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GROUPS WHOSE HOMOMORPHIC IMAGES HAVE A TRANSITIVE NORMALITY RELATION

BY

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ABSTRACT. A group G is a *T-group* if $H \lhd K \lhd G$ implies that $H \lhd G$, i.e. normality is transitive. A *just non-T-group* (*JNT*-group) is a group which is not a T-group but all of whose proper homomorphic images are T-groups. In this paper all soluble *JNT*-groups are classified; it turns out that these fall into nine distinct classes. In addition all soluble $JN\bar{T}$-groups and all finite $JN\bar{T}$-groups are determined; here a group G is a $\bar{T}$-group if $H \lhd K \lhd L \leq G$ implies that $H \lhd L$. It is also shown that a finitely generated soluble group which is not a T-group has a finite homomorphic image which is not a T-group.

1. Introduction and statement of results.

(1.1) **Definitions.** If P is a group theoretical property, a *just non-P-group* is a group which is not a P-group but all of whose proper homomorphic images are P-groups; for brevity we shall call these *JNP-groups*. For example, when P is commutativity, soluble *JNP*-groups have been studied by Newman ([16] and [17]) and also by Rosati [21].

Denote by T the property that *normality is transitive*; thus a group G has T if $H \lhd K \lhd G$ always implies that $H \lhd G$. Here we are concerned with *JNT-groups*; these can alternatively be defined as the groups whose nonnormal subnormal subgroups are core-free and form a nonempty set.

The principal object of this paper is to classify the soluble *JNT*-groups. This will be done by dividing the soluble *JNT*-groups into nine types (and eleven subtypes). While the descriptions of the different types vary in both complexity and precision, a rather clear picture of the structure of a soluble *JNT*-group emerges.

Notation.

$\langle X : \lambda \in \Lambda \rangle$: subgroup generated by the (subsets) X_λ, $\lambda \in \Lambda$.

X^Y: normal closure of X in Y, i.e., the subgroup generated by all conjugates $x^y = y^{-1}xy$, $(x \in X, y \in Y)$.

$[X, Y] = [X, {}_1Y]$: commutator subgroup generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ $(x \in X, y \in Y)$.

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$[X,_{i+1}Y] = [[X;\ _iY],\ Y]$.

$C_H(X)$: centraliser of X in H.

$\zeta(G)$: centre of G.

R^*: multiplicative group of units of R, a ring with unity.

FG: group algebra of a group G over a field F.

$\overset{\Omega}{\simeq}$: isomorphism of Ω-operator groups.

X^+: additive subgroup of a ring generated by X.

(1.2) **The classification of soluble JNT-groups.** We shall now describe the nine types of groups which can occur.

I. The nonabelian nilpotent groups all of whose proper homomorphic images are abelian(2), with the exception of the quaternion group of order 8.

II. Let L be an abelian group of type 2^∞ with generators $a_1, a_2, \cdots$ and relations $a_1^2 = 1$ and $a_{i+1}^2 = a_i$. Let $D = \langle d\rangle$ be a group of order 2. Define G to be the group generated by the direct product $L \times D$ and a cyclic group $\langle t\rangle$ of order 2 or 8 where $a^t = a^{-1}$, $d^t = a_1 d$ and $t^2 = 1$ or $a_2 d$ for all $a \in L$.

III. Let L and D be as in II except that D may now have order 1. Let X be an extra-special 2-group generated by elements of order 2 and let C be the direct product of $L \times D$ and X in which $\langle a_1\rangle$ and the centre of X are amalgamated. Choose an element σ from $\mathrm{Hom}(X/\langle a_1\rangle, \langle a_1\rangle)$ and let $\langle t\rangle$ be a cyclic group of order 2 or 8. Define G to be the group generated by C and $\langle t\rangle$ where for all $a \in L$ and $x \in X$,

$$a^t = a^{-1}, \qquad d^t = a_1 d \quad (\text{if } d \neq 1), \qquad x^t = x(x\langle a_1\rangle)^\sigma,$$

and

$$t^2 = 1 \quad \text{or} \quad a_2 d \quad (\text{if } d \neq 1).$$

Moreover, if $d \neq 1$ we can take $\sigma = 0$.

IV. Let A be an elementary abelian p-group of order p^2 where p is an odd prime, and let X be the group of automorphisms of A determined by a diagonal but nonscalar subgroup of $\mathrm{GL}(2, p)$. Define G to be the holomorph of A by X.

V. Let P be an extra-special p-group of exponent p, an odd prime, and let n be an integer lying strictly between 1 and p. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a basis for P modulo its centre and define an automorphism t of G by the rule $x_\lambda^t = x_\lambda^n$, $(\lambda \in \Lambda)$. Define G to be the holomorph of P by $\langle t\rangle$.

VI. Let A be a group of type p^∞ with generators $a_1, a_2, \cdots$ and relations $a_{i+1}^p = a_i$ and $a_1^p = 1$. Let Γ be a nonperiodic group of p-adic integers all of which are congruent to 1 modulo p.

(a) Form an extension W of A by Γ using the natural coupling of Γ to A. Let D be a nontrivial elementary abelian p-group of automorphisms of W which

(2) Newman [17] calls these *JN2-groups*, however this conflicts with our present terminology.

acts trivially on $\langle a_1 \rangle$ and $W/\langle a_1 \rangle$. Define G to be the holomorph of W by D. If $p = 2$ and $-1 \in \Gamma$, then in addition require that $t^2 = 1$ or a_1 where $-1 \to tA$ in the isomorphism of Γ with W/A. If $[D, t] \neq 1$, the possibility $t^2 = a_1$ can be dispensed with.

(b) Let $p = 2$ and $-1 \in \Gamma$, and choose a group $\langle u \rangle$ of order 2. Form an extension W of $F = A \times \langle u \rangle$ by Γ using the natural coupling of Γ to A, supplemented by $\alpha \to (u \to u)$ if $\alpha \equiv 1 \bmod 4$ and $\alpha \to (u \to a_1 u)$ if $\alpha \not\equiv 1 \bmod 4$; moreover, require that $W' = A$ and $t^2 = a_2 u$ where t is as in (a). Finally, form G as in (a) except that D is allowed to be 1 and the centre of G should not contain any element of order 2 except a_1. [This last condition is automatically satisfied in VI(a).]

VII. Let A and Γ be as in VI and let X be an extra-special p-group with generators of order p. Write E for the direct product of A with X in which $\langle a_1 \rangle$ and the centre of X are amalgamated.

(a) Form an extension W of E by Γ in which $W' = A$, using the natural coupling of Γ to A supplemented by causing each α in Γ to correspond to an (outer) automorphism of X which acts trivially on $\langle a_1 \rangle$ and $X/\langle a_1 \rangle$. Let D be an elementary abelian p-group of automorphisms of W which acts trivially on $\langle a_1 \rangle$ and $W/\langle a_1 \rangle$. Define G to be the holomorph of W by D and suppose D is chosen so that the centre of G contains no elements of order p outside $\langle a_1 \rangle$. If $p = 2$ and $-1 \in \Gamma$, require also that $t^2 = 1$ where $-1 \to tE$ in the isomorphism of Γ with W/E.

(b) Let $p = 2$ and $-1 \in \Gamma$, and choose a group $\langle u \rangle$ of order 2. Form an extension W of $F = E \times \langle u \rangle$ by Γ using the coupling of Γ to E described in (a), supplemented by $\alpha \to (u \to u)$ if $\alpha \equiv 1 \bmod 4$ and $\alpha \to (u \to a_1 u)$ if $\alpha \not\equiv 1 \bmod 4$; moreover require that $W' = A$ and $t^2 = a_2 u$ where $-1 \to tF$ in the isomorphism of Γ with W/F. Finally form G as in (a).

VIII. Let X be a soluble T-group and let A be a noncyclic abelian group which is faithful and irreducible as an X-module (so that A contains no proper nontrivial submodules). Define G to be the natural semidirect product of A by X.

IX. Let p be any prime and let F be a subfield of the field of p-adic numbers; denote by Q the field of rational numbers. Choose X to be a group of p-adic integer units in F such that $X \neq \langle -1 \rangle$ and $X^+ < Q + X^+ = F$. Define G to be the natural semidirect product of F (as an additive group) by X.

Our principal conclusion is, then,

Theorem 1. *A group is a soluble JNT-group if and only if it is isomorphic with a group of type* I *to* IX.

(1.21) Power automorphisms and soluble T-groups. The proof of Theorem 1 occupies §§3 to 7. Not surprisingly, considerable use will be made of the theory of soluble T-groups. We present next a summary of the relevant facts from this theory.

A fundamental concept is that of a *power automorphism* of a group; this is an automorphism which leaves every subgroup of the group invariant, and so maps each element to a power of itself. A crucial result is that in an abelian group a power automorphism maps elements of the same order to the same power; moreover if an element of infinite order is present in the group, the only nontrivial power automorphism is the involution $a \longrightarrow a^{-1}$ ([7], [18, §4.1]). An extensive study of power automorphisms of nonabelian groups has been made by Cooper [2].

If G is a soluble T-group, then it is metabelian [18, Theorem 2.3.1]. Also $L = [G', G]$ is the last term of the lower central series, G/L is a *Dedekind group* (i.e. every subgroup is normal) and $C_G(G') = C_G(L)$ is the Fitting subgroup of G.

Noncommutative soluble T-groups are divided into three classes:

(a) periodic groups,

(b) nonperiodic groups of *type* I, i.e. groups in which the centraliser C of the derived subgroup is nonperiodic,

(c) nonperiodic groups of *type* II, i.e. groups in which C is periodic.

If G is a periodic soluble T-group and $L = [G', G]$, then L and G/L do not contain elements with the same odd prime order ([7], [18, Theorem 4.2.2]): also the 2-component of L is radicable.

If G is a soluble T-group of type I and $C = C_G(G')$, then C is abelian and $G = \langle t, C\rangle$ where $|G:C| = 2$, $c^t = c^{-1}$ for all $c \in C$, and $\langle t^2, C^2\rangle = \langle t^2, C^4\rangle$ [18, Theorem 3.1.1].

If G is a soluble T-group of type II, somewhat less is known of its structure. However $C = C_G(G')$ is abelian, G' is radicable and $C = G' \times B$ where B lies in the centre of G. If G' contains an element of prime order p, the p-component of B has finite exponent, say $p^{n(p)}$; if $x \in G$, then x induces in the p-component of C the power automorphism $a \longrightarrow a^{\alpha}$ where α is a p-adic integer unit satisfying $\alpha \equiv 1 \bmod p^{n(p)}$; here, of course, a^{α} is understood to mean a^{α_1} where α_1 is an integer congruent to α modulo the order of a [18, Theorem 4.3.1].

(1.3) **Remarks on the classification.**

(1.31) **Nilpotent just nonabelian groups.** These groups—which include the extraspecial groups of Hall and Higman [10]—occur in our classification and deserve comment.

Let G be a nilpotent just nonabelian group and let Z be its centre. Then, as Newman has shown in [17], G is a p-group, Z is either cyclic or quasi-cyclic and G/Z is elementary abelian; moreover, G' lies in Z and has order p. Let $\{x_\lambda Z: \lambda \in \Lambda\}$ be a basis for G/Z and let $G' = \langle a\rangle$. Then

$$[x_\lambda, x_\mu] = a^{f(\lambda,\mu)}. \tag{1}$$

If we regard G/Z as a vector space over GF(p), then f is a nondegenerate alternating bilinear form. Also $x_\lambda^p \in Z$ and it may be shown that x_λ can be chosen

so that either $x_\lambda^p = 1$ or x_λ^p generates Z [17, Lemma 3]. Indeed one can write down generators and relations for G by utilising (1), the position of the x_λ^p and relations sufficient to make $[G', G] = 1$. Newman also proves that if G is countable, it is a central product involving three rather simple kinds of groups [17, Theorem 5].

It should be mentioned that soluble just nonabelian groups which are not nilpotent have also been discussed by Newman [16]; these occur under our type VIII heading unless they are metacyclic.

(1.32) **Faithful irreducible representations.** In connection with groups of type VIII it is desirable to know which soluble T-groups X can act faithfully and irreducibly on an abelian group A. Of course in a situation of this kind A is necessarily isomorphic with the additive group of a vector space, i.e., it is either an elementary abelian p-group or a direct product of copies of the additive group of rational numbers. In the former event the problem reduced to the case where X is abelian. This is because of

Lemma 1. *Let F be a field and let X be a soluble T-group.*

(i) *If X is abelian, it has a faithful irreducible representation over F if and only if there is an extension field E of F such that $X \simeq Y \leq E^*$ and $E = FY$.*

(ii) *In general, X has a faithful irreducible representation over F if and only if the centre of its Fitting subgroup has such a representation.*

Proof. (i) The proof is well known and we omit it.

(ii) We recall first that the Fitting subgroup N of any T-group X (whether soluble or not) is nilpotent and coincides with $C_X(X')$ [18, Lemma 2.2.2]. Write C for the centre of N.

Suppose first that there exists a faithful irreducible (right) FX-module M. Let $0 \neq a \in M$ and $D = C_N(a)$. Since N is nilpotent, D is subnormal in X and hence $D \lhd X$. Therefore D fixes ax for all $x \in X$, which shows that D acts trivially on M and $D = 1$ since M is irreducible and faithful. Consequently M is fixed-point-free with respect to each nonunit element of N.

Assume now that N is nonperiodic. If X is abelian, $C = N = X$ and M is already a faithful irreducible FC-module. Let X be nonabelian—and thus a soluble T-group of type I. It follows from the structure theory of these groups (§1.21) that $N = C$ is abelian and $X = \langle x, C \rangle$ where $a^x = a^{-1}$ if $a \in C$, and $x^2 \in C$. Suppose that M_1 is a proper nonzero FC-submodule of M; then M_1x is also an FC-submodule because $C \lhd X$. Since $x^2 \in C$, we see that $M_1 + M_1x$ and $M_1 \cap M_1x$ are FX-submodules of M; hence $M_1 \cap M_1x = 0$ and $M = M_1 \oplus M_1x$ by the irreduciblility of M. If M_2 is a proper nonzero FC-submodule of M_1, then $M = M_2 \oplus M_2x$ by the same argument, with the result that $M_1 = M_1 \cap (M_2 \oplus M_2x) = M_2$. Therefore M_1

is an irreducible FC-module; it is faithful because M is fixed-point-free.

Now suppose that N is periodic. Let Y be a finitely generated—and therefore finite-subgroup of C. Choose $a \neq 0$ from M; then $(a)FY$ has finite dimension over F, so it contains an irreducible FY-submodule, say L. Since the action of N is fixed-point-free, L is faithful. Consequently, by the first part of this lemma, Y is isomorphic with a finite subgroup of an extension of the field F, which implies that Y is cyclic and C locally cyclic. Let $\overline{F}$ be the algebraic closure of F. Then the torsion-subgroup of $\overline{F}^*$ is a direct product of p^∞-groups, one for each p not equal to the characteristic of F. We can identify C with a subgroup of $\overline{F}^*$. Having done this, define E to be the subfield of $\overline{F}$ generated by C. Since E is algebraic over F, an FC-submodule of E is an ideal. Thus E is an irreducible FC-module, and it is obviously faithful.

Conversely, suppose M_1 is a right FC-module which is faithful and irreducible; the problem is to construct an FX-module that is faithful and irreducible. First we form the induced FX-module

$$I = M_1 \otimes_{FC} (FX)$$

and then we choose an FX-composition series for I; here the term *composition series* (or *system*) is used in the general sense of Kuroš [14, vol. 2, §56]. The composition factors are irreducible FX-modules, so we can assume that none is faithful. Now refine the series to an FC-composition series. If $1 \neq S \lhd X$, then $S \cap C \neq 1$; for $S \cap X' \leq S \cap C$ (since X is metabelian) and $S \cap X' = 1$ implies $S \leq C$. It follows that none of the FC-composition factors can be faithful as FC-modules. It is straightforward to show that an irreducible FC-submodule of I is FC-isomorphic with one of the factors of the FC-composition series, and hence is not faithful.

However, if $\{g_\lambda : \lambda \in \Lambda\}$ is a transversal to C in X, with say $g_{\lambda_0} = 1$, then

$$I \overset{FC}{\simeq} \operatorname*{Dr}_{\lambda \in \Lambda} (M_1 \otimes_{FC} (FC)g_\lambda)$$

and $M_1 \otimes_{FC} (FC)g_{\lambda_0} \overset{FC}{\simeq} M_1$, which gives the contradiction that M_1 is not faithful. This completes the proof.

Returning to the discussion of groups of type VIII, we see that Lemma 1 (with $F = \mathrm{GF}(p)$) gives a complete description of the possible groups X when A is an elementary abelian p-group, although not in group theoretical terms if A is infinite; when A is finite the only restriction on the soluble T-group X is that the centre of $C_X(X')$ be cyclic of order prime to p.

However, when A is a direct product of copies of the additive group of rational numbers (Q), the situation is less clear. Here the following theorem of Baer [1, Proposition] is relevant: *a locally finite group cannot act as an irreducible group of automorphisms of a torsion-free abelian group* $A \neq 1$. This tells us, at least, that X *cannot be periodic.*

(1.33) **Groups of type IX.** While it would probably be difficult to give a purely group theoretic characterisation of G in this case, some possibilities can easily be obtained.

Let F be a subfield of F_p, the field of p-adic numbers, let R_p be the ring of p-adic integers and define $R = F \cap R_p$; now define $X = R^*$. Clearly $X^+ \leq R$. Let $r \in R$; if $r \not\equiv 0 \bmod p$, then $r \in R^* = X$; if $r \equiv 0 \bmod p$, then $1 + r \in R^* = X$ and $r \in X^+$. Therefore $X^+ = R$. Also R is not a field because $1/p \notin R$, so $X^+ \neq F$. Finally, let $f \in F$; we can write $f = q + u$ where $q \in Q$ (the field of rational numbers) and $u \in R_p$. Since $Q \leq F$, we have $u = f - q \in F$, so $u \in F \cap R_p = R$. Consequently, $F = Q + X^+$. The natural semidirect product of F with X is of type IX. For example, we could take $F = F_p$ and $X = R_p^*$.

On the other hand, *in a group of type* IX *the subgroup* X *cannot be periodic.* For suppose that X is periodic; then $\langle -1 \rangle \neq X \leq R_p^*$ and the structure of R_p^* show that X is cyclic of order $m > 2$ dividing $p - 1$ where p is odd. Thus $F = Q + X^+$ is a finite extension of Q. If $X = \langle x \rangle$, the irreducible polynomial of x is cyclotomic. Hence each element of F can be written in the form $r + \sum_{i=1}^{n-1} n_i x^i$ where r is rational, n_i integral and $n = \phi(m) > 1$. But x/p has no such representation since $1, x, \cdots, x^{n-1}$ are linearly independent.

2. **Preliminary results.** In this section we shall collect results of a general nature about JNT-groups as well as some technical lemmas necessary for the classification.

(2.1) **Some properties of JNT-groups.** Our first result indicates the complex subnormal structure of JNT-groups in general.

Lemma 2. *To each group G there corresponds a JNT-group G^* with subgroups H and K such that $K \lhd H$ and H is subnormal in G^* with subnormal index ≤ 4 while $G \simeq H/K$.*

Proof. The first step is to embed G in a nonunit perfect group G_1 such that G is subnormal in G_1 with subnormal index ≤ 2; that this is possible is a theorem of Dark [3]. Write Z for the set of integers in their natural order and form the standard wreath power $G_2 = \mathrm{Wr}\, G_1^Z$. The order-automorphism $n \longrightarrow n + 1$ of Z gives rise to an automorphism t of G_2 permuting the copies of G_1 in the same way. Finally, let G^* be the holomorph of G_2 by $\langle t \rangle$.

Let Z_1 and Z_2 be the sets of negative and nonnegative integers in natural order. Then

$$G_2 \simeq (\mathrm{Wr}\, G_1^{Z_1}) \wr (\mathrm{Wr}\, G_1^{Z_2}) \tag{2}$$

where now not all the wreath products are standard; for these and other results about wreath products see P. Hall [9]. The base group L of the wreath product (2) has G_1 as a homomorphic image, so that $G_1 \simeq L/K$ for some $K \lhd L$. Hence

there exists a subgroup H of L such that $G \simeq H/K$ and H is subnormal in L with subnormal index ≤ 2. Clearly H is subnormal in G^* with subnormal index ≤ 4.

To show that G^* is a JNT-group observe first that G_2—and hence G^*—is not a T-group. Also it follows by arguments of P. Hall [9, p. 183] that G^* is monolithic(3) with G_2' as its monolith; here G_2 is perfect since G_1 is, so $G_2 = G_2'$. Hence every proper homomorphic image of G^* is abelian.

Corollary. *There exist JNT-groups with unbounded subnormal indices.*

In contrast to this there is

Lemma 3. *A JNT-group G has all its subnormal indices ≤ 2 if one of the following conditions is satisfied.*

(i) *G contains a nontrivial normal abelian subgroup A.*

(ii) *G contains a minimal normal subgroup N which itself contains a minimal normal subgroup N_1.*

Proof. Let H be a nonnormal subnormal subgroup of G. Suppose that (i) is valid. Since $AH \lhd G$, we have

$$[A, H]^g \leq [A, AH] = [A, H]$$

for all $g \in G$. Hence $[A, H] \lhd G$ and $[A, {}_iH] \lhd G$ for each $i \geq 1$. Therefore $H[A, {}_iH] \lhd G$ provided $[A, {}_iH] \neq 1$. Now if s is the subnormal index of H in G, then $[A, {}_sH] \leq H$, so that $[A, {}_sH] = 1$. Let i be the least integer for which $[A, {}_iH] = 1$. Then $i > 0$ and $H[A, {}_{i-1}H] \lhd G$; but also $H \lhd H[A, {}_{i-1}H]$, so the result follows.

Now suppose that (ii) is valid. Clearly N is the direct product of N_1 and certain of its conjugates. Therefore N_1 is simple. If N_1 is abelian, so is N and the result follows from (i). If N_1 is nonabelian, a theorem of Wielandt [22] shows that N normalises H, so $H \lhd HN \lhd G$.

These may be compared with the (more elementary) fact that a non-T-group whose proper subgroups are all T-groups has its subnormal indices ≤ 2 [20].

We shall now consider the possibilities for minimal normal subgroups in JNT-groups.

Lemma 4. *Let G be a JNT-group:*

(i) *the number of minimal normal subgroups of G equals* 0, 1 *or* 2;

(ii) *if this number equals* 2, *both minimal normal subgroups are cyclic of the same prime order;*

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(3) A group is *monolithic* if the intersection of all its nontrivial normal subgroups is nontrivial; this intersection is then called the *monolith*.

(iii) *if G has at least one minimal normal subgroup, each nontrivial normal subgroup of G contains a minimal normal subgroup of G.*

Proof. Throughout H denotes a nonnormal subnormal subgroup of G. Let M be a minimal normal subgroup of G and let N be a nontrivial normal subgroup of G not containing M; thus $M \cap N = 1$. Clearly $M \overset{G}{\simeq} MN/N$ and, since G/N is a T-group, this shows that M is simple. Now $M \leq H$ would imply that $H \lhd G$; therefore $H \cap M = 1$. Also $(H \cap N)M \lhd G$, so that

$$[H \cap N,\ G] \leq ((H \cap N)M) \cap N = H \cap N$$

and $H \cap N \lhd G$. Therefore $H \cap N = 1$. If M is not abelian, a theorem of Wielandt [22] shows that $[H, M] = 1$. In this case

$$H \cap (M \times N) \leq C_{M \times N}(M) = N,$$

and $H \cap (M \times N) = 1$ since $H \cap N = 1$. However this implies that $H = (HM) \cap (HN)$, and since $HM \lhd G$ and $HN \lhd G$, the contradiction $H \lhd G$ is obtained. Thus M is abelian and therefore cyclic of prime order, say p. As we have just seen, $H_1 = H \cap (M \times N)$ cannot be trivial; obviously, H_1 is subnormal in G. Since $H \cap N = 1$,

$$H_1 \simeq H_1N/N \leq (M \times N)/N \simeq M,$$

which shows that H_1 has order p. Next $H \cap M = 1$, so

$$H_1 \simeq H_1M/M \leq (M \times N)/M \simeq N,$$

which shows that N contains a subnormal subgroup N_1 of order p. But $MN_1 \lhd G$, so $[N_1, G] \leq (MN_1) \cap N = N_1$ and $N_1 \lhd G$. Hence N contains a minimal normal subgroup of G, namely N_1. Moreover, if N itself is minimal normal in G, then $N = N_1$ and $|M| = p = |N|$. Thus (ii) and (iii) have been proved.

Finally, let L, M and N be three distinct minimal normal subgroups of G. All three must have the same prime order p. Let $g \in G$; then g induces in the elementary abelian p-group $(M \times N)L/L$ a power automorphism of the form $x \longrightarrow x^n$. Since $M \overset{G}{\simeq} LM/L$ and $N \overset{G}{\simeq} NL/L$, it follows that g induces $x \longrightarrow x^n$ in M and in N. Therefore every subgroup of $M \times N$ is normal in G and, in particular, $H \cap (M \times N) \lhd G$. Hence $H \cap (M \times N) = 1$, which is a contradiction. Thus G can have no more than two minimal normal subgroups.

For example, soluble JNT-groups of type IX possess no minimal normal subgroups, those of types I–III and V–VIII have one (and so are monolithic) and those of type IV have two.

A JNT-group with two minimal normal subgroups, and any soluble JNT-group, has its subnormal indices ≤ 2; these statements follow from Lemmas 3 and 4.

Next we record two technical lemmas about JNT-groups which will prove valuable.

Lemma 5. *Let M and N be normal subgroups of a JNT-group G. If one of the following conditions holds, then either M or N is trivial.*

(i) *M and N are periodic and do not contain elements with the same prime order.*

(ii) *M is periodic, N is torsion-free and G/N is periodic.*

Proof. Let H denote a nonnormal subnormal subgroup of G and let $M \neq 1$ and $N \neq 1$. Both (i) and (ii) imply that $M \cap N = 1$; thus $(H \cap M)N \lhd G$ implies that $[H \cap M, G] \leq H \cap M$ and so $H \cap M = 1$. Similarly $H \cap N = 1$. If (i) is valid, $H \cap (M \times N) = (H \cap M) \times (H \cap N) = 1$, which gives $H = (HM) \cap (HN) \lhd G$. If (ii) is valid, then H is periodic since $H \simeq HN/N$; therefore HM is periodic and $(HM) \cap N = 1$, which implies that $(HM) \cap (HN) = H$ and $H \lhd G$.

Lemma 6. *Let $N \lhd G$ where N is abelian and each of its primary components is either elementary abelian or of infinite exponent. Assume that every subgroup of $C_G(N)$ is normal in G. Then G is not a JNT-group.*

Proof. Suppose G is a JNT-group and let H be a nonnormal subnormal subgroup of G. Then $H \not\leq C_G(N)$, so there exists $h \in H$ such that $[N, h] \neq 1$. The element h induces a nontrivial power automorphism in N, since $N \leq C_G(N)$. If N is not periodic, $a^h = a^{-1}$ for all $a \in N$ (see §1.21) and $[N, {}_i h] = N^{2^i}$. If s is the subnormal index of H in G, then $H \geq N^{2^s} > 1$, which implies that $H \lhd G$. Thus N is periodic and for some prime p the p-component P is not centralised by h. There is a p-adic integer α such that $a^h = a^\alpha$ for all $a \in P$ [18, Lemma 4.1.2]. With s as before, $H \geq P^{(\alpha-1)^s}$ and consequently $P^{(\alpha-1)^s} = 1$. However this implies that either P is elementary abelian and $\alpha \equiv 1 \bmod p$ or P has infinite exponent and $\alpha = 1$; in each case $[P, h] = 1$.

(2.2) **Splitting criteria.** The following partial generalisation of the Schur-Zassenhaus theorem is well known—see for example [4, Theorem 3] or [18, Lemma 5.1.1]. It will only be required, of course, in situations where G is soluble.

Lemma 7. *Let $N \lhd G$ where G is locally finite and G/N is countable. Suppose that N and G/N do not contain elements with the same prime order. Then N has a complement in G.*

However, we shall encounter situations where this splitting criterion is inadequate. The following result is particularly useful for splitting groups over noncentral minimal normal subgroups; it is based on an idea of M. F. Newman [16].

Lemma 8. *Let A and N be normal subgroups of a group G such that $A \leq N$ and $[A, N] \neq 1$. Assume in addition that N is metabelian and that every subgroup of N/A is normal in G/A. Suppose that A is isomorphic with the additive*

group of a vector space over a prime field and that A contains no proper non-trivial G-invariant subspaces. Then A has a complement in G.

Proof. We show first that A has a complement in N. Let $C = C_N(A)$; since $[A, N] \neq 1$, we can find a $g \in N \backslash C$. Since every subgroup of N/A is normal in G/A, there is for each $x \in G$ an integer n such that $g^x \equiv g^n \bmod A$. Since A is abelian, $[a, g]^x = [a^x, g^n] \in [A, g]$. It follows that $[A, g] \lhd G$, and for a similar reason $C_A(g) \lhd G$. If A is regarded as a vector space over a prime field, then $[A, g]$ and $C_A(g)$ are subspaces invariant under G. Hence

$$C_A(g) = 1 \quad \text{and} \quad [A, g] = A.$$

The mapping $a \longrightarrow [a, g]$ is therefore an automorphism of A.

Next, define for any $g \in N \backslash C$

$$X_g = \{x\colon x \in N,\ [x, g, g] = 1\}.$$

Let x and y belong to X_g. Since N is metabelian,

$$[xy, g, g] = [[x, g]^y[y, g], g] = [x, g, g^{y^{-1}}]^y = [x, g, g]^y = 1$$

and

$$[x^{-1}, g, g] = [[x, g]^{-x^{-1}}, g] = [x, g, g]^{-x^{-1}} = 1.$$

Thus X_g is a subgroup. Indeed X_g is a complement of A in N. For let $x \in N$ and $a \in A$; then $[ax, g, g] = [[a, g]^x[x, g], g] = [a, g, g]^x[x, g, g]$. Now N/A is a Dedekind group, so it is nilpotent of class ≤ 2 and $[x, g, g] \in A$. Since $a \longrightarrow [a, g]$ is an automorphism of A, it follows that given x in N we can choose a from A so that $ax \in X_g$. Consequently $N = AX_g$. If $x \in A \cap X_g$, then $[x, g, g] = 1$, which implies that $x = 1$. Thus $A \cap X_g = 1$ and X_g is a complement of A in N.

Denote by K any complement of A in N; we shall prove that $K = X_g$ for some $g \in N \backslash C$. Since A is abelian and $N = AK$, there exists a g in K such that $[A, g] \neq 1$. Now $K \simeq N/A$, which is nilpotent of class ≤ 2; hence $K \leq X_g$. Since X_g is also a complement of A, we obtain $K = X_g$.

Next, all complements of A in N are conjugate in N. For consider two such complements X_g and X_h where g and h come from $N \backslash C$. The mapping $a \longrightarrow [g, a, h, h]$ is an automorphism A. Since $[g, h, h] \in A$, there exists an a in A such that

$$[g, h, h]^{-1} = [g, a, h, h]: \tag{3}$$

with this a we compute

$$[g^a, h, h] = [g[g, a], h, h] = [[g, h][g, a, h], h] = [g, h, h][g, a, h, h] = 1$$

by (3). Hence $g^a \in X_h$. Also X_h is nilpotent of class ≤ 2, so $X_h \leq X_{g^a} = (X_g)^a$. Since X_h and X_g^a are both complements of A in N, they are equal.

Finally, we shall show that A has a complement in G. To this end, let K be a complement of A in N and let $g \in G$. Then $K^{g^{-1}}$ is also a complement of A in N, so $K^{g^{-1}} = K^h$ for some $h \in N$. Hence $hg \in N_G(K)$ and

$$G = N(N_G(K)) = (AK)N_G(K) = AN_G(K).$$

But $A \cap N_G(K) = N_A(K) = C_A(K)$, since $A \cap K = 1$. Hence $A \cap N_G(K) = 1$ and $N_G(K)$ is a complement of A in G.

3. **Nilpotent JNT-groups.** Let G be a nilpotent JNT-group; we shall prove that G is of type I. Suppose first that g is an element of infinite order in G such that $\langle g\rangle \lhd G$. Then $\langle g^{2^i}\rangle \lhd G$ and if $i > 2$, the group $G/\langle g^{2^i}\rangle$ is Dedekind and contains elements of order $2^i > 4$. Hence $G/\langle g^{2^i}\rangle$ is abelian and this causes G to be abelian. Consequently the centre of G is periodic and by Lemma 5 it is a p-group for some prime p. Hence there is a minimal normal subgroup N of G which lies in the centre and has order p. Suppose that gN is an element in G/N with infinite order. Then $L = \langle g, N\rangle \lhd G$ and $L = \langle g\rangle \times N$. Thus $L^p = \langle g^p\rangle$ is an infinite cyclic group and $L^p \lhd G$. This is impossible, so G/N—and hence G—is periodic. Therefore G is a p-group.

Assume next that there exists $M \lhd G$ such that $M \neq 1$ and $N \not\leq M$. Then $M \cap N = 1$ and at least one of G/M and G/N is hamiltonian, from which it follows that $p = 2$ and G is a 2-group. Let H be a nonnormal subnormal subgroup of G and choose from H an element h of order 2. Then hM and hN generate normal subgroups of G/M and G/N respectively, in each case with order 1 or 2. Therefore h belongs to the centre of G and $\langle h\rangle \lhd G$, which shows that $H \lhd G$. We conclude that G is monolithic with monolith N.

If G/N is abelian, G is just nonabelian and therefore of type I. Assume G/N to be hamiltonian. Then again $p = 2$ and G is a 2-group and $|N| = 2$. Write

$$G/N = (Q/N) \times (E/N)$$

where Q/N is a quaternion group of order 8 and E/N is an elementary abelian 2-group. Let iN, jN and kN be a canonical set of generators for Q/N. Thus $j^i = j^{-1}$ or $j^{-1}a$ where $a \in N$. Since N lies in the centre of G, we obtain $j^{i^2} = j$ in either case. By the same reasoning i^2 commutes with k, with the result that i^2 is in the centre of $Q = \langle i, j, k, N\rangle$. Let $e \in E$; since $[Q, E] \leq N$, the mapping $xN \longrightarrow [x, e]$ is a homomorphism of Q/N into N. The kernel must be nontrivial, so it contains i^2N. Thus $[i^2, E] = 1$ and because $G = QE$ the element i^2 lies in the centre of G. Since $i^2 \neq 1$ and N is the monolith of G, we can conclude that $N < \langle i^2\rangle$. Moreover iN has order 4, so i has order 8 and $N = \langle i^4\rangle$. Hence $i^j = i^{-1} = i^7$ or $i^j = i^{-1}i^4 = i^3$; but $i^2 = (i^2)^j = (i^j)^2 = i^6$ in either case, giving the contradiction $i^4 = 1$. This case cannot, therefore, arise.

4. **Soluble JNT-groups without minimal normal subgroups.** Let G be a soluble JNT-group which is not nilpotent and write $L = [G', G]$. Then $L \neq 1$, so G/L is a Dedekind group; L is also the limit of the lower central series of G. We shall assume throughout this section that L *contains no minimal normal subgroups of* G, our aim being to show that G is of type IX. This will be achieved by means of the following programme:

(i) L is torsion-free and abelian.

(ii) L is *rationally irreducible* with respect to G, i.e. every nontrivial normal subgroup of G that is contained in L has periodic factor group in L.

(iii) L is radicable.

(iv) G is of type IX.

If $1 \neq N \lhd G$, then G/N is a soluble T-group and hence is metabelian. Thus $G'' \leq N$ and G'' is either 1 or the monolith of G; the latter is impossible since $G'' \leq L$. Consequently G is metabelian, so G'—and hence L—is abelian.

If L is not torsion-free, there is a prime p such that the subgroup $P = \{a: a \in L, a^p = 1\}$ is not 1. Clearly $P \lhd G$, and since P cannot contain a minimal normal subgroup of G, there is an infinite chain of nontrivial normal subgroups of G,

$$P = P_1 > P_2 > \cdots > P_\alpha > \cdots, \qquad (\alpha < \beta),$$

such that

$$\bigcap_{\alpha<\beta} P_\alpha = 1; \tag{4}$$

here β is necessarily a limit ordinal. Let $g \in G$; since G/P_α is a T-group, g induces in P/P_α a power automorphism. Should g centralise P/P_2, it will centralise every P/P_α and hence P; for g must induce in P/P_α an automorphism $a \to a^n$ where n is independent of α, since P is elementary abelian. Therefore $C_G(P/P_2) = C_G(P)$. Consequently $G/C_G(P)$ is cyclic of order dividing $p - 1$, and, if $1 \neq a \in P$, then a^G is finitely generated and therefore finite. However this would imply that a^G contained a minimal normal subgroup of G, contrary to hypothesis.

Next we prove that L is rationally irreducible with respect to G. If this is false, there surely exists an infinite chain of nontrivial normal subgroups of G,

$$L = L_1 > L_2 > \cdots > L_\alpha > \cdots, \qquad (\alpha < \beta),$$

such that L/L_2 is not periodic and

$$\bigcap_{\alpha<\beta} L_\alpha = 1; \tag{5}$$

again β is a limit ordinal. Let $\alpha \geq 2$; now $(G/L_\alpha)' = G'/L_\alpha \geq L/L_\alpha$, so G/L_α is a soluble T-group of type I. Let $C = C_G(L)$; then certainly C centralises L/L_α. Now if H is any T-group, $C_H(H') = C_H([H', H])$ [18, Lemma 2.2.2]. From this we deduce that $C \leq C_G(G'/L_\alpha)$. Let $g \in G \backslash C$—note that $C \neq G$ because G is not

nilpotent. Then g does not centralise G'/L_α if α is large enough. From the structure of soluble T-groups of type I (§1.21) it follows that C/L_α is abelian and g induces in C/L_α the automorphism $a \to a^{-1}$ for each $\alpha \geq 2$. By (5), C is abelian and $c^g = c^{-1}$ for all $c \in C$ and $g \in G \backslash C$. This implies that every subgroup of C is normal in G and Lemma 6 yields the contradiction that G is not a JNT-group.

We wish now to establish that L is radicable. Supposing this to be false, we can find a prime p such that $L^p < L$. Here p must be odd, for L/L^2 is radicable by Lemma 2.4.1 of [18]. Now G/L^p is a nonabelian soluble T-group in which the elements of finite order form a subgroup (since L/L^p is periodic and G/L nilpotent). If G/L is not periodic, G/L^p is soluble T of type II [18, Corollary 2, Theorem 3.1.1] and L/L^p is radicable [18, Theorem 4.3.1]. This is impossible, so G/L is periodic.

If L^{p^ω} is the intersection of all the subgroups L^{p^i}, then L/L^{p^ω} is torsion-free since L is; the rational irreduciblility of L now implies that $L^{p^\omega} = 1$. If $x \in D = C_G(L/L^p)$, then x induces in each L/L^{p^i} an automorphism with order a power of p. Since G/L is periodic, we conclude that x induces in L an automorphism with order a power of p. If $C = C_G(L)$, then D/C is a p-group. However G/L^p is a periodic soluble T-group and therefore G/L can have no elements of order p [18, Theorem 4.2.2]. Since $L \leq C$, it follows that $C = D$. Therefore G/C is a cyclic group of order dividing $p - 1$. Let $1 \neq a \in L$ and set $A = a^G$. Then A is free abelian of finite rank. Suppose now that G/L contains an element with odd prime order q. Then G/A^q is periodic and G/L and L/A^q both contain an element of order q, contradicting Theorem 4.2.2 of [18]. Consequently G/L is a 2-group.

Let g be any element of $G \backslash C$. Then g induces in A/A^{3^i} a power automorphism whose order is a power of 2 and divides $\phi(3^i) = 2 \cdot 3^{i-1}$; hence g must induce the identity or $a \to a^{-1}$, the latter being the only power automorphism of order 2. The intersection of all the A^{3^i} is 1 since A is free abelian; thus $a_1^g = a_1^{-1}$ for all a_1 in A unless $[A, g] = 1$. Since L/A is periodic and L is torsion-free, it follows that

$$a^g = a^{-1} \qquad (a \in L,\ g \in G \backslash C). \tag{6}$$

Next $L = L^2$; thus L/A^2 has a subgroup of type 2^∞. Let i be an integer > 2 and let P/A^{2^i} denote the subgroup of all elements of L/A^{2^i} which have odd order. Then $P < L$ and L/P is a radicable abelian 2-group. C/A^{2^i} centralises L/A^{2^i}, and hence G'/A^{2^i}, so it is Dedekind; but C/A^{2^i} also has a factor of type 2^∞, which causes it to be abelian. Therefore C is abelian. Let us write

$$C/A^{2^i} = (P/A^{2^i}) \times (E/A^{2^i}) \tag{7}$$

where E/A^{2^i} is a 2-group—recall here that G/L is a 2-group. G/P is a 2-group which is not Dedekind. Let $g \in G\backslash C$; then g cannot centralise G'/P since elements of L/P are transformed by g into their inverses and L/P is radicable. From the structure of soluble 2-groups with T [18, Lemma 4.2.1] we know that g^2P belongs to the centre of G/P and hence g^2 centralises E/A^{2^i}. Also g^2 centralises L by (6), so g^2 centralises P. By (7), C/A^{2^i} is centralised by g^2 for each i; thus g^2 centralises C. In addition, g induces a nontrivial power automorphism in C/P since g does not centralise L/P. Once again we invoke the structure of soluble 2-groups with T and conclude that every g in $G\backslash C$ induces $a \longrightarrow a^{-1}$ in C/P. If $c \in C$ and $g \in G\backslash C$, then $c^g = c^{-1}a$ where $a \in P$; therefore

$$c = c^{g^2} = (c^g)^{-1}a^{-1} = (c^{-1}a)^{-1}a^{-1} = ca^{-2}$$

since C is abelian. Since L is torsion-free, $a = 1$ and $c^g = c^{-1}$. It follows that every subgroup of C is normal in G. Lemma 6 provides the contradiction that G is not a JNT-group. L is therefore radicable.

Lemma 8 can now be applied with $A = L$ and $N = G$; observe here that by rational irreducibility there are no proper nontrivial radicable subgroups of L that are normal in G. Hence L has a complement in G, say X;

$$G = LX \quad \text{and} \quad L \cap X = 1.$$

Suppose that X does not act faithfully on L, i.e. $D = C_X(L) \neq 1$. Notice that $D \lhd LX = G$ and that G/D is a T-group. Now $G' \not\leq D$ and $G'D/D \geq LD/D \simeq L$, so G/D is a soluble T-group of type I. If $g \in G\backslash C$, then $a^g = a^{-1}$ for all a in L in view of the structure of soluble T-groups of type I and because L and LD/D are isomorphic as G-operator groups. Rational irreducibility now forces L to be isomorphic with Q, the additive group of rational numbers. Let M be a subgroup of L such that L/M is isomorphic with Q/Z, where Z is the subgroup of all integers. Then $M \lhd G$ and C/M is abelian. An element g of $G\backslash C$ induces $a \longrightarrow a^{-1}$ in L and thus in L/M; also g induces a power automorphism in C/M which must agree with $a \longrightarrow a^{-1}$ on L/M. Since L/M has elements of every finite order and power automorphisms map elements of the same order to the same power, g must induce $a \longrightarrow a^{-1}$ also in C/M. The intersection of all subgroups like M is 1, which shows that C is abelian and $a^g = a^{-1}$ for all $a \in C$. Hence every subgroup of C is normal in G. Lemma 6 now gives a contradiction, so X acts faithfully on L. Since the group G is metabelian, $[L, X'] = 1$; therefore X is abelian and $G' = L$.

Let Q denote the field of rational numbers. Choose $a \neq 1$ from L; then L is an irreducible QX-module and $L = a^{QX}$. The mapping $r \longrightarrow a^r$ is a homomorphism of QX-modules from QX onto L with kernel K, a maximal ideal of QX;

thus QX/K is a field. Let $A = a^X$; then $A \lhd G$ and G/A is a T-group. If $x \in X$ and n is a nonzero integer, then

$$(a^{1/n}A)^x = a^{m/n}A$$

for some integer m. Therefore $(1/n)x - m/n \in ZX + K$, where ZX is the integral group ring of X, and

$$QX = Q + ZX + K. \tag{8}$$

Since L is not a minimal normal subgroup of G, there is a prime p and a normal subgroup P of G such that P is properly contained in L and L/P is a p-group. Let I be the intersection of all such P. If $I \neq 1$, then L/I is a p-group since it is periodic. Hence $I = I^q$ for all primes $q \neq p$; but $I = I^p$ by minimality of I, so in fact I is radicable. Thus $I = 1$.

Next let L/P_1 and L/P_2 be nontrivial p-groups where $P_1 \lhd G$ and $P_2 \lhd G$. An element x of X induces in L/P_1 and L/P_2 power automorphisms that can be described by p-adic integers α_1 and α_2. But $L/P_1 \cap P_2$ is also a p-group and x induces in it a power automorphism describable by a p-adic integer α_3. Clearly $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_3$, so $\alpha_1 = \alpha_2$. It follows that to each x in X there corresponds a unique p-adic integer unit α_x such that $b^x P = (bP)^{\alpha_x}$ for all b in L and all $P \lhd G$ with $P \leq L$ and L/P a p-group. Moreover, $\alpha_x = 1$ if and only if $x = 1$ since $I = 1 = C_X(L)$.

This enables us to construct a mapping α from QX to F_p, the field of p-adic numbers, as follows:

$$\left(\sum_{x \in X} r_x x\right)\alpha = \sum_{x \in X} r_x \alpha_x \qquad (r_x \in Q).$$

α is a ring homomorphism because $\alpha_{xy} = \alpha_x \alpha_y$; also it is easy to verify that $\operatorname{Ker} \alpha = K$, using the fact that $I = 1$. Let F be the image of QX under α; then

$$QX/K \simeq F \leq F_p,$$

the isomorphism being of rings. Therefore F is a subfield of F_p. Define Y to be the image of X under α; then Y consists of p-adic integer units. Let $\bar{G}$ be the semidirect product of F (*qua* additive group) by Y. Then the mapping $a^r x \to ((r)\alpha, \alpha_x)$ is easily seen to be an isomorphism of G with $\bar{G}$.

Finally, $\bar{G}$ is of type IX. For $(ZX)\alpha = Y^+$, the additive subgroup generated by Y, and by (8), $F = Q + Y^+$; on the other hand, $F = Y^+$ would imply that L is minimal normal in G. Also $Y \neq \langle -1 \rangle$ since otherwise $F = Q$ and G would be a T-group.

5. **Nonperiodic soluble JNT-groups which contain a minimal normal subgroup.** Throughout this section G will denote a nonnilpotent, nonperiodic, soluble JNT-group such that $L = [G', G]$ contains a minimal normal subgroup of G, say N.

Of course, N is either an elementary abelian p-group or a direct product of copies of the additive group of rational numbers.

(5.1) **Case N torsion-free.** Observe first that G is monolithic with monolith N. For otherwise there is a normal subgroup $M \neq 1$ such that $M \cap N = 1$. Since $N \overset{G}{\simeq} MN/M$, the subgroup MN/M is minimal normal in the soluble T-group G/M and hence is simple; therefore N is cyclic of prime order, contrary to assumption.

Suppose that G splits over N and that X is a complement of N. Then $C_X(N) \lhd G$ and, G being monolithic, this implies that $C_X(N) = 1$ and X acts faithfully on N. Finally X is a soluble T-group and N is not cyclic, so G is of type VIII.

Consider next what happens if G does not split over N. In this situation Lemma 8 shows that

$$[N, G'] = 1. \tag{9}$$

G/N cannot be abelian; for if it were, G' would equal N and, since $L = [G', G] \neq 1$, Lemma 8 would imply that G splits over N. Also, $G/C_G(N)$ is essentially an irreducible group of automorphisms of N; since N is torsion-free, a theorem of Baer [1, Proposition] shows that $G/C_G(N)$—and hence G/N—cannot be periodic. If G/N were a soluble T-group of type I, then G/G' would be periodic [18, 3.1] and, by (9) so would $G/C_G(N)$. It follows that G/N is a soluble T-group of type II and G'/N is periodic. Commutation with a fixed element of G' gives rise to a homomorphism of G'/N into N since $[N, G'] = 1$. However $\mathrm{Hom}(G'/N, N) = 0$ because G'/N is periodic and N is torsion-free. Consequently G' is abelian. Now, N being radicable, we can write $G' = N \times R$ where $R \simeq G'/N$. Clearly R is the subgroup of all elements in G' with finite order and $R \lhd G$. But N is the monolith of G, so $R = 1$ and $G' = N$, contradicting the noncommutativity of G/N. Hence this case cannot arise.

(5.2) **Case N an elementary abelian p-group and $[N, G'] \neq 1$.** Here Lemma 8 can be applied directly to show that G splits over N; let X be a complement of N and suppose that $C = C_X(N) > 1$. Since $C \lhd G$, we can assert that G/C is a soluble T-group. Moreover $N \overset{G}{\simeq} NC/C$, so N is cyclic of order p. It follows that G' centralises N. This is impossible, so $C = 1$ and G is of type VIII—note that N is not cyclic.

(5.3) **Case N an elementary abelian p-group and $[N, G'] = 1$.** This case leads to several different types of groups and will accordingly be analysed under several subheadings. Since N, but not G, is periodic, G/N is abelian or soluble T of type I or soluble T of type II.

(5.31) **Case G/N abelian.** Here $G' = N$ and G splits over N by Lemma 8. Let X be a complement of N in G and suppose that $C = C_X(N) \neq 1$. Then NC/C—and hence N—is cyclic of order p. Therefore X/C is finite and C must be

nonperiodic. Let F be a maximal torsion-free subgroup of C; then $F \lhd G$ since $[N, F] = 1$ and X is abelian. Also, C/F is periodic by maximality of F; consequently G/F is periodic. However, Lemma 5 yields the contradiction $N = 1$ or $F = 1$. It follows that $C = 1$. If N were cyclic, X would be cyclic with order dividing $p - 1$ and G would be finite. Hence N is not cyclic and G is of type VIII.

(5.32) **Case G/N a soluble T-group of type I.** From the structure of soluble T-groups of type I it is seen that G/G' is periodic and G'/N nonperiodic and abelian. Choose a maximal torsion-free subgroup F/N of G'/N; then $F \lhd G$ because G/N is a T-group; also G/F is periodic. By hypothesis $[N, G'] = 1$; thus $F' \leq N \leq \zeta(F)$. Therefore, for any x, y in F, $1 = [x, y]^p = [x^p, y]$ and $F^p \leq \zeta(F)$; in particular, F^p is abelian. Since F/N is torsion-free and N elementary abelian, $H = (F^p)^p$ is torsion-free. But clearly $H \lhd G$ and G/H is periodic. Lemma 5 once again gives a contradiction, showing that this case cannot arise.

(5.33) **Case G/N a soluble T-group of type II.** Here G'/N is a nontrivial, periodic radicable group; moreover, if $C = C_G(G'/N)$, then C/N is periodic and abelian. Two possibilities must now be distinguished.

(5.331) **Case $[N, C] \neq 1$.** By Lemma 8 the group G splits over N; let X be a complement of N and assume that $D = C_X(N) \neq 1$. Then G/D is a soluble T-group, so it is metabelian; also $N \cap D = 1$, showing that G is metabelian. Next $(G')^p \neq 1$ since G'/N is radicable; therefore $G/(G')^p$ is a soluble T-group in which the elements of finite order form a proper subgroup; such a group cannot be of type I. Consequently, either $G' = (G')^p$ or $G/(G')^p$ is soluble T of type II, in which event $G'/(G')^p$ is radicable, an obvious absurdity. It follows that $G' = (G')^p$. Also G' is periodic and abelian; with the aid of Lemma 5, we deduce that G' is a radicable abelian p-group. If $c \in C$, the mapping $a \to [a, c]$ is a homomorphism of G' into N since $[G', C] \leq N$ and $[N, G'] = 1$. But $\mathrm{Hom}(G', N) = 0$ because G' is radicable and N elementary abelian. Hence $[G', C] = 1$ and $[N, C] = 1$, which is contrary to hypothesis. Therefore $D = 1$ and G is again of type VIII.

(5.332) **Case $[N, C] = 1$.** We shall show that G is of type VI or VII by pursuing the following programme:

(i) G' is a radicable abelian p-group.

(ii) G is monolithic with monolith N.

(iii) Properties of $C = C_G(G'/N)$.

(iv) C is abelian and G is of type VI.

(v) C is nilpotent of class 2 and G is of type VII.

Since C/N is abelian, C is nilpotent of class ≤ 2. Lemma 5 implies that C is also a p-group. Commutation with a fixed element of C produces a homomorphism of G'/N into N since $[G', C] \leq N$ and $[N, C] = 1$. But $\mathrm{Hom}(G'/N, N) = 0$, so that

$$[G', C] = 1 \quad \text{and} \quad C = C_G(G'). \tag{10}$$

In particular G' is abelian. Furthermore $G' = (G')^p$ since otherwise $G/(G')^p$ would be soluble T of type II. Therefore G' is a radicable abelian p-group.

Let $g \in G$ and denote by τ_g the automorphism induced by g in G'. Now g induces in G'/N a power automorphism which can be described by a p-adic integer unit, say α_g. Writing θ for the power automorphism $a \to a^{\alpha_g}$ of G', we see that $\tau_g^{-1}\theta$ acts trivially on G'/N. Therefore $\tau_g^{-1}\theta - 1 \in \mathrm{Hom}(G', N) = 0$ and $\tau_g = \theta$. Consequently

$$a^g = a^{\alpha_g} \qquad (a \in G', g \in G). \tag{11}$$

The next point to establish is that G is monolithic with monolith N; since (11) shows every subgroup of G' to be normal in G, it will then follow that G' is a p^∞-group. Suppose there exists $M \lhd G$ such that $1 \neq M$ and $N \cap M = 1$. If $C \cap M = 1$, then $[G', M] \leq C \cap M = 1$ and $M \leq C$. This cannot be, so $C \cap M \neq 1$ and there is no loss in assuming that $M \leq C$. Now G/C—and hence G/M—is nonperiodic and G/M is not abelian since $N \leq G'$. Hence G/M is also a soluble T-group of type II. It follows that C/M—and hence C—is abelian. Let $g \in G$; then g induces in G'/N—*and therefore in* C/N—the automorphism $a \to a^{\alpha_g}$; here we use the radicability of G'/N. But g also induces $a \to a^{\alpha_g}$ in G' by (11), and therefore in $G'M/M$ and C/M. Hence $a^g = a^{\alpha_g}$ for all $a \in C$. Since $C = C_G(G')$ and G' has infinite exponent, Lemma 6 provides a contradiction. Thus we conclude that G is monolithic with monolith N.

Let Z be the centre of C; then $G' \leq Z$ by (10) and since G' is radicable,

$$Z = G' \times D_1 \tag{12}$$

for some $D_1 \leq Z$. Clearly $D_1N \lhd C \lhd G$, so $D_1N \lhd G$ and $[D_1, G] \leq G' \cap (D_1N)$, which shows that

$$[D_1, G] \leq N. \tag{13}$$

If $d \in D_1$ and $g \in G$, then $1 = [d, g]^p = [d^p, g]$; therefore $D_1^p \leq \zeta(G)$ and $D_1^p \lhd G$; since N is the monolith of G, it follows that $D_1^p = 1$ and D_1 is an elementary abelian p-group.

Write T/G' for the torsion-subgroup of G/G'; then $C \leq T$ since C/N is periodic. Moreover $C \neq G'$ by Lemma 6. The structure of soluble T-groups of type II provides the following information: C/G' has finite exponent p^e and $\alpha_g \equiv 1 \bmod p^e$ for all $g \in G$ [18, Theorem 4.3.1]. Observe that $e > 0$, so that

$$\alpha_g \equiv 1 \bmod p. \tag{14}$$

Now suppose that $g \in T$; since α_g is a p-adic integer unit of finite order satisfying (14), there are the following possibilities: either p is odd and $\alpha_g = 1$ or $p = 2$ and $\alpha_g = \pm 1$. Thus T/C has order 1 or 2 and we can write

$$T = \langle t, C \rangle \tag{15}$$

where either $t = 1$ or $\alpha_t = -1$; in either case $t^2 \in C$ and t^2N has order 1 or 2; therefore t^2 has order dividing 4. Next we show that t^2 *is in the centre of* G; let $t \neq 1$. If $g \in G$ then $t^g = ta$ where $a \in G'$. Hence $(t^2)^g = (ta)^2 = t^2$ since $a^t = a^{-1}$. In particular $\langle t^2 \rangle \lhd G$. If $t^2 \neq 1$, then $N \leq \langle t^2 \rangle$. Let $p = 2$ and $G' = \langle a_1, a_2, \cdots \rangle$ where $a_{i+1}^2 = a_i$ and $a_1^2 = 1$; then $N = \langle a_1 \rangle$. Clearly $t^2 \in Z$; let us consider the position of t^2 in Z. If $t^2 \in G'$, then $t^2 = 1$ or a_1 since $a_2^t = a_2^{-1}$ if $t \neq 1$. Suppose $t^2 \notin G'$; then since t^2 has order 2 or 4, it belongs to $\langle a_2 \rangle \times D_1$, and since $a_1 \in \langle t^2 \rangle$, we can assume that $t^2 = a_2 u$ where $1 \neq u \in D_1$. Thus the possibilities for t^2 are 1, a_1 and $a_2 u$.

The group T/G' has finite exponent and this is well known to imply that T/G' is a direct factor of G/G' (see [13, Theorem 8]); let

$$G/G' = (T/G') \times (Y/G'). \tag{16}$$

From (16) and (15) we obtain

$$G = \langle t, C, Y \rangle. \tag{17}$$

Consider the case when C *is abelian*. Here $C = Z = G' \times D_1$ and (17) becomes $G = \langle t, D_1, Y \rangle$. Now set $W = \langle t, Y \rangle$. Since D_1 is elementary abelian, we can write $D_1 = (\langle t, G' \rangle \cap D_1) \times D$, say. Hence $G = \langle t, D, Y \rangle = WD$—observe that $W \lhd G$ since $G' \leq Y \leq W$. Since $T \cap Y = G'$, we have $W \cap D \leq \langle t, G' \rangle \cap D = 1$. Hence $G = WD$ and $W \cap D = 1$.

Next we analyse the structure of W. First

$$W/W \cap C \simeq WC/C = G/C. \tag{18}$$

The map $gC \longrightarrow \alpha_g$ is an isomorphism of G/C with a nonperiodic group Γ of p-adic integers all of which are congruent to 1 modulo p. Now $W \cap C = W \cap (G' \times D_1) = G' \times (W \cap D_1)$. Also

$$W \cap D_1 = \langle t, Y \rangle \cap D_1 = \langle t, G' \rangle \cap D_1 = \langle t^2, G' \rangle \cap D_1.$$

If $t^2 \in G'$, then $W \cap D_1 = 1$ by the last equation; otherwise $t^2 = a_2 u$ and $\langle t^2, G' \rangle \cap D_1 = \langle u \rangle$, so $W \cap D_1 = \langle u \rangle$. Hence $W \cap C = G'$ or $G' \times \langle u \rangle$ according as $t^2 \in G'$ or $t^2 \notin G'$. Also, $W/W \cap C \simeq \Gamma$ by (18). Suppose that $t^2 = a_1$; if there exists a d in D such that $t^d \neq t$, then $d^t = da_1$ by (13); thus $(td)^2 = t^2 a_1 = 1$, and, replacing t by td, we can assume that $t^2 = 1$. In other words, we can exclude $t^2 = a_1$ unless $[D, t] = 1$. Suppose $t^2 \notin G'$; then $p = 2$ and $t^2 = a_2 u, (u \in D_1)$. Let $g \in G$; since t^2 belongs to the centre of G, we have $(a_2 u)^g = a_2 u$ and $u^g = a_2^{1-\alpha_g} u$. Thus $u^g = u$ or $a_1 u$ according as $\alpha_g \equiv 1 \bmod 4$ or $\alpha_g \not\equiv 1 \bmod 4$.

Finally, D centralises N and W/N. If $t^2 \in G'$, then in addition $D \neq 1$; for $W' = G'$ and $C_W(W') = W \cap C = G'$; thus W is not a JNT-group by Lemma 6 and $D \neq 1$. The centre of G contains no element of order p except a_1 since N is the monolith. G is of type VI(a) or (b) according as $t^2 \in G'$ or $t^2 \notin G'$.

The remaining possibility under heading (5.332) is that C *is nilpotent of class exactly* 2. The centre of C is Z and C/N is abelian; thus

$$N = C' < G' \leq Z < C.$$

If $x, y \in C$, then $1 = [x, y]^p = [x^p, y]$, showing that $C^p \leq Z$ and C/Z is an elementary abelian p-group.

Let us prove next that $C^p = G'$. Now $G' \leq C^p$ is immediate, and since C^p is abelian, we can write $C^p = G' \times F$. Suppose x is a nontrivial element of F; then $x = c^p a$ for some $c \in C$ and $a \in N$; for C/N is abelian. Let g be any element of G; then $c^g = c^{a_g} b$ for some $b \in N$. Hence

$$(c^p)^g = (c^{a_g} b)^p = (c^p)^{a_g}$$

and $\langle c^p \rangle \lhd G$. Now $a \neq x$ since $G' \cap F = 1$; thus $c^p \neq 1$ and $a \in \langle c^p \rangle$. Therefore $x = c^p a \in \langle c^p \rangle$ and $\langle x \rangle \lhd G$. This gives the contradiction $N \leq \langle x \rangle$. Hence $F = 1$ and $C^p = G'$.

Let $\{x_\lambda Z : \lambda \in \Lambda\}$ be a basis for the elementary abelian p-group C/Z and set $X = \langle x_\lambda : \lambda \in \Lambda \rangle$. Then $C = XZ = XG'D_1$. Let $y \in (XG') \cap D_1$ and write $y = x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} a$ where $a \in G'$, the n_i are integers and the λ_i are distinct elements of Λ. Then $x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} \in Z$ and the linear independence of the $x_{\lambda_i} Z$ implies that $p | n_i$ for $i = 1, \cdots, r$. Hence $y \in C^p G' = G'$ and $y \in G' \cap D_1 = 1$. Writing $E = XG'$ we obtain $C = E \times D_1$.

Suppose that $x_\lambda^p \neq 1$. Since $G' = C^p$, there is an element a of G' such that $x_\lambda^p = a^p$. This implies that $(x_\lambda a^{-1})^p = 1$. Replacing x_λ by $x_\lambda a^{-1}$ we may assume that $x_\lambda^p = 1$ for all $\lambda \in \Lambda$. This implies that $X \cap Z = N$: for let $y \in X \cap Z$; since $X' \leq N$, we can write $y \equiv x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} \bmod N$ where the λ_i are distinct. Thus $p | n_i$ for all i and $y \in N$ as required. Next N is actually the centre of X since $\zeta(X) \leq X \cap Z = N$. Thus $X' = \zeta(X) = N$ and $X/N \simeq C/Z$. Consequently X is an extraspecial p-group. Since $X \cap G' = N = \langle a_1 \rangle$, the group E is a direct product of X and G' in which the centre of X and $\langle a_1 \rangle$ are amalgamated.

Consider now the position of t^2; if $t^2 \notin G'$, then $p = 2$ and $t^2 = a_2 u$ where $u \in D_1$. If $t^2 \in G'$ and $t^2 \neq 1$, then $p = 2$ and $t^2 = a_1$. Suppose that $[X, t] \neq 1$; then $[x_\lambda, t] \neq 1$ for some $\lambda \in \Lambda$. Since $x_\lambda N$ has order 2, it is centralised by t and $x_\lambda^t = x_\lambda a_1$; thus $(tx_\lambda)^2 = a_1^2 x_\lambda^2 = 1$. If, however, $[X, t] = 1$, write $a_1 = x^2$ for some $x \in X$ (note $N = X^2$); then $(tx)^2 = 1$. Therefore, if $t^2 \in G'$, we may assume that $t^2 = 1$.

From $C = XZ$ and equations (12) and (17) we obtain $G = \langle t, X, Y, D_1 \rangle$. Now define $W = \langle t, X, Y \rangle$. Writing

$$D_1 = (\langle t, G' \rangle \cap D_1) \times D,$$

we have $G = WD$ and $W \lhd G$. Suppose that $w \in W \cap D$ and, using $G' \leq Y$, write

$w = t^i xy$ where $x \in X$ and $y \in Y$; then $y \in T \cap Y = G'$ and $t^i \in C$, which shows that i may be assumed even. Since $t^2 \in Z$, it follows that $x \in X \cap Z = N$ and $w \in \langle t, G' \rangle \cap D = 1$. Hence $W \cap D = 1$.

We turn now to the structure of W. First $W \cap C = W \cap (E \times D_1) = E \times (W \cap D_1)$; moreover, as above,

$$W \cap D_1 = \langle t, X, G' \rangle \cap D_1 = \langle t^2, X, G' \rangle \cap D_1 = \langle t^2, G' \rangle \cap D_1.$$

If $t^2 \in G'$, then $W \cap D_1 = 1$; otherwise $t^2 = a_2 u$ and $W \cap D_1 = \langle u \rangle$. Thus $W \cap C = E$ or $E \times \langle u \rangle$ according as $t^2 \in G'$ or $t^2 \notin G'$. Moreover $W/W \cap C \simeq G/C$ and G/C is isomorphic with Γ, a group of p-adic integers all of which are congruent to 1 modulo p. If $t^2 \notin G'$ then one shows (as in the case C abelian) that $u^g = u$ or $a_1 u$ according as $\alpha_g \equiv 1 \bmod 4$ or $\alpha_g \not\equiv 1 \bmod 4$. Since $\alpha_g \equiv 1 \bmod p$ for all g in G, the factor X/N is central in G and $[X, G] \leq N$.

Finally D acts faithfully on W and centralises W/N and N. Therefore G is of type VII(a) or (b) according to whether t^2 is or is not in G'.

6. **Periodic soluble JNT-groups.** Throughout this section G will denote a non-nilpotent JNT-group which is both soluble and periodic; N is a minimal normal subgroup of G contained in $L = [G', G]$. Thus N is an elementary abelian p-group for some prime p.

(6.1) **Case** $[N, G'] \neq 1$. Here G is of type VIII; the argument is that of (5.2).

(6.2) **Case** $[N, G'] = 1$. Since G'/N is abelian, G' is nilpotent in this case; hence G' is a p-group by Lemma 5. Consequently G has a unique Sylow p-subgroup P containing G'.

We shall require the equation

$$[N, C_P(G'/N)] = 1. \tag{19}$$

To prove this let x in P centralise G'/N. Since $C_G(G'/N)/N$ is nilpotent, $\langle x, N \rangle \lhd G$, which shows that $[N, x] \lhd G$. Assuming that $[N, x] \neq 1$, we find that $N = [N, x]$ since N is minimal normal in G. If p^r is the order of x, then $N = [N, {}_{p^r}x] = [N, x^{p^r}] = 1$, a contradiction.

(6.21) **Case** $[N, G'] = 1$ **and** P/N **abelian.** Here P centralises G'/N since $G' \leq P$; therefore

$$[N, P] = 1 \tag{20}$$

by (19). Also $P' \leq N$ and (20) shows that P is nilpotent of class at most 2. Since G/P is abelian, $[(C_G(P))', C_G(P)] = 1$ and $C_G(P)$ is nilpotent. Lemma 5 shows that $C_G(P)$ is a p-group; hence

$$C_G(P) \leq P. \tag{21}$$

Now it is necessary to distinguish two subcases.

(6.211) **Case** P **abelian.** Suppose that P contains an element of order p^2

and define P_1 to be the subgroup of all a in P such that $a^{p^2} = 1$. Thus $1 \neq P_1^p \lhd G$ and G/P_1^p is a T-group. Let $g \in G$; then g induces a power automorphism $a \longrightarrow a^\alpha$ in P/P_1^p—here α is a p-adic integer unit. Writing ξ for the automorphism of P induced by g and η for the power automorphism $a \longrightarrow a^\alpha$ of P, we see that $\theta = \xi^{-1}\eta$ is an automorphism of P which acts trivially on P/P_1^p. Therefore, for any $a \in P$ we have $a^\theta = ab^p$ for some $b \in P_1$; hence $(a^p)^\theta = (ab^p)^p = a^p$. Therefore $(b^p)^\theta = b^p$; from this and $a^\theta = ab^p$ it follows that $a^{\theta^p} = ab^{p^2} = a$. Hence $1 = \theta^p = \xi^{-p}\eta^p$ since a power automorphism commutes with every automorphism of P. Now by (21) we have $C_G(P) = P$, so the order of ξ is prime to p. Consequently $\xi^p = \eta^p$ implies that $\xi \in \langle\eta\rangle$ and ξ is a power automorphism of P. By Lemma 5.2.2 of [18] the group G is a T-group. In view of this contradiction P is an elementary abelian p-group.

Next it will be shown that G splits over P. By hypothesis $C_G(N) \geq G'$; therefore $G/C_G(N)$ is abelian. $G/C_G(N)$ can be regarded as an irreducible group of linear transformations of N *qua* vector space. This implies that $G/C_G(N)$ is isomorphic with a periodic subgroup of the multiplicative group of a field. Therefore $G/C_G(N)$ is locally cyclic and, in particular, countable (see [6, p. 296]). Since elements of G induce power automorphisms in the elementary abelian p-group P/N, the group $C_G(N)/(C_G(N) \cap C_G(P/N))$ is cyclic of order dividing $p - 1$. If g centralises both N and P/N, then g induces in P an automorphism of order 1 or p. But $P = C_G(P)$, so $g \in P$. Thus

$$C_G(N) \cap C_G(P/N) = C_G(P) = P \tag{22}$$

and G/P is countable. Lemma 7 shows that G splits over P, say $G = PX$ and $P \cap X = 1$. Moreover X acts faithfully on P in view of (22).

Consider the situation when N is the monolith of G. Assume that $N \neq P$ and let $a \in P\backslash N$. If a^G were finite, it would be a direct product of minimal normal subgroups of G by Maschke's theorem—observe that X has no elements of order p. This is consistent only with $a^G = N$ because N is the monolith. Therefore a^G must be infinite—and hence so is X. Since $X/C_X(P/N)$ is finite, $C_X(P/N) \neq 1$. Also $C_X(P/N) \lhd X$ and X is a soluble T-group, so $C_X(P/N) \cap C_X(X') \neq 1$; let x be a nonunit element of this intersection. Then $\langle x\rangle \lhd C_X(X') \lhd X$, so that $\langle x\rangle \lhd X$; consequently $[P, x] \lhd G$ and $C_P(x) \lhd G$. If $C_P(x) \neq 1$, then $N \leq C_P(x)$ and x belongs to $C_G(P/N) \cap C_G(N) = P$ by (22); thus $x \in P \cap X = 1$. Therefore $C_P(x) = 1$. Now x centralises P/N and consequently $[P, x] \leq N$. Evidently $[N, x] \lhd G$ and $[N, x] \neq 1$, which shows that $N = [N, x] = [P, x]$. Thus $[P, x] = [P, x, x]$. Let $a \in P$; then $[a, x] = [b, x, x]$ for some $b \in P$, and $C_P(x) = 1$ implies that $a = [b, x]$. Thus $P \leq [P, x] = N$. Consequently $N = P$ and G is of type VIII.

We are left with the following situation; there exists a nontrivial $M \lhd G$ with $M \cap N = 1$. If $M \cap P = 1$, then $M \leq C_G(P) = P$; therefore $M \cap P \neq 1$ and we can

assume that $M \leq P$. Also $M \overset{G}{\simeq} MN/N$ shows that every subgroup of M is normal in G and we can assume M to have order p. Also N has order p since $N \overset{G}{\simeq} NM/M$. Suppose now that $P > M \times N$ and choose $a \in P \setminus (M \times N)$. The automorphism groups induced by G in P/M and P/N are both finite. Therefore $G/C_G(P)$ is finite, from which it follows that a^G is finite. Maschke' theorem implies that a^G is a direct product of minimal normal subgroups of G; therefore there exists a minimal normal subgroup L of G contained in P such that $L \not\leq M \times N$. Then $LM/M \overset{G}{\simeq} LN/N$. Let $g \in G$; then g induces in P/M and P/N power automorphisms which both have the form $a \longrightarrow a^n$ since they must agree on LM/M and LN/N. Hence $a^g = a^n$ for all $a \in P$, a situation we have already seen to be impossible (by Lemma 5.2.2 of [18]).

Hence $P = M \times N$ and X is isomorphic with a subgroup of $\mathrm{GL}(2, p)$ which is diagonal because X induces power automorphism groups in M and N; this subgroup X is not scalar since it does not induce a group of power automorphisms in P. Clearly p is odd and G is of type IV.

(6.212) **Case P nilpotent of class 2.** Since P/N is abelian, $P' = N$. Consider the centraliser of P/N. If $g \in C_G(P/N)$, then $[g, P, P] = 1$ by (20); therefore, by the Three Subgroup Lemma, $[g, P'] = 1$, i.e. $[g, N] = 1$. Now let $a \in P$; then $a^g = ab$ where $b \in N$. Hence $a^{g^p} = ab^p = a$ since $[g, N] = 1$. It follows that $C_G(P/N)/C_G(P)$ is a p-group. But $C_G(P) \leq P$ by (21); therefore $C_G(P/N) \leq P$ and

$$C_G(P/N) = P. \tag{23}$$

Next, if $p = 2$, a periodic group of power automorphisms of P/N has order a power of 2 and equation (23) yields $P = G$, i.e. G is nilpotent. Thus p is an odd prime and therefore P is a regular p-group. Also N is the monolith of G; for suppose that $1 \neq M \lhd G$ and $M \cap N = 1$; then PM/M and PN/N are abelian, being Dedekind groups without elements of order 2, and therefore P is abelian, contrary to hypothesis.

Since $P \neq G$, we can find $g \in G \setminus P$ and (23) shows that g cannot centralise P/N. Let g induce in P/N the power automorphism $a \longrightarrow a^\alpha$; here α is a p-adic integer unit $\neq 1$. If $a, b \in P$, then $a^g \equiv a^\alpha \bmod N$ and $b^g \equiv b^\alpha \bmod N$; hence

$$[a, b]^g = [a^\alpha, b^\alpha] = [a, b]^{\alpha^2} \tag{24}$$

since $[N, P] = 1$. From this it follows that $\langle [a, b] \rangle \lhd G$. Since N is the monolith of G, the order of N is p; let $N = \langle a \rangle$, say.

Suppose that $P^p > 1$; then $N \leq P^p$ and consequently $a = b^p$ for some $b \in P$ since P is regular. Now $b^g = b^\alpha c$ for some $c \in N$, and $a^g = (b^g)^p = b^{\alpha p} = a^\alpha$. But equation (24) shows that $a^g = a^{\alpha^2}$; therefore $\alpha^2 \equiv \alpha \bmod p$ and $\alpha \equiv 1 \bmod p$. If P/N has finite exponent p^e, the congruence $\alpha^{p^{e-1}} \equiv 1 \bmod p^e$ implies that g induces in P/N an automorphism of order a power of p; therefore $g \in P$ by (23).

If, however, P/N has infinite exponent, α must have finite order; this, together with $\alpha \equiv 1 \bmod p$ and $p > 2$, implies that $\alpha = 1$. These arguments indicate that

$$P^p = 1. \tag{25}$$

Next the centre of P will be identified; call this Z. Clearly $N \leq Z$. Let $1 \neq a_1 \in Z$; then $\langle a_1, N\rangle \lhd G$, so $a_1^G \leq \langle a_1, N\rangle$, which implies that a_1^G is finite. Now $C_G(a_1^G) \geq P$; therefore Maschke's theorem can be applied to a_1^G; in the usual way it follows that $a_1^G = N$. Thus $Z = N$.

Now P/N is elementary abelian by (25); hence P is an extra-special p-group. Choose a basis for P/N, say $\{x_\lambda N: \lambda \in \Lambda\}$. Then

$$[x_\lambda, x_\mu] = a^{f(\lambda,\mu)} \tag{26}$$

where f is a nondegenerate alternating bilinear form. Now G/P is cyclic with order q dividing $p - 1$ since $P = C_G(P/N)$. Hence there is an element g with order q such that $G = P\langle g\rangle$ and $P \cap \langle g\rangle = 1$. Moreover $\langle g\rangle$ acts faithfully on P.

g induces in the elementary abelian group P/N a power automorphism of the form $x \longrightarrow x^n$ where $1 < n < p$. Thus

$$x_\lambda^g = x_\lambda^n a^{n_\lambda}, \quad (\lambda \in \Lambda), \tag{27}$$

for certain integers n_λ satisfying $0 \leq n_\lambda < p$. A suitable change of basis will simplify these equations. Suppose that $n_\lambda \neq 0$. Since f is nondegenerate, there is a $\mu \in \Lambda$ such that $f(\lambda, \mu) \not\equiv 0 \bmod p$. We shall replace x_λ by a suitable element of the form $\bar{x}_\lambda = x_\lambda^s x_\mu^t$. A brief computation using (26) and (27) yields $\bar{x}_\lambda^g = \bar{x}_\lambda^n a^u$ where $u = sn_\lambda + tn_\mu + st\binom{n}{2}\cdot f(\lambda, \mu)$. We wish to show that $u \equiv 0 \bmod p$ can be solved for s and t with $s \not\equiv 0 \bmod p$ and $t \not\equiv 0 \bmod p$. This amounts to solving

$$xn_\lambda + yn_\mu + z \equiv 0 \bmod p \tag{28}$$

for $x \not\equiv 0 \bmod p$ and $y \not\equiv 0 \bmod p$ where $z = \binom{n}{2}f(\lambda, \mu)$; notice that $z \not\equiv 0 \bmod p$. Since $n_\lambda \not\equiv 0 \bmod p$, we need only look for a y such that $yn_\mu + z \not\equiv 0 \bmod p$. If $n_\mu = 0$, any $y \not\equiv 0$ will do; if $n_\mu \neq 0$, we can choose y so that $1 \leq y < p$ and $yn_\mu + z \not\equiv 0 \bmod p$ since $p > 2$. Consequently (28) has a solution of the required sort.

Now replace x_λ by $\bar{x}_\lambda$, observing that we retain a basis for P/N. Performing this operation whenever necessary, we arrive at a basis for which $x_\lambda^g = x_\lambda^n$ for all $\lambda \in \Lambda$. Thus G is of type V.

(6.22) **Case $[N, G'] = 1$ and P/N nonabelian.** If a soluble p-group has the property T and is not abelian, then $p = 2$ [18, Lemma 4.2.1]. Thus P is a 2-group. Also (by Lemma 2.4.1 of [18]) L/N is a radicable abelian 2-group where, as usual, $L = [G', G]$. Define $C = C_G(G'/N)$ and note that C/N is nilpotent of class ≤ 2. Moreover G/C has order 1 or 2 because ± 1 are the only 2-adic integers with finite order.

Let $x \in C \cap P$; then $aN \longrightarrow [a, x]$ is a homomorphism of L/N into N since $[N, C \cap P] = 1$ (see equation (19)). But $\mathrm{Hom}(L/N, N) = 0$; therefore

$$[L, C \cap P] = 1. \tag{29}$$

Since $L \leq G' \leq C \cap P$, it follows that L is abelian.

Our next aim is to prove that G is a 2-group. Since $|G:C| = 1$ or 2, all elements of G with odd order belong to C. Now C/N is a Dedekind group, so the elements in C/N which have odd order form an abelian subgroup, say Q/N. Assume that $Q \neq N$. From this it follows that $[N, Q] \neq 1$; for if $[N, Q] = 1$, the group Q is nilpotent, and, since $Q \lhd G$, we deduce from Lemma 5 that Q is a 2-group and $Q = N$. Now $[N, Q] \neq 1$ implies that $L = N$. For suppose $L > N$; then L/N is a nontrivial radicable abelian 2-group and $L^2 \neq 1$; therefore L/L^2 is radicable, which shows that L is radicable. Now commutation with a fixed element of C induces a homomorphism of L into N, and yet $\mathrm{Hom}(L, N) = 0$; thus $[L, C] = 1$ and in particular $[N, Q] = 1$, a contradiction. It follows that $L = N$ and G/N is Dedekind; thus $C = C_G(G'/N) = G$ and (29) becomes $[L, P] = 1$. Now $[N, Q] = 1$ implies that G splits over N, by Lemma 8; say $G = NX$ and $N \cap X = 1$. Therefore $P = P \cap (NX) = N(P \cap X)$. Since P/N is not abelian, $P \cap X \neq 1$. Also $[N, P \cap X] \leq [L, P] = 1$, so $P \cap X \lhd NX = G$. Thus $G/P \cap X$ is a T-group and the isomorphism $N \overset{G}{\simeq} N(P \cap X)/P \cap X$ shows that $|N| = 2$. Consequently $N \leq \zeta(G)$ which implies that G is nilpotent. This contradiction establishes that G is a 2-group. Equation (29) now yields

$$[L, C] = 1 \quad \text{and} \quad C = C_G(L). \tag{30}$$

Observe that G/N is not a Dedekind group; for if it were, $L = N$ and $C = G$, so that (30) would become $[L, G] = 1$, i.e. G is nilpotent. Also L is radicable; for $L > N$ and this, as has already been seen, implies that L is radicable.

G/N is a soluble 2-group with the property T and it is also nonnilpotent. By Lemma 4.2.1 of [18] this means that one can write $G = \langle C, t\rangle$ where t transforms each element of C/N into its inverse and $t^2 \in C$; also, of course, C/N is abelian and of infinite exponent. Equation (30), together with the commutativity of C/N, implies that C *is nilpotent of class at most* 2. Let σ be the automorphism $a \longrightarrow a^{-1}$ of L and write τ for the automorphism of L induced by t. Then $\tau^{-1}\sigma$ is trivial on L/N and $\tau^{-1}\sigma - 1 \in \mathrm{Hom}(L, N) = 0$. Therefore $\tau = \sigma$ and

$$a^t = a^{-1}, \quad (a \in L), \tag{31}$$

which shows that t centralises N. Since $G = \langle C, t\rangle$, equation (30) permits us to conclude that $[N, G] = 1$.

Suppose there exists $M \lhd G$ with $M \neq 1$ and $M \cap N = 1$. Since $M \overset{G}{\simeq} MN/N$, one can assume that M has order 2. Also $L \leq M$, so G/M is not a Dedekind group and its structure is similar to that of G/N. In particular CM/M—and hence C—is abelian. Now t transforms elements of L into their inverses. It follows that $a^t =$

a^{-1} for all $a \in C$, which is impossible by Lemma 6. Hence G is monolithic with monolith equal to N. This indicates that L is of type 2^∞ and N has order 2.

Define Z to be the centre of C. Then $L \leq Z$ and, L being radicable, we may write $Z = L \times D$. Suppose that D contains an element d of order 4; then $d^t = d^{-1}a$ for some a in N. Hence $(d^2)^t = (d^{-1}a)^2 = d^{-2} = d^2$. Since $d^2 \in Z$, it follows that d^2 is in the centre of G and $1 \neq \langle d^2 \rangle \lhd G$; this is impossible since $\langle d^2 \rangle \cap N = 1$. Thus D is elementary abelian. This implies that DN/N lies in the centre of G/N, so that

$$[D, G] \leq N. \tag{32}$$

If $d \in D$, the mapping $xC \longrightarrow [x, d]$ is a homomorphism of G/C into N since $[C, D] = 1 = [N, G]$. Now $C_D(G) \lhd G$, so $C_D(G) = 1$, and, since $\mathrm{Hom}(G/C, N)$ has order 2, we must have $|D| = 1$ or 2. Write $D = \langle d \rangle$.

Consider next the position of t^2. Let $a_1, a_2, \cdots$ be a canonical set of generators for the 2^∞-group L. If $c \in C$, then $c^t = c^{-1}a$ for some a in N. Hence $c^{t^2} = (c^{-1}a)^{-1}a = c$. On account of $G = \langle C, t \rangle$ it follows that $t^2 \in \zeta(G)$; in particular $\langle t^2 \rangle \lhd G$. Therefore, either $t^2 = 1$ or $N \leq \langle t^2 \rangle$. Also $t^2 \in Z$ and since t^2N is centralised by t, the element t^2 has order dividing 4. Thus $t^2 \in \langle a_2 \rangle \times \langle d \rangle$ and the possibilities for t^2 are 1, a_1 or a_2d (if $d \neq 1$); for if $d \neq 1$, then $d^t = a_1d$ by (32) since $\langle d \rangle$ cannot be normal in G.

(6.221) **Case C abelian.** Here $C = Z = L \times D$, and $d \neq 1$ by Lemma 6. Since $d^t = a_1d$, we have $(td)^2 = t^2a_1d^2 = t^2a_1$. Hence $t^2 = a_1$ implies that $(td)^2 = 1$. Therefore we can assume that either $t^2 = 1$ or $t^2 = a_2d$; the order of t is 2 or 8. Thus G is of type II.

(6.222) **Case C nilpotent of class 2.** Since C/N is abelian, $C' = N \leq Z$. If x and y belong to C, then $1 = [x, y]^2 = [x^2, y]$, showing that C/Z is elementary abelian. Choose a basis for C/Z, say $\{x_\lambda Z : \lambda \in \Lambda\}$; then $x_\lambda^2 = ad^i$ where $a \in L$ and $i = 0$ or 1. Now $a = b^2$ for some $b \in L$ and $(x_\lambda b^{-1})^2 = x_\lambda^2 a^{-1} = d^i$. Write $\bar{x}_\lambda = x_\lambda b^{-1}$; then $\bar{x}_\lambda^t = \bar{x}_\lambda^{-1}c$ for some c in N and $(\bar{x}_\lambda^2)^t = (\bar{x}_\lambda^{-1}c)^2 = \bar{x}_\lambda^{-2}$. It follows that $\langle d^i \rangle \lhd G$, which can only mean that $d^i = 1$ and $\bar{x}_\lambda^2 = 1$. In short, we can assume that

$$x_\lambda^2 = 1 \tag{33}$$

for all $\lambda \in \Lambda$.

Define $X = \langle x_\lambda : \lambda \in \Lambda \rangle$. From (33) it follows that $X^2 = X'$; also $C = XZ$, so $N = C' = X'$ and $N = X' = X^2$. Suppose that $u \in X \cap Z$ and write $u = x_{\lambda_1}^{n_1} \cdots x_{\lambda_r}^{n_r} a$ where $a \in N$, the n_i are integers and the λ_i are distinct elements of Λ. The independence of the $x_{\lambda_i} Z$ indicates that each n_i is even; thus $u \in X^2N = N$. Consequently

$$X \cap Z = N. \tag{34}$$

Therefore $\zeta(X) \leq X \cap \zeta(C) = X \cap Z = N$, and $\zeta(X) = N$. Also, $X/N \simeq C/Z$, an elementary abelian 2-group. We conclude that X is an extra-special 2-group generated by elements of order 2. Clearly the group C is a direct product of X and Z in which $\zeta(X)$ and $\langle a_1 \rangle$ are amalgamated.

It has been remarked that $t^2 = 1$, a_1 or $a_2 d$ (if $d \neq 1$); in fact the second possibility can be discarded if t is chosen suitably. The argument for this has already been given in the last part of (5.332).

Since t acts trivially on both X/N and N, the map $\sigma: xN \longrightarrow [x, t]$ is an element of $\mathrm{Hom}(X/N, N)$. If $d \neq 1$, one can assume that $\sigma = 0$ and $[X, t] = 1$. For in this case if $x_\lambda^t = x_\lambda a_1$, we obtain $(x_\lambda d)^t = x_\lambda d$ while $(x_\lambda d)^2 = 1$. Thus G is of type III.

[In conclusion, observe that even if $d = 1$ one can still take $\sigma = 0$ at the expense of losing $x_\lambda^2 = 1$; for $(x_\lambda a_2)^t = x_\lambda a_2$ if $x_\lambda^t \neq x_\lambda$.]

7. **Proof of Theorem 1 concluded.** It remains to show that a group G of types I to IX is a JNT-group; in each case G is obviously soluble. One first observes that in no case is G a T-group. For types II, IV and VIII this is clear. For types I, III, V and VII it follows from the structure of Dedekind groups. If G is of type VI(a), then D is a nonnormal subnormal subgroup, as is $\langle u \rangle$ if G is of type VI(b). If G is of type IX, then 1_F generates a nonnormal subnormal subgroup since $X \neq \langle -1_F \rangle$.

Next it must be shown that every proper factor group of G is a T-group. If G is of type I or VIII this is clear. All types except IV and IX are monolithic and if N is the monolith one merely has to verify that G/N is a T-group. If G is of type II or III, then $N = \langle a_1 \rangle$ and $G/\langle a_1 \rangle$ fits the prescription for a soluble 2-group with the property T (see [18, Theorem 3.1.1]). If G is of type V, then $N = \zeta(P)$ and G/N is a T-group by Lemma 5.2.2 of [18]. If G is of type VI or VII, then $N = \langle a_1 \rangle$ and $H = G/N$ has a normal p^∞-subgroup K such that H/K is abelian and all subgroups of $C_H(K)$ are normal in H; a subnormal subgroup S of H either contains K or lies in $C_H(K)$; hence $S \lhd H$.

Turning to the nonmonolithic groups, we see that in type IV a nontrivial normal subgroup N of G contains one of the two normal subgroups of order p; hence G/N is a T-group by Lemma 5.2.2 of [18].

This leaves us with the case where G is of type IX. Let $1 \neq N \lhd G$; then certainly $N \cap F$ is nontrivial. Let $0 \neq f \in N \cap F$. Since $F = Q + X^+$, we can write

$$f^{-1} = \sum_{x \in X} r_x x, \quad (r_x \in Q).$$

Choose a positive integer n such that each nr_x is integral and observe that N contains the element

$$\sum_{x \in X} (nr_x) xf = nf^{-1}f = n.$$

Hence $nX^+ \leq N$. We shall show that G/nX^+ is a T-group. $F = Q + X^+$ is divisible as an additive abelian group and $F/(Q + nX^+)$ has finite exponent. Thus $F = Q + nX^+$ and F/nX^+ is isomorphic with a factor group of $Q/\langle 1\rangle$. Hence every automorphism of F/nX^+ is a power automorphism and every subgroup of F/nX^+ is normal in G/nX^+. If $x \neq 1$, then $F(x-1) = F$, which is easily seen to imply that a subnormal subgroup of G/nX^+ either contains F/nX^+ or is contained in it. This shows that G/nX^+ is a T-group.

8. **Finitely generated soluble JNT-groups.** The main result of this section is

Theorem 2. *A finitely generated hyperabelian group which is not a T-group has a finite homomorphic image which is not a T-group.*

Recall here that a group is *hyperabelian* if it possesses an ascending series of normal subgroups whose factors are all abelian; this is equivalent to requiring each nontrivial homomorphic image to have a nontrivial normal abelian subgroup.

Proof. Let G be a finitely generated hyperabelian group which is not a T-group. Suppose that $\{N_\alpha : \alpha \in A\}$ is a chain of normal subgroups of G such that no G/N_α is a T-group; write N for the union of the chain. Assume that G/N is nevertheless a T-group. Now a hyperabelian T-group is soluble because soluble T-groups are metabelian; moreover, a finitely generated soluble T-group is either finite or abelian [18, Theorem 3.3.1], and therefore is certainly finitely presented. Thus G/N is finitely presented. By a well-known principle this implies that $N = \langle a_1, \cdots, a_n\rangle^G$ for a certain finite set of a_i's. Hence $N = N_\alpha$ for some α. By this contradiction G/N is not a T-group. Zorn's Lemma shows that there exists a normal subgroup M of G which is maximal subject to G/M not being a T-group. Clearly G/M is a JNT-group. Moreover G/M contains a nontrivial normal abelian subgroup, being hyperabelian. Thus G/M is soluble and Theorem 2 will follow from

Lemma 9. *A finitely generated soluble JNT-group is finite (and hence of type* I, IV, V *or* VIII).

Proof. One can, of course, verify directly that no soluble JNT-group on our list can be both finitely generated and infinite. However, it is more economical to proceed independently as follows.

Let G be a finitely generated soluble JNT-group which is infinite. First of all observe that G cannot be nilpotent; for if G were nilpotent, the initial argument of §3 would show that G is periodic and this, as is well known, implies that G is finite.

Denote by A a nontrivial normal abelian subgroup of G. Suppose that G/A is infinite. If $1 < B \leq A$ and $B \lhd G$, then G/B is abelian and $G' \leq B$. Hence G' is minimal normal in G and lies in A. Therefore $G/C_G(G')$ is a finitely gener-

ated abelian group and by a theorem of P. Hall [8, Theorem 5.1], G' is a finite elementary abelian p-group for some prime p.

Now write $C = C_G(G')$; then G/C is finite. Also $C' \leq G' \leq \zeta(C)$; thus if x, $y \in C$, we have $1 = [x, y]^p = [x^p, y]$. Hence $C^p \leq \zeta(C)$ and C^p is abelian. Now G/C^p is periodic and hence finite, so C^p is finitely generated and infinite. Hence for some integer n the group $N = (C^p)^n$ is torsion-free and nontrivial, while G/N is finite. This contradicts Lemma 5.

Therefore G/A is finite, which shows that A is finitely generated and infinite; there is no loss in assuming A to be free abelian. Let $L = [G', G]$ and observe that $L \neq 1$. If $L \cap A \neq 1$, then $G/L \cap A$ is finite, by the first part of the proof, and, replacing A by $L \cap A$, we may assume that $A \leq L$. Let p be a prime dividing $|G : L|$; then G/A^p is a finite soluble T-group; however the prime p divides both $|G : L|$ and $|L : A^p|$, which is impossible [7]. Thus $L \cap A = 1$ and $L \simeq LA/A$, which shows that L is finite and abelian; therefore G/L is infinite. However this situation has been shown to be impossible.

Lemma 9 may be compared with B. H. Neumann's theorem that a finitely generated soluble just nonabelian group is finite [15, Theorem 6.3]—see also Rosati [21]. It is not difficult to show that a soluble just nonabelian group cannot be a T-group if it is infinite. Thus Neumann's theorem is a special case of Lemma 9.

9. **$JN\overline{T}$-groups.** A group G has the property $\overline{T}$ if $H \lhd K \lhd L \leq G$ always implies that $H \lhd L$. Thus $\overline{T}$-groups form the largest subgroup-closed subclass of the class of T-groups.

A $JN\overline{T}$-group is either a JNT-group or a T-group. There exist finite $JN\overline{T}$-groups which are T-groups, for example the symmetric group S_n where $n \geq 5$; but this phenomenon cannot occur in the soluble case. In fact we shall prove

Theorem 3. *A group is a soluble $JN\overline{T}$-group if and only if it is isomorphic with a group of type* I, IV, V *or* VIII (*with X a $\overline{T}$-group in the last case*).

Proof. Let G be a soluble $JN\overline{T}$-group. First observe that G is not a T-group; for suppose this is wrong. If G is a T-group of type I, then $L = [G', G]$ contains an element a of infinite order; therefore $L/\langle a^4\rangle$ has an element of order 4 and $G/\langle a^4\rangle$ is not abelian, which precludes $G/\langle a^4\rangle$ from being a $\overline{T}$-group (for this and other results about soluble $\overline{T}$-groups see [18, Theorem 6.1.1]). If G is a T-group of type II, then G' is radicable and if $1 \neq a \in G'$, the group $G/\langle a\rangle$ is also a T-group of type II and therefore not a $\overline{T}$-group. Finally, if G is periodic, L has an element of order 2 since G would otherwise be a $\overline{T}$-group. Therefore there is a normal subgroup $N \neq 1$ of G such that L/N is of type 2^∞; however this prevents G/N from being a $\overline{T}$-group.

Therefore G is a JNT-group and appears on our list. It remains to discard those JNT-groups G which have a factor group G/N such that $N \neq 1$ and G/N

is not a $\overline{T}$-group. Here one must keep in mind that a soluble $\overline{T}$-group H is either periodic or abelian and that $[H', H]$ cannot have an element of order 2. This excludes types II, III, VI, VII and IX—recall that in type IX the group X cannot be periodic. There is no difficulty in verifying that the remaining types are $JN\overline{T}$-groups (in type VIII one must assume that X is a soluble $\overline{T}$-group).

Locally finite $JN\overline{T}$-groups. A good deal more is known about $\overline{T}$-groups than T-groups and one would hope for correspondingly more information about $JN\overline{T}$-groups. For example, *a locally finite $\overline{T}$-group is soluble and therefore metabelian.* This is an easy corollary of the well-known theorem of Huppert that a finite group with all of its proper subgroups supersoluble is soluble [12, Satz 22]; see also [19].

We shall outline a method of describing the locally finite $JN\overline{T}$-groups that are insoluble; this is based on the Fitting-Gol'berg theory of semisimple groups (see [14, §61). Let G be a locally finite $JN\overline{T}$-group which is insoluble. Then every proper factor group of G is metabelian and $M = G''$ is the monolith of G. Then G acts *irreducibly* on M, i.e., M has no proper nontrivial subgroups that are G-admissible; in particular, M is characteristically simple. Clearly M is perfect and its centre is 1. Thus $C_G(M) = 1$, from which it follows that there is an isomorphism of G with a subgroup of Aut M, the full automorphism group of M, in which M is mapped onto the group of inner automorphisms Inn M. Thus one may assume that

$$\text{Inn } M < G \leq \text{Aut } M.$$

Conversely, let M be a nonabelian, characteristically simple group which is locally finite; let G be a subgroup of Aut M which contains Inn M and acts irreducibly on Inn M; assume also that $G/\text{Inn } M$ is a $\overline{T}$-group. Then in fact G is a $JN\overline{T}$-group. To prove this let $1 \neq N \lhd G$; if $N \cap (\text{Inn } M) = 1$, then $[N, \text{Inn } M] = 1$ and $[N, M] \leq \zeta(M) = 1$, which shows that $N = 1$. Thus Inn $M \leq N$ by irreducibility and G/N is a $\overline{T}$-group. On the other hand G is not a $\overline{T}$-group since if it were, Inn M—and therefore M—would be soluble. Also Inn M is the monolith of G.

Suppose that G_1 and G_2 are two isomorphic groups obtained in this way from groups M_1 and M_2; then evidently $M_1 \simeq M_2$. If we identify M_1 and M_2 and write α for the isomorphism of G_1 with G_2, then α determines by restriction an automorphism α^* of M_1. It is routine to check that $g^\alpha = (\alpha^*)^{-1} g^{\alpha^*}$, $(g \in G_1)$. This is summed up in

Theorem 4. *There is a one-one correspondence between isomorphism classes of insoluble, locally finite, $JN\overline{T}$-groups with given monolith M and conjugacy classes of irreducible, locally finite $\overline{T}$-subgroups of* Out M *(the group of outer automorphisms of M).*

Of course it is by no means clear which groups M can arise here. However one obvious candidate—and the only one if M is finite or merely possesses a

minimal normal subgroup—is a direct power of a nonabelian simple group H. For simplicity of presentation suppose we are dealing with *finite* $JN\overline{T}$-groups. Let M be the direct product of n copies of H. Then, as was shown by Fitting [5, Satz 12], $\mathrm{Aut}\, M \simeq (\mathrm{Aut}\, H) \wr S_n$ and $\mathrm{Out}\, M \simeq (\mathrm{Out}\, H) \wr S_n$; in these wreath products $\mathrm{Aut}\, H$ and $\mathrm{Out}\, H$ are in their regular representations and the symmetric group S_n is in its natural permutation representation. The irreducible subgroups of $\mathrm{Out}\, M$ correspond in the isomorphism to subgroups of $(\mathrm{Out}\, H) \wr S_n$ which map onto transitive subgroups of S_n in the canonical homomorphism of $(\mathrm{Out}\, H) \wr S_n$ onto S_n.

Thus *a finite* $JN\overline{T}$*-group is either soluble—and therefore of type* I, IV, V or VIII—*or insoluble, in which case it corresponds to a conjugacy class of irreducible* $\overline{T}$*-subgroups of* $(\mathrm{Out}\, H) \wr S_n$ *where* H *is a finite nonabelian simple group and* n *a positive integer* > 1. However the problem of adequately describing all finite JNT-groups remains open.

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