups are cyclic of the same prime order;

(3) A group is monolithic if the intersection of all its nontrivial normal subgroups is nontrivial; this intersection is then called the monolith.
(iii) if $G$ has at least one minimal normal subgroup, each nontrivial normal subgroup of $G$ contains a minimal normal subgroup of $G$.

Proof. Throughout $H$ denotes a nonnormal subnormal subgroup of $G$. Let $M$ be a minimal normal subgroup of $G$ and let $N$ be a nontrivial normal subgroup of $G$ not containing $M$; thus $M \cap N = 1$. Clearly $M \subseteq MN/N$ and, since $G/N$ is a $T$-group, this shows that $M$ is simple. Now $M \leq H$ would imply that $H \triangleleft G$; therefore $H \cap M = 1$. Also $(H \cap N)M \triangleleft G$, so that

$$[H \cap N, G] \leq ((H \cap N)M) \cap N = H \cap N$$

and $H \cap N \triangleleft G$. Therefore $H \cap N = 1$. If $M$ is not abelian, a theorem of Wielandt [22] shows that $[H, M] = 1$. In this case

$$H \cap (M \times N) \leq C_{M \times N}(M) = N,$$

and $H \cap (M \times N) = 1$ since $H \cap N = 1$. However this implies that $H = (HM) \cap (HN)$, and since $HM \triangleleft G$ and $HN \triangleleft G$, the contradiction $H \triangleleft G$ is obtained. Thus $M$ is abelian and therefore cyclic of prime order, say $p$. As we have just seen, $H_1 = H \cap (M \times N)$ cannot be trivial; obviously, $H_1$ is subnormal in $G$. Since $H \cap N = 1$,

$$H_1 \simeq H_1N/N \leq (M \times N)/N \simeq M,$$

which shows that $H_1$ has order $p$. Next $H \cap M = 1$, so

$$H_1 \simeq H_1M/M \leq (M \times N)/M \simeq N,$$

which shows that $N$ contains a subnormal subgroup $N_1$ of order $p$. But $MN_1 \triangleleft G$, so $[N_1, G] \leq (MN_1) \cap N = N_1$ and $N_1 \triangleleft G$. Hence $N$ contains a minimal normal subgroup of $G$, namely $N_1$. Moreover, if $N$ itself is minimal normal in $G$, then $N = N_1$ and $|M| = p = |N|$. Thus (ii) and (iii) have been proved.

Finally, let $L$, $M$ and $N$ be three distinct minimal normal subgroups of $G$. All three must have the same prime order $p$. Let $g \in G$; then $g$ induces in the elementary abelian $p$-group $(M \times N)L/L$ a power automorphism of the form $x \mapsto x^n$. Since $M \trianglelefteq LM/L$ and $N \trianglelefteq NL/L$, it follows that $g$ induces $x \mapsto x^n$ in $M$ and in $N$. Therefore every subgroup of $M \times N$ is normal in $G$ and, in particular, $H \cap (M \times N) \triangleleft G$. Hence $H \cap (M \times N) = 1$, which is a contradiction. Thus $G$ can have no more than two minimal normal subgroups.

For example, soluble $JNT$-groups of type IX possess no minimal normal subgroups, those of types I–III and V–VIII have one (and so are monolithic) and those of type IV have two.

A $JNT$-group with two minimal normal subgroups, and any soluble $JNT$-group, has its subnormal indices $\leq 2$; these statements follow from Lemmas 3 and 4.
Next we record two technical lemmas about \( JNT \)-groups which will prove valuable.

**Lemma 5.** Let \( M \) and \( N \) be normal subgroups of a \( JNT \)-group \( G \). If one of the following conditions holds, then either \( M \) or \( N \) is trivial.

(i) \( M \) and \( N \) are periodic and do not contain elements with the same prime order.

(ii) \( M \) is periodic, \( N \) is torsion-free and \( G/N \) is periodic.

**Proof.** Let \( H \) denote a nonnormal subnormal subgroup of \( G \) and let \( M/1 \) and \( N/1 \). Both (i) and (ii) imply that \( M \cap N = 1 \); thus \( (H \cap M)N < G \) implies that \( [H \cap M, G] \leq H \cap M \) and so \( H \cap M = 1 \). Similarly \( H \cap N = 1 \). If (i) is valid, \( H \cap (M \times N) = (H \cap M) \times (H \cap N) = 1 \), which gives \( H = (HM) \cap (HN) \triangleleft G \). If (ii) is valid, then \( H \) is periodic since \( H \cong HN/N \); therefore \( HM \) is periodic and \( (HM) \cap N = 1 \), which implies that \( (HM) \cap (HN) = H \) and \( H \triangleleft G \).

**Lemma 6.** Let \( N \triangleleft G \) where \( N \) is abelian and each of its primary components is either elementary abelian or of infinite exponent. Assume that every subgroup of \( C_G(N) \) is normal in \( G \). Then \( G \) is not a \( JNT \)-group.

**Proof.** Suppose \( G \) is a \( JNT \)-group and let \( H \) be a nonnormal subnormal subgroup of \( G \). Then \( H \not\leq C_G(N) \), so there exists \( b \in H \) such that \( [N, b] \neq 1 \). The element \( b \) induces a nontrivial power automorphism in \( N \), since \( N \leq C_G(N) \). If \( N \) is not periodic, \( a^b = a^{-1} \) for all \( a \in N \) (see §1.21) and \( [N, b] = N^{2^s} \). If \( s \) is the subnormal index of \( H \) in \( G \), then \( H \geq N^{2^s} > 1 \), which implies that \( H \triangleleft G \). Thus \( N \) is periodic and for some prime \( p \) the \( p \)-component \( P \) is not centralised by \( b \).

There is a \( p \)-adic integer \( a \) such that \( a^b = a^a \) for all \( a \in P \) [18, Lemma 4.1.2]. With \( s \) as before, \( H \geq P^{(a-1)^s} \) and consequently \( P^{(a-1)^s} = 1 \). However this implies that either \( P \) is elementary abelian and \( a \equiv 1 \mod p \) or \( P \) has infinite exponent and \( a = 1 \); in each case \( [P, b] = 1 \).

(2.2) Splitting criteria. The following partial generalisation of the Schur-Zassenhaus theorem is well known—see for example [4, Theorem 3] or [18, Lemma 5.1.1]. It will only be required, of course, in situations where \( G \) is soluble.

**Lemma 7.** Let \( N \triangleleft G \) where \( G \) is locally finite and \( G/N \) is countable. Suppose that \( N \) and \( G/N \) do not contain elements with the same prime order. Then \( N \) has a complement in \( G \).

However, we shall encounter situations where this splitting criterion is inadequate. The following result is particularly useful for splitting groups over non-central minimal normal subgroups; it is based on an idea of M. F. Newman [16].

**Lemma 8.** Let \( A \) and \( N \) be normal subgroups of a group \( G \) such that \( A \leq N \) and \( [A, N] \neq 1 \). Assume in addition that \( N \) is metabelian and that every subgroup of \( N/A \) is normal in \( G/A \). Suppose that \( A \) is isomorphic with the additive
group of a vector space over a prime field and that \( A \) contains no proper non-
trivial \( G \)-invariant subspaces. Then \( A \) has a complement in \( G \).

**Proof.** We show first that \( A \) has a complement in \( N \). Let \( C = C_N(A) \); since
\([A, N] \neq 1\), we can find a \( g \in N \setminus C \). Since every subgroup of \( N/A \) is normal in
\( G/A \), there is for each \( x \in G \) an integer \( n \) such that \( g^n x = g^n \mod A \). Since \( A \) is
abelian, \([a, g]^x = [a^x, g^n] \in [A, g] \). It follows that \([A, g] \triangleleft G \), and for a similar
reason \( C_A(g) \triangleleft G \). If \( A \) is regarded as a vector space over a prime field, then
\([A, g] \) and \( C_A(g) \) are subspaces invariant under \( G \). Hence
\[
C_A(g) = 1 \quad \text{and} \quad [A, g] = A.
\]
The mapping \( a \to [a, g] \) is therefore an automorphism of \( A \).

Next, define for any \( g \in N \setminus C \)
\[
X_g = \{ x; x \in N, [x, g, g] = 1 \}.
\]
Let \( x \) and \( y \) belong to \( X_g \). Since \( N \) is metabelian,
\[
[x, y, g, g] = [[x, g]^y[y, g], g] = [x, g, g^y] = 1
\]
and
\[
[x^{-1}, g, g] = [[x, g]^{-x^{-1}}, g] = [x, g, g]^{-x^{-1}} = 1.
\]
Thus \( X_g \) is a subgroup. Indeed \( X_g \) is a complement of \( A \) in \( N \). For let \( x \in N \) and
\( a \in A \); then \([ax, g, g] = [(a, g]^x[x, g], g] = [a, g]x[x, g, g] \). Now \( N/A \) is a Dede-
kind group, so it is nilpotent of class \( \leq 2 \) and \([x, g, g] \in A \). Since \( a \to [a, g] \) is
an automorphism of \( A \), it follows that given \( x \in N \) we can choose \( a \) from \( A \) so that
\( ax \in X_g \). Consequently \( N = AX_g \). If \( x \in A \cap X_g \), then \([x, g, g] = 1 \), which
implies that \( x = 1 \). Thus \( A \cap X_g = 1 \) and \( X_g \) is a complement of \( A \) in \( N \).

Denote by \( K \) any complement of \( A \) in \( N \); we shall prove that \( K = X_g \) for
some \( g \in N \setminus C \). Since \( A \) is abelian and \( N = AK \), there exists a \( g \) in \( K \) such that
\([A, g] \neq 1 \). Now \( K \cong N/A \), which is nilpotent of class \( \leq 2 \); hence \( K \leq X_g \). Since
\( X_g \) is also a complement of \( A \), we obtain \( K = X_g \).

Next, all complements of \( A \) in \( N \) are conjugate in \( N \). For consider two such
complements \( X_g \) and \( X_b \) where \( g \) and \( b \) come from \( N \setminus C \). The mapping \( a \to [g, a, b, b] \) is an automorphism \( A \). Since \([g, b, b] \in A \), there exists an \( a \) in \( A \) such that
\[
[g, b, b]^{-1} = [g, a, b, b];
\]
with this \( a \) we compute
\[
[g^a, b, b] = [g[g, a], b, b] = [(g, b)[g, a, b], b] = [g, b, b][g, a, b, b] = 1
\]
by (3). Hence \( g^a \in X_b \). Also \( X_b \) is nilpotent of class \( \leq 2 \), so \( X_b \trianglelefteq X_g^a = (X_g)^a \).
Since \( X_b \) and \( X_g^a \) are both complements of \( A \) in \( N \), they are equal.
Finally, we shall show that $A$ has a complement in $G$. To this end, let $K$ be a complement of $A$ in $N$ and let $g \in G$. Then $K^{g^{-1}}$ is also a complement of $A$ in $N$, so $K^{g^{-1}} = K^b$ for some $b \in N$. Hence $bg \in N_G(K)$ and

$$G = N(N_G(K)) = (AK)N_G(K) = AN_G(K).$$

But $A \cap N_G(K) = N_A(K) = C_A(K)$, since $A \cap K = 1$. Hence $A \cap N_G(K) = 1$ and $N_G(K)$ is a complement of $A$ in $G$.

3. Nilpotent $J_1 T$-groups. Let $G$ be a nilpotent $J_1 T$-group; we shall prove that $G$ is of type I. Suppose first that $g$ is an element of infinite order in $G$ such that $\langle g \rangle < G$. Then $\langle g^{2^i} \rangle < G$ and if $i > 2$, the group $G/\langle g^{2^i} \rangle$ is Dedekind and contains elements of order $2^i > 4$. Hence $G/\langle g^{2^i} \rangle$ is abelian and this causes $G$ to be abelian. Consequently the centre of $G$ is periodic and by Lemma 5 it is a $p$-group for some prime $p$. Hence there is a minimal normal subgroup $N$ of $G$ which lies in the centre and has order $p$. Suppose that $gN$ is an element in $G/N$ with infinite order. Then $L = \langle g, N \rangle \triangleleft G$ and $L = \langle g \rangle \times N$. Thus $L^p = \langle g^p \rangle$ is an infinite cyclic group and $L^p \triangleleft G$. This is impossible, so $G/N$—and hence $G$—is periodic. Therefore $G$ is a $p$-group.

Assume next that there exists $M \triangleleft G$ such that $M \neq 1$ and $N \ntriangleleft M$. Then $M \cap N = 1$ and at least one of $G/M$ and $G/N$ is hamiltonian, from which it follows that $p = 2$ and $G$ is a 2-group. Let $H$ be a nonnormal subnormal subgroup of $G$ and choose from $H$ an element $b$ of order 2. Then $bM$ and $bN$ generate normal subgroups of $G/M$ and $G/N$ respectively, in each case with order 1 or 2. Therefore $b$ belongs to the centre of $G$ and $\langle b \rangle \triangleleft G$, which shows that $H \triangleleft G$. We conclude that $G$ is monolithic with monolith $N$.

If $G/N$ is abelian, $G$ is just nonabelian and therefore of type I. Assume $G/N$ to be hamiltonian. Then again $p = 2$ and $G$ is a 2-group and $|N| = 2$. Write

$$G/N = (Q/N) \times (E/N)$$

where $Q/N$ is a quaternion group of order 8 and $E/N$ is an elementary abelian 2-group. Let $iN, jN$ and $kN$ be a canonical set of generators for $Q/N$. Thus $i^i = j^{-1}$ or $j^{-1}a$ where $a \in N$. Since $N$ lies in the centre of $G$, we obtain $j^{i^2} = j$ in either case. By the same reasoning $i^2$ commutes with $k$, with the result that $i^2$ is in the centre of $Q = \langle i, j, k, N \rangle$. Let $e \in E$; since $[Q, E] \leq N$, the mapping $xN \mapsto [x, e]$ is a homomorphism of $Q/N$ into $N$. The kernel must be nontrivial, so it contains $i^2N$. Thus $[i^{i^2}, E] = 1$ and because $G = QE$ the element $i^2$ lies in the centre of $G$. Since $i^2 \neq 1$ and $N$ is the monolith of $G$, we can conclude that $N < \langle i^2 \rangle$. Moreover $iN$ has order 4, so $i$ has order 8 and $N = \langle i^4 \rangle$. Hence

$$i^i = i^{-1} = i^7 \quad \text{or} \quad i^i = i^{-1}i^4 = i^3; \quad \text{but} \quad i^2 = (i^{i^2})^i = (i^i)^2 = i^6 \quad \text{in either case, giving the contradiction} \ i^4 = 1.$$
4. Soluble JNT-groups without minimal normal subgroups. Let \( G \) be a soluble JNT-group which is not nilpotent and write \( L = [G', G] \). Then \( L \neq 1 \), so \( G/L \) is a Dedekind group; \( L \) is also the limit of the lower central series of \( G \). We shall assume throughout this section that \( L \) contains no minimal normal subgroups of \( G \), our aim being to show that \( G \) is of type IX. This will be achieved by means of the following programme:

(i) \( L \) is torsion-free and abelian.

(ii) \( L \) is rationally irreducible with respect to \( G \), i.e. every nontrivial normal subgroup of \( G \) that is contained in \( L \) has periodic factor group in \( L \).

(iii) \( L \) is radicable.

(iv) \( G \) is of type IX.

If \( 1 \neq N <_G G \), then \( G/N \) is a soluble \( T \)-group and hence is metabelian. Thus \( G'' \leq N \) and \( G'' \) is either 1 or the monolith of \( G \); the latter is impossible since \( G'' \leq L \). Consequently \( G \) is metabelian, so \( G' \)-and hence \( L \)-is abelian.

If \( L \) is not torsion-free, there is a prime \( p \) such that the subgroup \( P = \{ a; a \in L, a^p = 1 \} \) is not 1. Clearly \( P \triangleleft G \), and since \( P \) cannot contain a minimal normal subgroup of \( G \), there is an infinite chain of nontrivial normal subgroups of \( G \),

\[
P = P_1 > P_2 > \cdots > P_\alpha > \cdots, \quad (\alpha < \beta),
\]

such that

\[
\bigcap_{\alpha < \beta} P_\alpha = 1;
\]

here \( \beta \) is necessarily a limit ordinal. Let \( g \in G \); since \( G/P_\alpha \) is a \( T \)-group, \( g \) induces in \( P/P_\alpha \) a power automorphism. Should \( g \) centralise \( P/P_2 \), it will centralise every \( P/P_\alpha \) and hence \( P \); for \( g \) must induce in \( P/P_\alpha \) an automorphism \( a \to a^n \) where \( n \) is independent of \( \alpha \), since \( P \) is elementary abelian. Therefore \( C_G(P/P_2) = C_G(P) \). Consequently \( G/C_G(P) \) is cyclic of order dividing \( p - 1 \), and, if \( 1 \neq a \in P \), then \( a^G \) is finitely generated and therefore finite. However this would imply that \( a^G \) contained a minimal normal subgroup of \( G \), contrary to hypothesis.

Next we prove that \( L \) is rationally irreducible with respect to \( G \). If this is false, there surely exists an infinite chain of nontrivial normal subgroups of \( G \),

\[
L = L_1 > L_2 > \cdots > L_\alpha > \cdots, \quad (\alpha < \beta),
\]

such that \( L/L_2 \) is not periodic and

\[
\bigcap_{\alpha < \beta} L_\alpha = 1;
\]

again \( \beta \) is a limit ordinal. Let \( \alpha \geq 2 \); now \( (G/L_\alpha)' = G'/L_\alpha \geq L/L_\alpha \), so \( G/L_\alpha \) is a soluble \( T \)-group of type I. Let \( C = C_G(L) \); then certainly \( C \) centralises \( L/L_\alpha \). Now if \( H \) is any \( T \)-group, \( C_H(H') = C_H([H', H]) \) [18, Lemma 2.2.2]. From this we deduce that \( C \leq C_G(G'/L_\alpha) \). Let \( g \in G \backslash C \)—note that \( C \neq G \) because \( G \) is not
nilpotent. Then $g$ does not centralise $G'/L_\alpha$ if $\alpha$ is large enough. From the structure of soluble $T$-groups of type I ($\S$1.21) it follows that $C/L_\alpha$ is abelian and $g$ induces in $C/L_\alpha$ the automorphism $a \to a^{-1}$ for each $\alpha \geq 2$. By (5), $C$ is abelian and $c^g = c^{-1}$ for all $c \in C$ and $g \in G\setminus C$. This implies that every subgroup of $C$ is normal in $G$ and Lemma 6 yields the contradiction that $G$ is not a JNT-group.

We wish now to establish that $L$ is radicable. Supposing this to be false, we can find a prime $p$ such that $L^p < L$. Here $p$ must be odd, for $L/L^2$ is radicable by Lemma 2.4.1 of [18]. Now $G/L^p$ is a nonabelian soluble $T$-group in which the elements of finite order form a subgroup (since $L/L^p$ is periodic and $G/L$ nilpotent). If $G/L$ is not periodic, $G/L^p$ is soluble $T$ of type II [18, Corollary 2, Theorem 3.1.1] and $L/L^p$ is radicable [18, Theorem 4.3.1]. This is impossible, so $G/L$ is periodic.

If $L^\omega$ is the intersection of all the subgroups $L^p$, then $L/L^\omega$ is torsion-free since $L$ is; the rational irreducibility of $L$ now implies that $L^\omega = 1$. If $x \in D = C_G(L/L^p)$, then $x$ induces in each $L/L^p$ an automorphism with order a power of $p$. Since $G/L$ is periodic, we conclude that $x$ induces in $L$ an automorphism with order a power of $p$. If $C = C_G(L)$, then $D/C$ is a $p$-group. However $G/L^p$ is a periodic soluble $T$-group and therefore $G/L$ can have no elements of order $p$ [18, Theorem 4.2.2]. Since $L \leq C$, it follows that $C = D$. Therefore $G/C$ is a cyclic group of order dividing $p - 1$. Let $1 \neq a \in L$ and set $A = a^G$. Then $A$ is free abelian of finite rank. Suppose now that $G/L$ contains an element with odd prime order $q$. Then $G/A^q$ is periodic and $G/L$ and $L/A^q$ both contain an element of order $q$, contradicting Theorem 4.2.2 of [18]. Consequently $G/L$ is a 2-group.

Let $g$ be any element of $G\setminus C$. Then $g$ induces in $A/A^{3i}$ a power automorphism whose order is a power of 2 and divides $\phi(3^i) = 2 \cdot 3^{i-1}$; hence $g$ must induce the identity or $a \to a^{-1}$, the latter being the only power automorphism of order 2. The intersection of all the $A^{3i}$ is 1 since $A$ is free abelian; thus $a_1^g = a_1^{-1}$ for all $a_1$ in $A$ unless $[A, g] = 1$. Since $L/A$ is periodic and $L$ is torsion-free, it follows that

$$a^g = a^{-1} \quad (a \in L, \ g \in G\setminus C).$$

Next $L = L^2$; thus $L/A^2$ has a subgroup of type $2^\infty$. Let $i$ be an integer $> 2$ and let $P/A^{2i}$ denote the subgroup of all elements of $L/A^{2i}$ which have odd order. Then $P < L$ and $L/P$ is a radicable abelian 2-group. $C/A^{2i}$ centralises $L/A^{2i}$, and hence $G'/A^{2i}$, so it is Dedekind; but $C/A^{2i}$ also has a factor of type $2^\infty$, which causes it to be abelian. Therefore $C$ is abelian. Let us write

$$C/A^{2i} = (P/A^{2i}) \times (E/A^{2i})$$

where $E/A^{2^i}$ is a 2-group—recall here that $G/L$ is a 2-group. $G/P$ is a 2-group which is not Dedekind. Let $g \in G\setminus C$; then $g$ cannot centralise $G'/P$ since elements of $L/P$ are transformed by $g$ into their inverses and $L/P$ is radicable. From the structure of soluble 2-groups with $T$ [18, Lemma 4.2.1] we know that $g^2 P$ belongs to the centre of $G/P$ and hence $g^2$ centralises $E/A^{2^i}$. Also $g^2$ centralises $L$ by (6), so $g^2$ centralises $P$. By (7), $C/A^{2^i}$ is centralised by $g^2$ for each $i$; thus $g^2$ centralises $C$. In addition, $g$ induces a nontrivial power automorphism in $C/P$ since $g$ does not centralise $L/P$. Once again we invoke the structure of soluble 2-groups with $T$ and conclude that every $g$ in $G\setminus C$ induces $a \mapsto a^{-1}$ in $C/P$. If $c \in C$ and $g \in G\setminus C$, then $c^g = c^{-1} a$ where $a \in P$; therefore \[ c = c^g^2 = (c^g)^{-1} a^{-1} = (c^{-1} a)^{-1} a^{-1} = ca^{-2} \] since $C$ is abelian. Since $L$ is torsion-free, $a = 1$ and $c^g = c^{-1}$. It follows that every subgroup of $C$ is normal in $G$. Lemma 6 provides the contradiction that $G$ is not a $J^N T$-group. $L$ is therefore radicable.

Lemma 8 can now be applied with $A = L$ and $N = G$; observe here that by rational irreducibility there are no proper nontrivial radicable subgroups of $L$ that are normal in $G$. Hence $L$ has a complement in $G$, say $X$; \[ G = LX \quad \text{and} \quad L \cap X = 1. \]

Suppose that $X$ does not act faithfully on $L$, i.e. $D = C_X(L) \neq 1$. Notice that $D \lhd LX = G$ and that $G/D$ is a $T$-group. Now $G'/D$ and $G'D/D \cong LD/D \cong L$, so $G/D$ is a soluble $T$-group of type I. If $g \in G\setminus C$, then $a^g = a^{-1}$ for all $a$ in $L$ in view of the structure of soluble $T$-groups of type I and because $L$ and $LD/D$ are isomorphic as $G$-operator groups. Rational irreducibility now forces $L$ to be isomorphic with $Q$, the additive group of rational numbers. Let $M$ be a subgroup of $L$ such that $L/M$ is isomorphic with $Q/Z$, where $Z$ is the subgroup of all integers. Then $M \lhd G$ and $C/M$ is abelian. An element $g$ of $G\setminus C$ induces $a \mapsto a^{-1}$ in $L$ and thus in $L/M$; also $g$ induces a power automorphism in $C/M$ which must agree with $a \mapsto a^{-1}$ on $L/M$. Since $L/M$ has elements of every finite order and power automorphisms map elements of the same order to the same power, $g$ must induce $a \mapsto a^{-1}$ also in $C/M$. The intersection of all subgroups like $M$ is 1, which shows that $C$ is abelian and $a^g = a^{-1}$ for all $a \in C$. Hence every subgroup of $C$ is normal in $G$. Lemma 6 now gives a contradiction, so $X$ acts faithfully on $L$. Since the group $G$ is metabelian, $[L, X'] = 1$; therefore $X$ is abelian and $C = L$.

Let $Q$ denote the field of rational numbers. Choose $a \neq 1$ from $L$; then $L$ is an irreducible $QX$-module and $L = a QX$. The mapping $r \mapsto a^r$ is a homomorphism of $QX$-modules from $QX$ onto $L$ with kernel $K$, a maximal ideal of $QX$;
thus $QX/K$ is a field. Let $A = a^X$; then $A \trianglelefteq G$ and $G/A$ is a $T$-group. If $x \in X$ and $n$ is a nonzero integer, then

$$(a^{1/n}A)^x = a^{m/n}A$$

for some integer $m$. Therefore $(1/n)x - m/n \in ZX + K$, where $ZX$ is the integral group ring of $X$, and

$$(8) \quad QX = Q + ZX + K.$$ 

Since $L$ is not a minimal normal subgroup of $G$, there is a prime $p$ and a normal subgroup $P$ of $G$ such that $P$ is properly contained in $L$ and $L/P$ is a $p$-group. Let $I$ be the intersection of all such $P$. If $I \neq 1$, then $L/I$ is a $p$-group since it is periodic. Hence $l = l^q$ for all primes $q \neq p$; but $l = l^p$ by minimality of $l$, so in fact $l$ is radicable. Thus $l = 1$.

Next let $L/P_1$ and $L/P_2$ be nontrivial $p$-groups where $P_1 \trianglelefteq G$ and $P_2 \trianglelefteq G$. An element $x$ of $X$ induces in $L/P_1$ and $L/P_2$ power automorphisms that can be described by $p$-adic integers $\alpha_1$ and $\alpha_2$. But $L/P_1 \cap P_2$ is also a $p$-group and $x$ induces in it a power automorphism describable by a $p$-adic integer $\alpha_3$. Clearly $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_3$, so $\alpha_1 = \alpha_2$. It follows that to each $x$ in $X$ there corresponds a unique $p$-adic integer unit $\alpha_x$ such that $b^{\alpha_x}P = (bP)^{\alpha_x}$ for all $b$ in $L$ and all $P \trianglelefteq G$ with $P \leq L$ and $L/P$ a $p$-group. Moreover, $\alpha_x = 1$ if and only if $x = 1$ since $l = 1 = C_X(L)$.

This enables us to construct a mapping $\alpha$ from $QX$ to $F_p$, the field of $p$-adic numbers, as follows:

$$\left( \sum_{x \in X} r_x x \right) \alpha = \sum_{x \in X} r_x \alpha_x x \quad (r_x \in Q).$$

$\alpha$ is a ring homomorphism because $\alpha_{xy} = \alpha_x \alpha_y$; also it is easy to verify that $\text{Ker } \alpha = K$, using the fact that $l = 1$. Let $F$ be the image of $QX$ under $\alpha$; then

$$QX/K \cong F \leq F_p,$$

the isomorphism being of rings. Therefore $F$ is a subfield of $F_p$. Define $Y$ to be the image of $X$ under $\alpha$; then $Y$ consists of $p$-adic integer units. Let $\overline{G}$ be the semi-direct product of $F$ (qua additive group) by $Y$. Then the mapping $a^x \rightarrow ((r)\alpha, a_x)$ is easily seen to be an isomorphism of $G$ with $\overline{G}$.

Finally, $\overline{G}$ is of type IX. For $(ZX)\alpha \simeq Y^+$, the additive subgroup generated by $Y$, and by (8), $F = Q + Y^+$; on the other hand, $F = Y^+$ would imply that $L$ is minimal normal in $G$. Also $Y \neq \langle -1 \rangle$ since otherwise $F = Q$ and $G$ would be a $T$-group.

5. Nonperiodic soluble $JNT$-groups which contain a minimal normal subgroup. Throughout this section $G$ will denote a nonnilpotent, nonperiodic, soluble $JNT$-group such that $L = [G', G]$ contains a minimal normal subgroup of $G$, say $N$.  

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Of course, $N$ is either an elementary abelian $p$-group or a direct product of copies of the additive group of rational numbers.

(5.1) Case $N$ torsion-free. Observe first that $G$ is monolithic with monolith $N$. For otherwise there is a normal subgroup $M \neq 1$ such that $M \cap N = 1$. Since $N \not\leq MN/M$, the subgroup $MN/M$ is minimal normal in the soluble $T$-group $G/M$ and hence is simple; therefore $N$ is cyclic of prime order, contrary to assumption.

Suppose that $G$ splits over $N$ and that $X$ is a complement of $N$. Then $C_X(N) \triangleleft G$ and, $G$ being monolithic, this implies that $C_X(N) = 1$ and $X$ acts faithfully on $N$. Finally $X$ is a soluble $T$-group and $N$ is not cyclic, so $G$ is of type VIII.

Consider next what happens if $G$ does not split over $N$. In this situation Lemma 8 shows that

$$[N, G'] = 1,$$

$G/N$ cannot be abelian; for if it were, $G'$ would equal $N$ and, since $L = [G', G] \neq 1$, Lemma 8 would imply that $G$ splits over $N$. Also, $G/C_G(N)$ is essentially an irreducible group of automorphisms of $N$; since $N$ is torsion-free, a theorem of Baer [1, Proposition] shows that $G/C_G(N)$--and hence $G/N$--cannot be periodic. If $G/N$ were a soluble $T$-group of type I, then $G/G'$ would be periodic [18, 3.1] and, by (9) so would $G/C_G(N)$. It follows that $G/N$ is a soluble $T$-group of type II and $G'/N$ is periodic. Commutation with a fixed element of $G'$ gives rise to a homomorphism of $G'/N$ into $N$ since $[N, G'] = 1$. However $\text{Hom}(G'/N, N) = 0$ because $G'/N$ is periodic and $N$ is torsion-free. Consequently $G'$ is abelian.

Now, $N$ being radicable, we can write $G = N \times R$ where $R = G'/N$. Clearly $R$ is the subgroup of all elements in $G'$ with finite order and $R \triangleleft G$. But $N$ is the monolith of $G$, so $R = 1$ and $G' = N$, contradicting the noncommutativity of $G/N$. Hence this case cannot arise.

(5.2) Case $N$ an elementary abelian $p$-group and $[N, G'] \neq 1$. Here Lemma 8 can be applied directly to show that $G$ splits over $N$; let $X$ be a complement of $N$ and suppose that $C = C_X(N) > 1$. Since $C \triangleleft G$, we can assert that $G/C$ is a soluble $T$-group. Moreover $N \not\leq NC/C$, so $N$ is cyclic of order $p$. It follows that $G'$ centralises $N$. This is impossible, so $C = 1$ and $G$ is of type VIII--note that $N$ is not cyclic.

(5.3) Case $N$ an elementary abelian $p$-group and $[N, G'] = 1$. This case leads to several different types of groups and will accordingly be analysed under several subheadings. Since $N$, but not $G$, is periodic, $G/N$ is abelian or soluble $T$ of type I or soluble $T$ of type II.

(5.31) Case $G/N$ abelian. Here $G' = N$ and $G$ splits over $N$ by Lemma 8. Let $X$ be a complement of $N$ in $G$ and suppose that $C = C_X(N) \neq 1$. Then $NC/C$--and hence $N$--is cyclic of order $p$. Therefore $X/C$ is finite and $C$ must be
nonperiodic. Let $F$ be a maximal torsion-free subgroup of $C$; then $F \triangleleft G$ since $[N, F] = 1$ and $X$ is abelian. Also, $C/F$ is periodic by maximality of $F$; consequently $G/F$ is periodic. However, Lemma 5 yields the contradiction $N = 1$ or $F = 1$. It follows that $C = 1$. If $N$ were cyclic, $X$ would be cyclic with order dividing $p - 1$ and $G$ would be finite. Hence $N$ is not cyclic and $G$ is of type VIII.

(5.32) Case $G/N$ a soluble $T$-group of type I. From the structure of soluble $T$-groups of type I it is seen that $G/G'$ is periodic and $G'/N$ nonperiodic and abelian. Choose a maximal torsion-free subgroup $F/N$ of $G'/N$; then $F \triangleleft G$ because $G/N$ is a $T$-group; also $G/F$ is periodic. By hypothesis $[N, G'] = 1$; thus $F' \leq N \leq 
abla(F)$. Therefore, for any $x, y$ in $F$, $1 = [x, y]^p = [x^p, y]$ and $F^p \leq \nabla(F)$; in particular, $F^p$ is abelian. Since $F/N$ is torsion-free and $N$ elementary abelian, $H = (F^p)^G$ is torsion-free. But clearly $H \triangleleft G$ and $G/H$ is periodic. Lemma 5 once again gives a contradiction, showing that this case cannot arise.

(5.33) Case $G/N$ a soluble $T$-group of type II. Here $G'/N$ is a nontrivial, periodic radicable group; moreover, if $C \cong C_{G'(G'/N)}$, then $C/N$ is periodic and abelian. Two possibilities must now be distinguished.

(5.331) Case $[N, C] \neq 1$. By Lemma 8 the group $G$ splits over $N$; let $X$ be a complement of $N$ and assume that $D = C_X(N) / 1$. Then $G/D$ is a soluble $T$-group, so it is metabelian; also $N \cap D = 1$, showing that $G$ is metabelian. Next $(G')^p / 1$ since $G'/N$ is radicable; therefore $G/(G')^p$ is a soluble $T$-group in which the elements of finite order form a proper subgroup; such a group cannot be of type I. Consequently, either $G' \cdot (G')^p$ or $G/(G')^p$ is soluble of type II, in which event $G'/G'/(G')^p$ is radicable, an obvious absurdity. It follows that $G' \cdot (G')^p$.

Also $G'$ is periodic and abelian; with the aid of Lemma 5, we deduce that $G'$ is a radicable abelian $p$-group. If $c \in C$, the mapping $a \mapsto [a, c]$ is a homomorphism of $G'$ into $N$ since $[G', C] \leq N$ and $[N, G'] = 1$. But $\text{Hom}(G', N) = 0$ because $G'$ is radicable and $N$ elementary abelian. Hence $[G', C] = 1$ and $[N, C] = 1$, which is contrary to hypothesis. Therefore $D = 1$ and $G$ is again of type VIII.

(5.332) Case $[N, C] = 1$. We shall show that $G$ is of type VI or VII by pursuing the following programme:

(i) $G'$ is a radicable abelian $p$-group.

(ii) $G$ is monolithic with monolith $N$.

(iii) Properties of $C = C_G(G'/N)$.

(iv) $C$ is abelian and $G$ is of type VI.

(v) $C$ is nilpotent of class 2 and $G$ is of type VII.

Since $C/N$ is abelian, $C$ is nilpotent of class $\leq 2$. Lemma 5 implies that $C$ is also a $p$-group. Commutation with a fixed element of $C$ produces a homomorphism of $G'/N$ into $N$ since $[G', C] \leq N$ and $[N, C] = 1$. But $\text{Hom}(G'/N, N) = 0$, so that

\[ [G', C] = 1 \quad \text{and} \quad C = C_G(G'). \]
In particular $G'$ is abelian. Furthermore $G' = (G')^p$ since otherwise $G/(G')^p$ would be soluble $T$ of type II. Therefore $G'$ is a radicable abelian $p$-group.

Let $g \in G$ and denote by $\tau_g$ the automorphism induced by $g$ in $G'$. Now $g$ induces in $G'/N$ a power automorphism which can be described by a $p$-adic integer unit, say $\alpha_g$. Writing $\theta$ for the power automorphism $a \rightarrow a^{\alpha_g}$ of $G'$, we see that $\tau_g^{-1}\theta$ acts trivially on $G'/N$. Therefore $\tau_g^{-1}\theta - 1 \in \text{Hom}(G', N) = 0$ and $\tau_g = \theta$.

Consequently

\begin{equation}
\alpha_g^a = \alpha_g^a \\
(a \in G', g \in G).
\end{equation}

The next point to establish is that $G$ is monolithic with monolith $N$; since (11) shows every subgroup of $G'$ to be normal in $G$, it will then follow that $G'$ is a $p^\infty$-group. Suppose there exists $M \triangleleft G$ such that $1 \neq M$ and $N \cap M = 1$. If $C \cap M = 1$, then $[G', M] \leq C \cap M = 1$ and $M \leq C$. This cannot be, so $C \cap M \neq 1$ and there is no loss in assuming that $M \leq C$. Now $G/C$—and hence $G/M$—is nonperiodic and $G/M$ is not abelian since $N \leq G'$. Hence $G/M$ is also a soluble $T$-group of type II. It follows that $C/M$—and hence $C$—is abelian. Let $g \in G$; then $g$ induces in $G'/N$—and therefore in $C/N$—the automorphism $a \rightarrow a^{\alpha_g}$; here we use the radicability of $G'/N$. But $g$ also induces $a \rightarrow a^{\alpha_g}$ in $G'$ by (11), and therefore in $G'M/M$ and $C/M$. Hence $a^{\alpha_g} = a^{\alpha_g}$ for all $a \in C$. Since $C = C_G(G')$ and $G'$ has infinite exponent, Lemma 6 provides a contradiction. Thus we conclude that $G$ is monolithic with monolith $N$.

Let $Z$ be the centre of $C$; then $G' \leq Z$ by (10) and since $G'$ is radicable,

\begin{equation}
Z = G' \times D_1
\end{equation}

for some $D_1 \leq Z$. Clearly $D_1N \triangleleft C \triangleleft G$, so $D_1N \triangleleft G$ and $[D_1, G] \leq G' \cap (D_1N)$, which shows that

\begin{equation}
[D_1, G] \leq N.
\end{equation}

If $d \in D_1$ and $g \in G$, then $1 = [d, g]^p = [d^p, g]$; therefore $D_1^p \leq \zeta(G)$ and $D_1^p \triangleleft G$; since $N$ is the monolith of $G$, it follows that $D_1^p = 1$ and $D_1$ is an elementary abelian $p$-group.

Write $T/G'$ for the torsion-subgroup of $G/G'$; then $C \leq T$ since $C/N$ is periodic. Moreover $C \neq G'$ by Lemma 6. The structure of soluble $T$-groups of type II provides the following information: $C/G'$ has finite exponent $p^e$ and $\alpha_g = 1 \mod p^e$ for all $g \in G$ [18, Theorem 4.3.1]. Observe that $e > 0$, so that

\begin{equation}
\alpha_g = 1 \mod p.
\end{equation}

Now suppose that $g \in T$; since $\alpha_g$ is a $p$-adic integer unit of finite order satisfying (14), there are the following possibilities: either $p$ is odd and $\alpha_g = 1$ or $p = 2$ and $\alpha_g = \pm 1$. Thus $T/C$ has order 1 or 2 and we can write

\begin{equation}
T = \langle t, C \rangle
\end{equation}
where either $t = 1$ or $t = -1$; in either case $t^2 \in C$ and $t^2N$ has order 1 or 2; therefore $t^2$ has order dividing 4. Next we show that $t^2$ is in the centre of $G$; let $t \neq 1$. If $g \in G$ then $t^6 = ta$ where $a \in G'$. Hence $(t^2)^6 = (ta)^2 = t^2$ since $a^t = a^{-1}$. In particular $(t^2) = G$. If $t^2 \neq 1$, then $N \leq (t^2)$. Let $p = 2$ and $G' = \langle a_1, a_2, \ldots \rangle$ where $a_{i+1} = a_i$ and $a_2 = 1$; then $N = \langle a_1 \rangle$. Clearly $t^2 \in Z$; let us consider the position of $t^2$ in $Z$. If $t^2 \in G'$, then $t^2 = 1$ or $a_1$ since $a_2 = a_1^{-1}$ if $t \neq 1$. Suppose $t^2 \notin G'$; then since $t^2$ has order 2 or 4, it belongs to $\langle a_2 \rangle \times D_1$, and since $a_1 \in \langle t^2 \rangle$, we can assume that $t^2 = a_2u$ where $1 \neq u \in D_1$. Thus the possibilities for $t^2$ are 1, $a_1$ and $a_2u$.

The group $T/G'$ has finite exponent and this is well known to imply that $T/G'$ is a direct factor of $G/G'$ (see [13, Theorem 8]); let

$$G/G' = (T/G') \times (Y/G').$$

From (16) and (15) we obtain

$$G = (t, C, Y).$$

Consider the case when $C$ is abelian. Here $C = Z = G' \times D_1$ and (17) becomes $G = (t, D_1, Y)$. Now set $W = (t, Y)$. Since $D_1$ is elementary abelian, we can write $D_1 = (\langle t, G' \rangle \cap D_1) \times D$, say. Hence $G = (t, D, Y) = WD$—observe that $W < G$ since $G' \leq Y \leq W$. Since $T \cap Y = G'$, we have $W \cap D \leq (t, G') \cap D = 1$. Hence $G = WD$ and $W \cap D = 1$.

Next we analyse the structure of $W$. First

$$W/W \cap C \simeq WC/C = G/C.$$

The map $gC \to a_g$ is an isomorphism of $G/C$ with a nonperiodic group $\Gamma$ of $p$-adic integers all of which are congruent to 1 modulo $p$. Now $W \cap C = W \cap (G' \times D_1) = G' \times (W \cap D_1)$. Also

$$W \cap D_1 = (t, Y) \cap D_1 = (t, G') \cap D_1 = \langle t^2, G' \rangle \cap D_1.$$  

If $t^2 \in G'$, then $W \cap D_1 = 1$ by the last equation; otherwise $t^2 = a_2u$ and $\langle t^2, G' \rangle \cap D_1 = \langle u \rangle$, so $W \cap D_1 = \langle u \rangle$. Hence $W \cap C = G'$ or $G' \times \langle u \rangle$ according as $t^2 \in G'$ or $t^2 \notin G'$. Also, $W/W \cap C \simeq \Gamma$ by (18). Suppose that $t^2 = a_1$; if there exists a $d$ in $D$ such that $t^d \neq t$, then $d^t = da_1$ by (13); thus $(td)^2 = t^2a_1^2 = 1$, and, replacing $t$ by $td$, we can assume that $t^2 = 1$. In other words, we can exclude $t^2 = a_1$ unless $[D, t] = 1$. Suppose $t^2 \notin G'$; then $p = 2$ and $t^2 = a_2u$, $(u \in D_1)$. Let $g \in G$; since $t^2$ belongs to the centre of $G$, we have $(a_2u)^g = a_2u$ and $u^g = a_2^{-1}u$. Thus $u^g = u$ or $a_1u$ according as $a_g = 1 \mod 4$ or $a_g \neq 1 \mod 4$.

Finally, $D$ centralises $N$ and $W/N$. If $t^2 \in G'$, then in addition $D \neq 1$; for $W' = G'$ and $C_W(W') = W \cap C = G'$; thus $W$ is not a JNT-group by Lemma 6 and $D \neq 1$. The centre of $G$ contains no element of order $p$ except $a_1$; since $N$ is the monolith, $G$ is of type $\text{VI}(a)$ or (b) according as $t^2 \in G'$ or $t^2 \notin G'$.  

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The remaining possibility under heading (5.332) is that $C$ is nilpotent of class exactly 2. The centre of $C$ is $Z$ and $C/N$ is abelian; thus

$$N = C' < G' < Z < C.$$ 

If $x, y \in C$, then $1 = [x, y]^p = [x^p, y]$, showing that $C^p \leq Z$ and $C/Z$ is an elementary abelian $p$-group.

Let us prove next that $C^p = G'$. Now $G' \leq C^p$ is immediate, and since $C^p$ is abelian, we can write $C^p = G' \times F$. Suppose $x$ is a nontrivial element of $F$; then $x = c^p a$ for some $c \in C$ and $a \in N$; for $C/N$ is abelian. Let $g$ be any element of $G$; then $c^g = c^a g b$ for some $b \in N$. Hence

$$(c^p)^g = (c^a g b)^p = (c^p)^g$$

and $(c^p) \not< G$. Now $a \neq x$ since $G' \cap F = 1$; thus $c^p \neq 1$ and $a \in \langle c^p \rangle$. Therefore $x = c^p a \in \langle c^p \rangle$ and $(x) \not< G$. This gives the contradiction $N \leq \langle x \rangle$. Hence $F = 1$ and $C^p = G'$.

Let $x, Z; \lambda \in \Lambda$ be a basis for the elementary abelian $p$-group $C/Z$ and set $X = \langle x_{\lambda}; \lambda \in \Lambda \rangle$. Then $C = XZ = XG'D_1$. Let $y \in (XG') \cap D_1$ and write

$$y = x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} a$$

where $a \in G'$, the $n_i$ are integers and the $\lambda_i$ are distinct elements of $\Lambda$. Then $x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} \in Z$ and the linear independence of the $x_{\lambda_i} Z$ implies that $p | n_i$ for $i = 1, \ldots, r$. Hence $y \in C^p G' = G'$ and $y \in G' \cap D_1 = 1$.

Writing $E = XG'$ we obtain $C = E \times D_1$.

Suppose that $x_{\lambda}^p \neq 1$. Since $G' = C^p$, there is an element $a$ of $G'$ such that $x_{\lambda}^p = a^p$. This implies that $x_{\lambda} a^{-1} p = 1$. Replacing $x_{\lambda}$ by $x_{\lambda} a^{-1}$ we may assume that $x_{\lambda}^p = 1$ for all $\lambda \in \Lambda$. This implies that $X \cap Z = N$; for let $y \in X \cap Z$; since $X' \leq N$, we can write $y = x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} \mod N$ where the $\lambda_i$ are distinct. Thus $p | n_i$ for all $i$ and $y \in N$ as required. Next $N$ is actually the centre of $X$ since $\zeta(X) \leq X \cap Z = N$. Thus $X' = \zeta(X) = N$ and $X/N \cong C/Z$. Consequently $X$ is an extra-special $p$-group. Since $X \cap G' = N = \langle a_1 \rangle$, the group $E$ is a direct product of $X$ and $G'$ in which the centre of $X$ and $\langle a_1 \rangle$ are amalgamated.

Consider now the position of $t^2$; if $t^2 \not< G'$, then $p = 2$ and $t^2 = a_2 u$ where $u \in D_1$. If $t^2 \in G'$ and $t^2 \neq 1$, then $p = 2$ and $t^2 = a_1$. Suppose that $[X, t] \neq 1$; then $[x_{\lambda}, t] \neq 1$ for some $\lambda \in \Lambda$. Since $x_{\lambda} N$ has order 2, it is centralised by $t$ and $x_{\lambda}^{t} = x_{\lambda} a_1$; thus $(tx_{\lambda})^2 = a_1^2 x_{\lambda}^2 = 1$. If, however, $[X, t] = 1$, write $a_1 = x^2$ for some $x \in X$ (note $N = X^2$); then $(tx)^2 = 1$. Therefore, if $t^2 \in G'$, we may assume that $t^2 = 1$.

From $C = XZ$ and equations (12) and (17) we obtain $G = \langle t, X, Y, D_1 \rangle$. Now define $W = \langle t, X, Y \rangle$. Writing

$$D_1 = ((t, G') \cap D_1) \times D,$$

we have $G = WD$ and $W \not< G$. Suppose that $w \in W \cap D$ and, using $G' \leq Y$, write
\[ w = t^i xy \] where \( x \in X \) and \( y \in Y \); then \( y \in T \cap Y = G' \) and \( t^i \in C \), which shows that \( i \) may be assumed even. Since \( t^2 \in Z \), it follows that \( x \in X \cap Z = N \) and \( w \in \langle t, G' \rangle \cap D = 1 \). Hence \( W \cap D = 1 \).

We turn now to the structure of \( W \). First \( W \cap C = W \cap (E \times D_1) = E \times (W \cap D_1) \); moreover, as above,

\[ W \cap D_1 = \langle t, X, G' \rangle \cap D_1 = \langle t^2, X, G' \rangle \cap D_1 = \langle t^2, G' \rangle \cap D_1. \]

If \( t^2 \in G' \), then \( W \cap D_1 = 1 \); otherwise \( t^2 = a_2 u \) and \( W \cap D_1 = \langle u \rangle \). Thus \( W \cap C = E \) or \( E \times \langle u \rangle \) according as \( t^2 \in G' \) or \( t^2 \notin G' \). Moreover \( W/W \cap C \cong G/C \) and \( G/C \) is isomorphic with \( \Gamma \), a group of \( p \)-adic integers all of which are congruent to 1 modulo \( p \). If \( t^2 \notin G' \) then one shows (as in the case \( C \) abelian) that \( u^k = u \) or \( a_2 u \) according as \( a_g \equiv 1 \mod 4 \) or \( a_g \not\equiv 1 \mod 4 \). Since \( a_g \equiv 1 \mod p \) for all \( g \in G \), the factor \( X/N \) is central in \( G \) and \( [X, G] \leq N \).

Finally \( D \) acts faithfully on \( W \) and centralises \( W/N \) and \( N \). Therefore \( G \) is of type VII(a) or (b) according to whether \( t^2 \) is or is not in \( G' \).

6. Periodic soluble \( JNT \)-groups. Throughout this section \( G \) will denote a non-nilpotent \( JNT \)-group which is both soluble and periodic; \( N \) is a minimal normal subgroup of \( G \) contained in \( L = [G', G] \). Thus \( N \) is an elementary abelian \( p \)-group for some prime \( p \).

(6.1) Case \( [N, G'] \neq 1 \). Here \( G \) is of type VIII; the argument is that of (5.2).

(6.2) Case \( [N, G'] = 1 \). Since \( G/N \) is abelian, \( G' \) is nilpotent in this case; hence \( G' \) is a \( p \)-group by Lemma 5. Consequently \( G \) has a unique Sylow \( p \)-subgroup \( P \) containing \( G' \).

We shall require the equation

\[ [N, C_P(G'/N)] = 1. \]

To prove this let \( x \) in \( P \) centralise \( G'/N \). Since \( C_G(G'/N)/N \) is nilpotent, \( \langle x, N \rangle < G \), which shows that \( [N, x] < G \). Assuming that \( [N, x] \neq 1 \), we find that \( N = [N, x] = [N, x^{p'}] = 1 \), a contradiction.

(6.21) Case \( [N, G'] = 1 \) and \( P/N \) abelian. Here \( P \) centralises \( G'/N \) since \( G' \leq P \); therefore

\[ [N, P] = 1 \]

by (19). Also \( P' \leq N \) and (20) shows that \( P \) is nilpotent of class at most 2. Since \( G/P \) is abelian, \( [(C_G(P))', C_G(P)] = 1 \) and \( C_G(P) \) is nilpotent. Lemma 5 shows that \( C_G(P) \) is a \( p \)-group; hence

\[ C_G(P) \leq P. \]

Now it is necessary to distinguish two subcases.

(6.211) Case \( P \) abelian. Suppose that \( P \) contains an element of order \( p^2 \)
and define $P_1$ to be the subgroup of all $a$ in $P$ such that $aP^2 = 1$. Thus $1 \neq P_1 \triangleleft G$ and $G/P_1$ is a $T$-group. Let $g \in G$; then $g$ induces a power automorphism $a \mapsto a^\alpha$ in $P/P_1$—here $\alpha$ is a $p$-adic integer unit. Writing $\xi$ for the automorphism of $P$ induced by $g$ and $\eta$ for the power automorphism $a \mapsto a^\alpha$ of $P$, we see that $\beta = \xi^{-1} \eta$ is an automorphism of $P$ which acts trivially on $P/P_1$. Therefore, for any $a \in P$ we have $a^\beta = ab^\beta$ for some $b \in P_1$; hence $(a^\beta)^\beta = (ab^\beta)^\beta = a^\beta$. Therefore $(b^\beta)^\beta = b^\beta$; from this and $a^\beta = ab^\beta$ it follows that $a^\beta = ab^\beta a = 1$. Hence $1 = \theta^\beta = \xi^{-p} \eta^p$ since a power automorphism commutes with every automorphism of $P$.

Now by (21) we have $C_G(P) = P$, so the order of $\xi$ is prime to $p$. Consequently $\xi^p = \eta^p$ implies that $\xi \in \langle \eta \rangle$ and $\xi$ is a power automorphism of $P$. By Lemma 5.2.2 of [18] the group $G$ is a $T$-group. In view of this contradiction $P$ is an elementary abelian $p$-group.

Next it will be shown that $G$ splits over $P$. By hypothesis $C_G(N) \not\supset G'$; therefore $G/C_G(N)$ is abelian. $G/C_G(N)$ can be regarded as an irreducible group of linear transformations of $N$ qua vector space. This implies that $G/C_G(N)$ is isomorphic with a periodic subgroup of the multiplicative group of a field. Therefore $G/C_G(N)$ is locally cyclic and, in particular, countable (see [6, p. 296]).

Since elements of $G$ induce power automorphisms in the elementary abelian $p$-group $P/N$, the group $C_G(N)/(C_G(N) \cap C_G(P/N))$ is cyclic of order dividing $p - 1$. If $g$ centralises both $N$ and $P/N$, then $g$ induces in $P$ an automorphism of order 1 or $p$. But $P = C_G(P)$, so $g \in P$. Thus

$$C_G(N) \cap C_G(P, N) = C_G(P) = P$$

and $G/P$ is countable. Lemma 7 shows that $G$ splits over $P$, say $G = PX$ and $P \cap X = 1$. Moreover $X$ acts faithfully on $P$ in view of (22).

Consider the situation when $N$ is the monolith of $G$. Assume that $N \neq P$ and let $a \in P \setminus N$. If $a^G$ were finite, it would be a direct product of minimal normal subgroups of $G$ by Maschke's theorem—observe that $X$ has no elements of order $p$. This is consistent only with $a^G = N$ because $N$ is the monolith. Therefore $a^G$ must be infinite—and hence so is $X$. Since $X/C_X(P/N)$ is finite, $C_X(P/N) \neq 1$.

Also $C_X(P/N) \triangleleft X$ and $X$ is a soluble $T$-group, so $C_X(P/N) \cap C_X(X') \neq 1$; let $x$ be a nonunit element of this intersection. Then $(x) \triangleleft C_X(X') \triangleleft X$, so that $(x) \triangleleft X$; consequently $[P, x] \triangleleft G$ and $C_P(x) \triangleleft G$. If $C_P(x) = 1$, then $N \subseteq C_P(x)$ and $x$ belongs to $C_G(P/N) \cap C_G(N) = P$ by (22); thus $x \in P \cap X = 1$. Therefore $C_P(x) = 1$. Now $x$ centralises $P/N$ and consequently $[P, x] \subseteq N$. Evidently $[N, x] \triangleleft G$ and $[N, x] \neq 1$, which shows that $N = [N, x] = [P, x]$. Thus $[P, x] = [P, x, x]$. Let $a \in P$; then $[a, x] = [b, x, x]$ for some $b \in P$, and $C_P(x) = 1$ implies that $a = [b, x]$. Thus $P \subseteq [P, x] = N$. Consequently $N = P$ and $G$ is of type VIII.

We are left with the following situation; there exists a nontrivial $M \triangleleft G$ with $M \cap N = 1$. If $M \cap P = 1$, then $M \leq C_G(P) = P$; therefore $M \cap P = 1$ and we can...
assume that $M \leq P$. Also $M \leq MN/N$ shows that every subgroup of $M$ is normal in $G$ and we can assume $M$ to have order $p$. Also $N$ has order $p$ since $N \leq NM/M$.

Suppose now that $P > M \times N$ and choose $a \in P \setminus (M \times N)$. The automorphism groups induced by $G$ in $P/M$ and $P/N$ are both finite. Therefore $G/C_G(P)$ is finite, from which it follows that $a^G$ is finite. Maschke's theorem implies that $a^G$ is a direct product of minimal normal subgroups of $G$; therefore there exists a minimal normal subgroup $L$ of $G$ contained in $P$ such that $L \not\leq M \times N$. Then $LM/M \leq LN/N$. Let $g \in G$; then $g$ induces in $P/M$ and $P/N$ power automorphisms which both have the form $a \mapsto a^n$ since they must agree on $LM/M$ and $LN/N$. Hence $a^g = a^n$ for all $a \in P$, a situation we have already seen to be impossible (by Lemma 5.2.2 of [18]).

Hence $P = M \times N$ and $X$ is isomorphic with a subgroup of $GL(2, p)$ which is diagonal because $X$ induces power automorphism groups in $M$ and $N$; this subgroup $X$ is not scalar since it does not induce a group of power automorphisms in $P$.

Clearly $p$ is odd and $G$ is of type IV.

(6.212) Case $P$ nilpotent of class 2. Since $P/N$ is abelian, $P' = N$. Consider the centraliser of $P/N$. If $g \in C_G(P/N)$, then $[g, P, P] = 1$ by (20); therefore, by the Three Subgroup Lemma, $[g, P'] = 1$, i.e. $[g, N] = 1$. Now let $a \in P$; then $a^g = ab$ where $b \in N$. Hence $a^g b^g = ab^g = a$ since $[g, N] = 1$. It follows that $C_G(P/N)/C_G(P)$ is a $p$-group. But $C_G(P) \leq P$ by (21); therefore $C_G(P/N) \leq P$ and

$$C_G(P/N) = P.$$  

Next, if $p = 2$, a periodic group of power automorphisms of $P/N$ has order a power of 2 and equation (23) yields $P = G$, i.e. $G$ is nilpotent. Thus $p$ is an odd prime and therefore $P$ is a regular $p$-group. Also $N$ is the monolith of $G$; for suppose that $1 \neq M \triangleleft G$ and $M \cap N = 1$; then $PM/M$ and $PN/N$ are abelian, being Dedekind groups without elements of order 2, and therefore $P$ is abelian, contrary to hypothesis.

Since $P \neq G$, we can find $g \in G \setminus P$ and (23) shows that $g$ cannot centralise $P/N$. Let $g$ induce in $P/N$ the power automorphism $a \mapsto a^\alpha$; here $\alpha$ is a $p$-adic integer unit $\neq 1$. If $a, b \in P$, then $a^g = a^\alpha \mod N$ and $b^g = b^\alpha \mod N$; hence

$$[a, b]^g = [a^\alpha, b^\alpha] = [a, b]^{\alpha^2}$$

since $[N, P] = 1$. From this it follows that $<[a, b]\triangleleft G$. Since $N$ is the monolith of $G$, the order of $N$ is $p$; let $N = \langle \alpha \rangle$, say.

Suppose that $P^p > 1$; then $N \leq P^p$ and consequently $a = b^\alpha$ for some $b \in P$ since $P$ is regular. Now $b^g = b^{a^\alpha}$ for some $c \in N$, and $a^g = (b^g)^{\alpha} = b^{a^\alpha} = a^\alpha$.

But equation (24) shows that $a^g = a^{\alpha^2}$; therefore $\alpha^2 = a \mod p$ and $\alpha = 1 \mod p$.

If $P/N$ has finite exponent $p^e$, the congruence $a^{p^{e-1}} \equiv 1 \mod p^e$ implies that $g$ induces in $P/N$ an automorphism of order a power of $p$; therefore $g \in P$ by (23).
If, however, $P/N$ has infinite exponent, $\alpha$ must have finite order; this, together with $\alpha \equiv 1 \mod p$ and $p > 2$, implies that $\alpha = 1$. These arguments indicate that

$$P^p = 1.$$ 

Next the centre of $P$ will be identified; call this $Z$. Clearly $N \leq Z$. Let $1 \neq a_1 \in Z$; then $\langle a_1, N \rangle < G$, so $a_1^G \leq \langle a_1, N \rangle$, which implies that $a_1^G$ is finite. Now $C_G(a_1^G) \geq P$; therefore Maschke's theorem can be applied to $a_1^G$; in the usual way it follows that $a_1^G = N$. Thus $Z = N$.

Now $P/N$ is elementary abelian by (25); hence $P$ is an extra-special $p$-group. Choose a basis for $P/N$, say $\{x_\lambda N : \lambda \in \Lambda \}$. Then

$$[x_\lambda, x_\mu] = a^f(\lambda, \mu)$$

where $f$ is a nondegenerate alternating bilinear form. Now $G/P$ is cyclic with order $q$ dividing $p - 1$ since $P = CG(P/N)$. Hence there is an element $g$ with order $q$ such that $G = P\langle g \rangle$ and $P \cap \langle g \rangle = 1$. Moreover $\langle g \rangle$ acts faithfully on $P$.

g induces in the elementary abelian group $P/N$ a power automorphism of the form $x \mapsto x^n$ where $1 < n < p$. Thus

$$x_\lambda^g = x_\lambda^n a_\lambda^n, \quad (\lambda \in \Lambda),$$

for certain integers $n_\lambda$ satisfying $0 \leq n_\lambda < p$. A suitable change of basis will simplify these equations. Suppose that $n_\lambda \neq 0$. Since $f$ is nondegenerate, there is a $\mu \in \Lambda$ such that $f(\lambda, \mu) \neq 0 \mod p$. We shall replace $x_\lambda$ by a suitable element of the form $\tilde{x}_\lambda = x_\lambda^{s_\lambda^t}$; A brief computation using (26) and (27) yields $\tilde{x}_\lambda^g = \tilde{x}_\lambda^{s_\lambda^t} u^\mu$ where $u = sn_\lambda + tn_\mu + st(\frac{r}{2})f(\lambda, \mu)$. We wish to show that $u \equiv 0 \mod p$ can be solved for $s$ and $t$ with $s \neq 0 \mod p$ and $t \neq 0 \mod p$. This amounts to solving

$$xn_\lambda + yn_\mu + z = 0 \mod p$$

for $x \neq 0 \mod p$ and $y \neq 0 \mod p$ where $z = (\frac{r}{2})f(\lambda, \mu)$; notice that $z \neq 0 \mod p$.

Since $n_\lambda \neq 0 \mod p$, we need only look for a $y$ such that $yn_\mu + z \neq 0 \mod p$. If $n_\mu = 0$, any $y \neq 0$ will do; if $n_\mu \neq 0$, we can choose $y$ so that $1 \leq y < p$ and $yn_\mu + z \neq 0 \mod p$ since $p > 2$. Consequently (28) has a solution of the required sort.

Now replace $x_\lambda$ by $\tilde{x}_\lambda$, observing that we retain a basis for $P/N$. Performing this operation whenever necessary, we arrive at a basis for which $x_\lambda = x_\lambda^n$ for all $\lambda \in \Lambda$. Thus $G$ is of type $V$.

(6.22) Case $[N, G'] = 1$ and $P/N$ nonabelian. If a soluble $p$-group has the property $T$ and is not abelian, then $p = 2$ [18, Lemma 4.2.1]. Thus $P$ is a 2-group. Also (by Lemma 2.4.1 of [18]) $L/N$ is a radicable abelian 2-group where, as usual, $L = [G', G]$. Define $C = C_G(G'/N)$ and note that $C/N$ is nilpotent of class $\leq 2$. Moreover $G/C$ has order 1 or 2 because $\pm 1$ are the only 2-adic integers with finite order.
Let $x \in C \cap P$; then $aN \rightarrow [a, x]$ is a homomorphism of $L/N$ into $N$ since $[N, C \cap P] = 1$ (see equation (19)). But $\text{Hom}(L/N, N) = 0$; therefore

$$[L, C \cap P] = 1.$$ 

Since $L \leq G' \leq C \cap P$, it follows that $L$ is abelian.

Our next aim is to prove that $G$ is a 2-group. Since $|G:C| = 1$ or 2, all elements of $G$ with odd order belong to $C$. Now $C/N$ is a Dedekind group, so the elements in $C/N$ which have odd order form an abelian subgroup, say $Q/N$.

Assume that $Q \neq N$. From this it follows that $[N, Q] \neq 1$; for if $[N, Q] = 1$, the group $Q$ is nilpotent, and, since $Q \vartriangleleft G$, we deduce from Lemma 5 that $Q$ is a 2-group and $Q = N$. Now $[N, Q] \neq 1$ implies that $L = N$. For suppose $L > N$; then $L/N$ is a nontrivial radicable abelian 2-group and $L^2 \neq 1$; therefore $L, L^2$ is radicable, which shows that $L$ is radicable. Now commutation with a fixed element of $C$ induces a homomorphism of $L$ into $N$, and yet $\text{Hom}(L, N) = 0$; thus $[L, C] = 1$ and in particular $[N, Q] = 1$, a contradiction. It follows that $L = N$ and $G/N$ is Dedekind; thus $C = C_G(G') = G$ and (29) becomes $[L, P] = 1$. Now $[N, Q] = 1$ implies that $G$ splits over $N$, by Lemma 8; say $G = NX$ and $N \cap X = 1$. Therefore $P = P \cap (NX) = N(\vartriangleleft P \cap X)$. Since $P/N$ is not abelian, $P \cap X \neq 1$. Also $[N, P \cap X] \leq [L, P] = 1$, so $P \cap X \vartriangleleft NX = G$. Thus $G/P \cap X$ is a $T$-group and the isomorphism $N \cong N(P \cap X)/P \cap X$ shows that $|N| = 2$. Consequently $N \leq \zeta(G)$ which implies that $G$ is nilpotent. This contradiction establishes that $G$ is a 2-group. Equation (29) now yields

$$[L, C] = 1 \quad \text{and} \quad C = C_G(L).$$

Observe that $G/N$ is not a Dedekind group; for if it were, $L = N$ and $C = G$, so that (30) would become $[L, G] = 1$, i.e. $G$ is nilpotent. Also $L$ is radicable; for $L > N$ and this, as has already been seen, implies that $L$ is radicable.

$G/N$ is a soluble 2-group with the property $T$ and it is also nonnilpotent. By Lemma 4.2.1 of [18] this means that one can write $G = \langle C, t \rangle$ where $t$ transforms each element of $C/N$ into its inverse and $t^2 \in C$; also, of course, $C/N$ is abelian and of infinite exponent. Equation (30), together with the commutativity of $C/N$, implies that $C$ is nilpotent of class at most 2. Let $\sigma$ be the automorphism $a \rightarrow a^{-1}$ of $L$ and write $\tau$ for the automorphism of $L$ induced by $t$. Then $\tau^{-1}\sigma$ is trivial on $L/N$ and $\tau^{-1}\sigma - 1 \in \text{Hom}(L, N) = 0$. Therefore $\tau = \sigma$ and

$$a^t = a^{-1}, \quad (a \in L),$$

which shows that $t$ centralises $N$. Since $G = \langle C, t \rangle$, equation (30) permits us to conclude that $[N, G] = 1$.

Suppose there exists $M \vartriangleleft G$ with $M \neq 1$ and $M \cap N = 1$. Since $M \cong MN/N$, one can assume that $M$ has order 2. Also $L \leq M$, so $G/M$ is not a Dedekind group and its structure is similar to that of $G/N$. In particular $CM/M$—and hence $C$—is abelian. Now $t$ transforms elements of $L$ into their inverses. It follows that $a^t =
a^{-1} for all a \in C, which is impossible by Lemma 6. Hence G is monolithic with monolith equal to N. This indicates that L is of type $2^\infty$ and N has order 2.

Define Z to be the centre of C. Then L \leq Z and, L being radicable, we may write Z = L \times D. Suppose that D contains an element d of order 8; then $d^4 = a^{-1}a$ for some a in N. Hence $(d^2)^i = (d^{-1}a)^2 = d^{-2} = d^2$. Since $d^2 \in Z$, it follows that $d^2$ is in the centre of G and $1 \neq \langle d^2 \rangle \triangleleft G$; this is impossible since $\langle d^2 \rangle \cap N = 1$. Thus D is elementary abelian. This implies that DN/N lies in the centre of G/N, so that

$[D, G] \leq N$.

If $d \in D$, the mapping $xC \rightarrow [x, d]$ is a homomorphism of G/C into N since $[C, D] = 1 = [N, G]$. Now $C_D(G) \triangleleft G$, so $C_D(G) = 1$, and, since Hom(G/C, N) has order 2, we must have $|D| = 1$ or 2. Write D = (d).

Consider next the position of t. Let $a_1, a_2, \ldots$ be a canonical set of generators for the $2^{\infty}$-group L. If c \in C, then $c = c^{-1}a$ for some a in N. Hence $c^2 = (c^{-1}a)^2 = c$. On account of $G = \langle C, t \rangle$ it follows that $t^2 \in \zeta(G)$; in particular $\langle t^2 \rangle \triangleleft G$. Therefore, either $t^2 = 1$ or $t^2 = t^2_1$. Also $t^2 \in Z$ and since $t^2 N$ is centralised by t, the element $t^2$ has order dividing 4. Thus $t^2 \in \langle a_1 \rangle \times \langle d \rangle$ and the possibilities for $t^2$ are 1, $a_1$, or $a_2 d$ (if $d \neq 1$); for if $d \neq 1$, then $d^2 = a_1 d$ by (32) since $\langle d \rangle$ cannot be normal in G.

(6.221) Case C abelian. Here $C = Z = L \times D$, and $d \neq 1$ by Lemma 6. Since $d^2 = a_1 d$, we have $(td)^2 = t^2 a_1 d^2 = t^2_1$. Hence $t^2 = a_1$ implies that $(td)^2 = 1$. Therefore we can assume that either $t^2 = 1$ or $t^2 = a_2 d$; the order of t is 2 or 8. Thus G is of type II.

(6.222) Case C nilpotent of class 2. Since C/N is abelian, $C' = N \leq Z$. If x and y belong to C, then $1 = [x^2, y] = [x, y]$, showing that C/Z is elementary abelian. Choose a basis for C/Z, say $\{x_\lambda : \lambda \in \Lambda\}$; then $x_\lambda^2 = a_\lambda d^i$ where $a_\lambda \in L$ and $i = 0$ or 1. Now $a = b^2$ for some $b \in L$ and $(x_\lambda b^{-1})^2 = x_\lambda^2 b^{-1} = d^i$. Write $x_\lambda = x_b^{-1}d^i$; then $x_{\lambda'} = x_{\lambda'}^{-1}c$ for some $c \in N$ and $(x_{\lambda'}^{-1})^i = (x_{\lambda'}^{-1}c)^2 = x_{\lambda'}^{-2}$. It follows that $\langle d^i \rangle \triangleleft G$, which can only mean that $d^i = 1$ and $x_{\lambda'}^2 = 1$. In short, we can assume that

$x_\lambda^2 = 1$

for all $\lambda \in \Lambda$.

Define $X = \langle x_\lambda : \lambda \in \Lambda \rangle$. From (33) it follows that $X^2 = X'$; also $C = XZ$, so $N = C' = X'$ and $N = X' = X^2$. Suppose that $u \in X \cap Z$ and write $u = x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} a$ where $a \in N$, the $n_i$ are integers and the $\lambda_i$ are distinct elements of $\Lambda$. The independence of the $x_{\lambda_i} Z$ indicates that each $n_i$ is even; thus $u \in X^2 N = N$. Consequently

$X \cap Z = N$. 

Therefore $\zeta(X) \leq X \cap \zeta(C) = X \cap Z = N$, and $\zeta(X) = N$. Also, $X/N \cong C/Z$, an elementary abelian 2-group. We conclude that $X$ is an extra-special 2-group generated by elements of order 2. Clearly the group $C$ is a direct product of $X$ and $Z$ in which $\zeta(X)$ and $\langle a_1 \rangle$ are amalgamated.

It has been remarked that $t^2 = 1$, $a_1$ or $a_2d$ (if $d \neq 1$); in fact the second possibility can be discarded if $t$ is chosen suitably. The argument for this has already been given in the last part of (5.332).

Since $t$ acts trivially on both $X/N$ and $N$, the map $\sigma: xN \mapsto [x, t]$ is an element of $\text{Hom}(X/N, N)$. If $d \neq 1$, one can assume that $\sigma = 0$ and $[X, t] = 1$. For in this case if $x^t_a = x^a$, we obtain $(x^a)^t = x^a$ while $(x^a)^d = 1$. Thus $G$ is of type III.

[in conclusion, observe that even if $d = 1$ one can still take $\sigma = 0$ at the expense of losing $x^2_A = 1$; for $(x^A_a)^2 = x^A_a$ if $x^A_a \neq x^A$.

7. Proof of Theorem 1 concluded. It remains to show that a group $G$ of types I to IX is a JNT-group; in each case $G$ is obviously soluble. One first observes that in no case is $G$ a T-group. For types II, IV and VIII this is clear. For types I, III, V and VII it follows from the structure of Dedekind groups. If $G$ is of type VI(a), then $D$ is a nonnormal subnormal subgroup, as is $\langle u \rangle$ if $G$ is of type VI(b). If $G$ is of type IX, then $1_F$ generates a nonnormal subnormal subgroup since $X \neq \langle -1_F \rangle$.

Next it must be shown that every proper factor group of $G$ is a T-group. If $G$ is of type I or VIII this is clear. All types except IV and IX are monolithic and if $N$ is the monolith one merely has to verify that $G/N$ is a T-group. If $G$ is of type II or III, then $N = \langle a_1 \rangle$ and $G/\langle a_1 \rangle$ fits the prescription for a soluble 2-group with the property $T$ (see [18, Theorem 3.1.1]). If $G$ is of type V, then $N = \zeta(P)$ and $G/N$ is a T-group by Lemma 5.2.2 of [18]. If $G$ is of type VI or VII, then $N = \langle a_1 \rangle$ and $H = G/N$ has a normal $p^\infty$-subgroup $K$ such that $H/K$ is abelian and all subgroups of $C_H(K)$ are normal in $H$; a subnormal subgroup $S$ of $H$ either contains $K$ or lies in $C_H(K)$; hence $S \lhd H$.

Turning to the nonmonolithic groups, we see that in type IV a nontrivial normal subgroup $N$ of $G$ contains one of the two normal subgroups of order $p$; hence $G/N$ is a T-group by Lemma 5.2.2 of [18].

This leaves us with the case where $G$ is of type IX. Let $1 \neq N \lhd G$; then certainly $N \cap F$ is nontrivial. Let $0 \neq f \in N \cap F$. Since $F = Q + X^+$, we can write

$$f^{-1} = \sum_{x \in X} r_x x, \quad (r_x \in Q).$$

Choose a positive integer $n$ such that each $nr_x$ is integral and observe that $N$ contains the element

$$\sum_{x \in X} (nr_x) x f = n f^{-1} = n.$$
Hence \( nX^+ \leq N \). We shall show that \( G/nX^+ \) is a T-group. \( F = Q + X^+ \) is divisible as an additive abelian group and \( F/(Q + nX^+) \) has finite exponent. Thus \( F = Q + nX^+ \) and \( F/nX^+ \) is isomorphic with a factor group of \( Q/\langle 1 \rangle \). Hence every automorphism of \( F/nX^+ \) is a power automorphism and every subgroup of \( F/nX^+ \) is normal in \( G/nX^+ \). If \( x \neq 1 \), then \( F(x - 1) = F \), which is easily seen to imply that a subnormal subgroup of \( G/nX^+ \) either contains \( F/nX^+ \) or is contained in it. This shows that \( G/nX^+ \) is a T-group.

8. Finitely generated soluble \( JNT \)-groups. The main result of this section is

**Theorem 2.** A finitely generated hyperabelian group which is not a T-group has a finite homomorphic image which is not a T-group.

Recall here that a group is hyperabelian if it possesses an ascending series of normal subgroups whose factors are all abelian; this is equivalent to requiring each nontrivial homomorphic image to have a nontrivial normal abelian subgroup.

**Proof.** Let \( G \) be a finitely generated hyperabelian group which is not a T-group. Suppose that \( \{N_\alpha; \alpha \in \mathcal{A}\} \) is a chain of normal subgroups of \( G \) such that no \( G/N_\alpha \) is a T-group; write \( N \) for the union of the chain. Assume that \( G/N \) is nevertheless a T-group. Now a hyperabelian T-group is soluble because soluble T-groups are metabelian; moreover, a finitely generated soluble T-group is either finite or abelian [18, Theorem 3.3.1], and therefore is certainly finitely presented. Thus \( G/N \) is finitely presented. By a well-known principle this implies that \( N = \langle a_1, \ldots, a_n \rangle \) for a certain finite set of \( a_i \)'s. Hence \( N = N_\alpha \) for some \( \alpha \). By this contradiction \( G/N \) is not a T-group. Zorn's Lemma shows that there exists a normal subgroup \( M \) of \( G \) which is maximal subject to \( G/M \) not being a T-group. Clearly \( G/M \) is a \( JNT \)-group. Moreover \( G/M \) contains a nontrivial normal abelian subgroup, being hyperabelian. Thus \( G/M \) is soluble and Theorem 2 will follow from

**Lemma 9.** A finitely generated soluble \( JNT \)-group is finite (and hence of type I, IV, V or VIII).

**Proof.** One can, of course, verify directly that no soluble \( JNT \)-group on our list can be both finitely generated and infinite. However, it is more economical to proceed independently as follows.

Let \( G \) be a finitely generated soluble \( JNT \)-group which is infinite. First of all observe that \( G \) cannot be nilpotent; for if \( G \) were nilpotent, the initial argument of \( \S 3 \) would show that \( G \) is periodic and this, as is well known, implies that \( G \) is finite.

Denote by \( A \) a nontrivial normal abelian subgroup of \( G \). Suppose that \( G/A \) is infinite. If \( 1 < B \leq A \) and \( B \triangleleft G \), then \( G/B \) is abelian and \( G' \leq B \). Hence \( G' \) is minimal normal in \( G \) and lies in \( A \). Therefore \( G/C_G(G') \) is a finitely gener-
ated abelian group and by a theorem of P. Hall [8, Theorem 5.1], $G'$ is a finite elementary abelian $p$-group for some prime $p$.

Now write $C = C_C(G')$; then $G/C$ is finite. Also $C' \leq G' \leq \zeta(C)$; thus if $x, y \in C$, we have $1 = [x, y]^p = [x^p, y]$. Hence $C^p \leq \zeta(C)$ and $C^p$ is abelian. Now $G/C^p$ is periodic and hence finite, so $C^p$ is finitely generated and infinite. Hence for some integer $n$ the group $N = (C^p)^n$ is torsion-free and nontrivial, while $G/N$ is finite. This contradicts Lemma 5.

Therefore $G/A$ is finite, which shows that $A$ is finitely generated and infinite; there is no loss in assuming $A$ to be free abelian. Let $L = [G', G]$ and observe that $L \neq 1$. If $L \cap A \neq 1$, then $G/L \cap A$ is finite, by the first part of the proof, and, replacing $A$ by $L \cap A$, we may assume that $A \leq L$. Let $p$ be a prime dividing $|G : L|$, then $G/A^p$ is a finite soluble $T$-group; however the prime $p$ divides both $|G : L|$ and $|L : A^p|$, which is impossible [7]. Thus $L \cap A = 1$ and $L \cong LA/A$, which shows that $L$ is finite and abelian; therefore $G/L$ is infinite. However this situation has been shown to be impossible.

Lemma 9 may be compared with B. H. Neumann's theorem that a finitely generated soluble just nonabelian group is finite [15, Theorem 6.3]—see also Rosati [21]. It is not difficult to show that a soluble just nonabelian group cannot be a $T$-group if it is infinite. Thus Neumann's theorem is a special case of Lemma 9.

9. $JNT$-groups. A group $G$ has the property $\bar{T}$ if $H \triangleleft K \triangleleft L \triangleleft G$ always implies that $H \triangleleft L$. Thus $\bar{T}$-groups form the largest subgroup-closed subclass of the class of $T$-groups.

A $JNT$-group is either a $JNT$-group or a $T$-group. There exist finite $JNT$-groups which are $T$-groups, for example the symmetric group $S_n$ where $n \geq 5$; but this phenomenon cannot occur in the soluble case. In fact we shall prove

**Theorem 3.** A group is a soluble $JNT$-group if and only if it is isomorphic with a group of type I, IV, V or VIII (with $X$ a $\bar{T}$-group in the last case).

**Proof.** Let $G$ be a soluble $JNT$-group. First observe that $G$ is not a $T$-group; for suppose this is wrong. If $G$ is a $T$-group of type I, then $L = [G', G]$ contains an element $a$ of infinite order; therefore $L/(a^4)$ has an element of order 4 and $G/(a^4)$ is not abelian, which precludes $G/(a^4)$ from being a $\bar{T}$-group (for this and other results about soluble $\bar{T}$-groups see [18, Theorem 6.1.1]). If $G$ is a $T$-group of type II, then $G'$ is radicable and if $1 \neq a \in G'$, the group $G/(a)$ is also a $T$-group of type II and therefore not a $\bar{T}$-group. Finally, if $G$ is periodic, $L$ has an element of order 2 since $G$ would otherwise be a $\bar{T}$-group. Therefore there is a normal subgroup $N \neq 1$ of $G$ such that $L/N$ is of type $2^\infty$; however this prevents $G/N$ from being a $\bar{T}$-group.

Therefore $G$ is a $JNT$-group and appears on our list. It remains to discard those $JNT$-groups $G$ which have a factor group $G/N$ such that $N \neq 1$ and $G/N$
is not a $T'$-group. Here one must keep in mind that a soluble $T'$-group $H$ is either periodic or abelian and that $[H', H]$ cannot have an element of order 2. This excludes types II, III, VI, VII and IX—recall that in type IX the group $X$ cannot be periodic. There is no difficulty in verifying that the remaining types are $JNT$-groups (in type VIII one must assume that $X$ is a soluble $T'$-group).

Locally finite $JNT$-groups. A good deal more is known about $T'$-groups than $T'$-groups and one would hope for correspondingly more information about $JNT$-groups. For example, a locally finite $T'$-group is soluble and therefore metabelian. This is an easy corollary of the well-known theorem of Huppert that a finite group with all of its proper subgroups supersoluble is soluble [12, Satz 22]; see also [19].

We shall outline a method of describing the locally finite $JNT$-groups that are insoluble; this is based on the Fitting-Gol'berg theory of semisimple groups (see [14, §61]). Let $G$ be a locally finite $JNT$-group which is insoluble. Then every proper factor group of $G$ is metabelian and $M = G''$ is the monolith of $G$. Then $G$ acts irreducibly on $M$, i.e., $M$ has no proper nontrivial subgroups that are $G$-admissible; in particular, $M$ is characteristically simple. Clearly $M$ is perfect and its centre is 1. Thus $G_G(M) = 1$, from which it follows that there is an isomorphism of $G$ with a subgroup of $Aut M$, the full automorphism group of $M$, in which $M$ is mapped onto the group of inner automorphisms $Inn M$. Thus one may assume that

$$Inn M < G < Aut M.$$

Conversely, let $M$ be a nonabelian, characteristically simple group which is locally finite; let $G$ be a subgroup of $Aut M$ which contains $Inn M$ and acts irreducibly on $Inn M$; assume also that $G/Inn M$ is a $T'$-group. Then in fact $G$ is a $JNT$-group. To prove this let $1 < N < G$; if $N \cap (Inn M) = 1$, then $[N, Inn M] = 1$ and $[N, M] < \zeta(M) = 1$, which shows that $G = N$. Thus $Inn M < N$ by irreducibility and $G/N$ is a $T'$-group. On the other hand $G$ is not a $T'$-group since if it were, $Inn M$—and therefore $M$—would be soluble. Also $Inn M$ is the monolith of $G$.

Suppose that $G_1$ and $G_2$ are two isomorphic groups obtained in this way from groups $M_1$ and $M_2$; then evidently $M_1 \cong M_2$. If we identify $M_1$ and $M_2$ and write $\alpha$ for the isomorphism of $G_1$ with $G_2$, then $\alpha$ determines by restriction an automorphism $\alpha^*$ of $M_1$. It is routine to check that $g^{\alpha^*} = (\alpha^*)^{-1} g^{\alpha^*}$, $g \in G_1$). This is summed up in

Theorem 4. There is a one-one correspondence between isomorphism classes of insoluble, locally finite, $JNT$-groups with given monolith $M$ and conjugacy classes of irreducible, locally finite $T'$-subgroups of $Out M$ (the group of outer automorphisms of $M$).

Of course it is by no means clear which groups $M$ can arise here. However one obvious candidate—and the only one if $M$ is finite or merely possesses a
minimal normal subgroup—is a direct power of a nonabelian simple group \( H \). For simplicity of presentation suppose we are dealing with finite JNT-groups. Let \( M \) be the direct product of \( n \) copies of \( H \). Then, as was shown by Fitting [5, Satz 12], \( \text{Aut} \ M \cong (\text{Aut} \ H) \wr S_n \) and \( \text{Out} \ M \cong (\text{Out} \ H) \wr S_n \); in these wreath products \( \text{Aut} \ H \) and \( \text{Out} \ H \) are in their regular representations and the symmetric group \( S_n \) is in its natural permutation representation. The irreducible subgroups of \( \text{Out} \ M \) correspond in the isomorphism to subgroups of \( (\text{Out} \ H) \wr S_n \) which map onto transitive subgroups of \( S_n \) in the canonical homomorphism of \( (\text{Out} \ H) \wr S_n \) onto \( S_n \).

Thus a finite JNT-group is either soluble—and therefore of type I, IV, V or VIII—or insoluble, in which case it corresponds to a conjugacy class of irreducible \( \bar{T} \)-subgroups of \( (\text{Out} \ H) \wr S_n \) where \( H \) is a finite nonabelian simple group and \( n \) a positive integer \( > 1 \). However the problem of adequately describing all finite JNT-groups remains open.

REFERENCES


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