

COMPARISON OF EIGENVALUES FOR LINEAR DIFFERENTIAL EQUATIONS OF ORDER $2n$

BY

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ABSTRACT. An abstract eigenvalue comparison theorem is proven for u_0 -positive linear operators in a Banach space equipped with a cone of "nonnegative" elements. This result is then applied to certain linear differential equations of order $2n$ in order to obtain eigenvalue comparison theorems of an "integral type."

I. Introduction. We shall be concerned with comparing the smallest positive eigenvalues, λ_0 and Λ_0 , respectively, of the following differential equations:

$$(1.1) \quad \begin{aligned} & [a(x)u^{(n)}]^{(n)} + \lambda(-1)^{n+1}c(x)u = 0, \\ & u(\alpha) = u'(\alpha) = \dots = u^{(n-1)}(\alpha) = u_1(\beta) = u'_1(\beta) = \dots = u_1^{(n-1)}(\beta) = 0; \end{aligned}$$

$$(1.2) \quad \begin{aligned} & [A(x)v^{(n)}]^{(n)} + \Lambda(-1)^{n+1}C(x)v = 0, \\ & v(\alpha) = v'(\alpha) = \dots = v^{(n-1)}(\alpha) = v_1(\beta) = v'_1(\beta) = \dots = v_1^{(n-1)}(\beta) = 0, \end{aligned}$$

where $a(x)$ and $A(x)$ are positive functions of class $C^{(n)}[\alpha, \beta]$, $c(x)$ and $C(x)$ are of class $C[\alpha, \beta]$, and $u_1(x)$ and $v_1(x)$ are defined by the following:

$$u_1(x) \equiv a(x)u^{(n)}(x), \quad v_1(x) \equiv A(x)v^{(n)}(x).$$

It is a well-known consequence of the Courant minimum principle that if the pointwise conditions $0 < A(x) \leq a(x)$ and $0 \leq c(x) \leq C(x)$ hold on $[\alpha, \beta]$, then $\Lambda_0 \leq \lambda_0$, with equality holding if and only if $A(x) \equiv a(x)$ and $c(x) \equiv C(x)$. However, in this paper we shall be interested in establishing eigenvalue comparison theorems of the "integral type." The following is an example of the kind of result we have obtained for equations of the form (1.1) and (1.2):

Theorem 1.1. *If*

$$0 < C(x) \quad \text{on } [\alpha, \beta],$$

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$$\int_x^\beta c(s) ds \leq \int_x^\beta C(s) ds \quad \text{on } [\alpha, \beta],$$

$$\int_a^x \frac{1}{a(s)} ds \leq \int_a^x \frac{1}{A(s)} ds \quad \text{on } [\alpha, \beta],$$

then $\Lambda_0 \leq \lambda_0$ with equality holding if and only if $a(x) \equiv A(x)$ and $c(x) \equiv C(x)$.

In [5], K. Kreith demonstrates that the separation properties characteristic of Sturmian theorems are a result of certain orderings between the resolvents of the differential operators involved. One would expect that theorems involving the comparison of eigenvalues would be true for essentially the same reason. The results below demonstrate that this is indeed the case.

In § 2 it will be shown that eigenvalue comparison theorems hold for general operator equations in a Banach space \mathcal{B} if the operators satisfy prescribed positivity requirements with respect to a cone \mathcal{P} . In § 4 this general theory will be used to prove eigenvalue comparison theorems for problems (1.1) – (1.2). However, before the theory can be applied, one must first establish certain positivity properties of the resolvents involved. This is done in § 3.

II. An abstract comparison theorem. Our abstract formulation of the eigenvalue comparison theorems will be valid in a real Banach space \mathcal{B} equipped with a cone \mathcal{P} of “nonnegative” elements.

Definition 2.1. A closed subset \mathcal{P} of \mathcal{B} is called a cone if it satisfies

- (i) $u \in \mathcal{P}$ and $v \in \mathcal{P} \Rightarrow u + v \in \mathcal{P}$;
- (ii) $u \in \mathcal{P}$ and $\alpha \geq 0 \Rightarrow \alpha u \in \mathcal{P}$;
- (iii) $u \in \mathcal{P}$ and $-u \in \mathcal{P} \Rightarrow u = \theta$.

A partial ordering can be introduced in a Banach space with a cone by defining $u \leq v$ if and only if $v - u \in \mathcal{P}$. We say the operator M is greater than the operator L and write $L \leq M$ if $Lu \leq Mu$ for all $u \in \mathcal{P}$.

Definition 2.2. We say that a bounded linear operator L is u_0 -positive if there exists a nonzero $u_0 \in \mathcal{P}$ with the following property: for every nonzero $u \in \mathcal{P}$ there exist positive numbers k_1 and k_2 such that $k_1 u_0 \leq Lu \leq k_2 u_0$.

The following two theorems are found in [4], which also has an extensive discussion of positive operators.

Theorem 2.1. *If L is compact and u_0 -positive, then L has exactly one (normalized) eigenvector in \mathcal{P} and the corresponding eigenvalue is positive and larger than the absolute value of any other eigenvalue.*

Proof. See [4, pp. 67, 76 and 81].

Theorem 2.2. *If L is compact and u_0 -positive and u_1 is the normalized positive eigenvector of L , then L is u_1 -positive.*

Proof. See [4, p. 76].

We are now ready to establish the abstract eigenvalue comparison theorem.

Theorem 2.3. *Let L and M be linear operators and assume that at least one of the operators is u_0 -positive. We further assume that $L \leq M$ and that*

$$(2.1) \quad Lu_1 = \lambda_1 u_1 \quad (u_1 \in \mathcal{P}, \lambda_1 > 0),$$

$$(2.2) \quad Mu_2 = \lambda_2 u_2 \quad (u_2 \in \mathcal{P}, \lambda_2 > 0).$$

Then $\lambda_1 \leq \lambda_2$ and if $\lambda_1 = \lambda_2$, u_1 is a scalar multiple of u_2 .

Proof. We shall follow a similar proof given by K. Kreith in [5]. First suppose that L is u_0 -positive. Define

$$\epsilon_0 = \sup \{ \epsilon \mid L(u_2 - \epsilon u_1) \in \mathcal{P} \}.$$

From $u_2 \in \mathcal{P}$ and the u_1 -positivity of L , it follows that $0 < \epsilon_0 < \infty$. Noting that

$$(2.3) \quad L(u_2 - \epsilon_0 u_1) = Lu_2 - \epsilon_0 Lu_1 \leq \lambda_2(u_2 - \epsilon_0(\lambda_1/\lambda_2)u_1),$$

we have $L(u_2 - \epsilon_0(\lambda_1/\lambda_2)u_1) \in \mathcal{P}$, from which it follows that $\lambda_1 \leq \lambda_2$. Suppose $\lambda_1 = \lambda_2$ and that $u_2 - \epsilon_0 u_1 \neq \theta$. Then (2.3) and the u_1 -positivity of L imply that there exists $k > 0$ such that $\lambda_2^{-1} k u_1 \leq \lambda_2^{-1} L(u_2 - \epsilon_0 u_1) \leq (u_2 - \epsilon_0 u_1)$. This contradicts the maximal property of ϵ_0 , and therefore $u_2 = \epsilon_0 u_1$ whenever $\lambda_1 = \lambda_2$. The case where M is u_0 -positive is similar to the above and will be omitted.

III. Some preliminary results.

Lemma 3.1. *The Green's function $G(x, s)$ for the operator*

$$(3.1) \quad \begin{aligned} L[u] &= (-1)^n [a(x)u^{(n)}]^{(n)}, \\ u(\alpha) &= u'(\alpha) = \dots = u^{(n-1)}(\alpha) = u_1(\beta) = u_1'(\beta) = \dots = u_1^{(n-1)}(\beta) = 0 \end{aligned}$$

is given by

$$(3.2) \quad G(x, s) = \begin{cases} \frac{1}{(n-1)!} \int_{\alpha}^x \int_{\alpha}^{x_n} \dots \int_{\alpha}^{x_2} \frac{(s-x_1)^{n-1}}{a(x_1)} dx_1 \dots dx_n, & \alpha \leq x \leq s \leq \beta, \\ \frac{1}{(n-1)!} \int_{\alpha}^x \int_{\alpha}^{x_n} \dots \int_{\alpha}^{x_2} \frac{(x-x_1)^{n-1}}{a(x_1)} dx_1 \dots dx_n; & \alpha \leq s \leq x \leq \beta. \end{cases}$$

Proof. It is well known that the Green's function $G(x, s)$ is uniquely defined by the following three conditions: (In the statements below all derivatives are taken with respect to x .)

1. $G, G', \dots, G^{(2n-2)}$ are continuous functions of (x, s) on the square $\alpha \leq x, s \leq \beta$.
2. $G^{(2n-1)}$ is a continuous function of (x, s) in each of the triangles $\alpha \leq x \leq s \leq \beta$ and $\alpha \leq s \leq x \leq \beta$, and $G^{(2n-1)}(s+, s) - G^{(2n-1)}(s-, s) = (-1)^n / a(s)$.
3. For each s , $G(x, s)$ satisfies the boundary conditions (3.1). $G(x, s)$ also satisfies that associated homogeneous equation $L[G] = 0$ everywhere except at the point $x = s$.

Using the appropriate variations of Liouville's formula for iterated integrals

$$\int_a^x \int_a^{x_n} \dots \int_a^{x_2} F(x_1) dx_1 \dots dx_n = \frac{1}{(n-1)!} \int_a^x (x-x_1)^{n-1} F(x_1) dx_1,$$

it can be shown that $G(x, s)$ defined in (3.2) satisfies 1-3.

It should be noted that the form of the Green's function given in (3.2) is a significant improvement over that found in Collatz's book on eigenvalue problems [1, p. 84] for the case $n = 2$.

Consider the operators (3.1) and

$$(3.3) \quad H[v] = [(-1)^n A(x)v^{(n)}]^{(n)},$$

$$v(\alpha) = v'(\alpha) = \dots = v^{(n-1)}(\alpha) = v_1(\beta) = v'_1(\beta) = \dots = v_1^{(n-1)}(\beta) = 0,$$

and let $G_a(x, s)$ and $G_A(x, s)$ be the Green's functions for (3.1) and (3.3), respectively.

Lemma 3.2. *If $\int_a^x (1/a(r)) dr \leq \int_a^x (1/A(r)) dr$ on $[\alpha, \beta]$ then $0 \leq G_a(x, s) \leq G_A(x, s)$ for $(x, s) \in [\alpha, \beta] \times [\alpha, \beta]$.*

Proof. Notice that for $\alpha \leq x \leq s \leq \beta$ we have

$$\begin{aligned} \int_a^x \frac{(s-x_1)^{n-1}}{a(x_1)} dx_1 &= \int_a^x \left[\int_a^{x_1} \frac{1}{a(r)} dr \right]' (s-x_1)^{n-1} dx_1 \\ &= \left[\int_a^x \frac{1}{a(r)} dr \right] (s-x)^{n-1} + (n-1) \int_a^x \left[\int_a^{x_1} \frac{1}{a(r)} dr \right] (s-x_1)^{n-2} dx_1 \\ &\leq \left[\int_a^x \frac{1}{A(r)} dr \right] (s-x)^{n-1} + (n-1) \int_a^x \left[\int_a^{x_1} \frac{1}{A(r)} dr \right] (s-x_1)^{n-2} dx_1 \\ &= \int_a^x \frac{(s-x_1)^{n-1}}{A(x_1)} dx_1. \end{aligned}$$

By a similar argument one can show that, for $\alpha \leq s \leq x \leq \beta$,

$$\int_a^{x_2} \frac{(x - x_1)^{n-1}}{a(x_1)} dx_1 \leq \int_a^{x_2} \frac{(x - x_1)^{n-1}}{A(x_1)} dx_1.$$

The result now follows from (3.2).

We would like to point out to the reader that Lemma 3.2 and most of the theorems that follow are valid under the weaker assumptions

$$\int_a^x \int_a^{x_n} \dots \int_a^{x_2} \frac{1}{a(x_1)} dx_1 \dots dx_n \leq \int_a^x \int_a^{x_n} \dots \int_a^{x_2} \frac{1}{A(x_1)} dx_1 \dots dx_n$$

and/or

$$\int_x^\beta \int_{x_n}^\beta \dots \int_{x_2}^\beta c(x_1) dx_1 \dots dx_n \leq \int_x^\beta \int_{x_n}^\beta \dots \int_{x_2}^\beta C(x_1) dx_1 \dots dx_n$$

where $n \geq 2$. However, for the sake of simplicity, we will leave it to the reader to make these generalizations.

We shall now consider the integral operators

$$(3.4) \quad M[u] = \int_a^\beta G_a(x, s)c(s)u(s) ds,$$

$$(3.5) \quad N[u] = \int_a^\beta G_A(x, s)C(s)u(s) ds$$

which will be defined on the Banach space

$$\mathcal{B} = \{u \in C^{2n}[\alpha, \beta] \mid u(\alpha) = u'(\alpha) = \dots = u^{(n-1)}(\alpha) = 0\}$$

with norm $\|u\| = \max_{x \in [\alpha, \beta]} \{|u(x)|, |u'(x)|, \dots, |u^{(2n)}(x)|\}$. Let $\mathcal{P}_1 = \{u \in \mathcal{B} \mid u(x) \geq 0 \text{ on } [\alpha, \beta]\}$ be the cone of nonnegative functions.

Lemma 3.3. *If $c(x) > 0$ on $[\alpha, \beta]$ then the operator (3.4) is u_0 -positive with respect to the cone \mathcal{P}_1 .*

Proof. We shall show that (3.4) is u_0 -positive where $u_0(x) = \int_a^\beta G_a(x, s)c(s) ds$. For any closed interval $I \subset [\alpha, \beta]$, let $b(x) = \int_I G_a(x, s)c(s) ds$. We notice that

$$u_0^{(n)}(x) = \frac{1}{a(x)(n-1)!} \int_x^\beta (s-x)^{n-1} c(s) ds,$$

$$b^{(n)}(x) = \frac{1}{a(x)(n-1)!} \int_D (s-x)^{n-1} c(s) ds,$$

where $D = I \cap [x, \beta]$, and that

$$(3.6) \quad \begin{aligned} u_0(\alpha) = u_0'(\alpha) = \dots = u_0^{(n-1)}(\alpha) = 0, & \quad u_0^{(n)}(\alpha) > 0, \\ b(\alpha) = b'(\alpha) = \dots = b^{(n-1)}(\alpha) = 0, & \quad b^{(n)}(\alpha) > 0. \end{aligned}$$

Since $u_0^{(n)}(a) > 0$ and $b^{(n)}(a) > 0$ we can find an $\epsilon_1 > 0$ and δ such that $a < \delta \leq \beta$ and $\epsilon_1 u_0^{(n)}(x) \leq b^{(n)}(x)$ for x in $[\alpha, \delta]$. Since $u_0^{(n)}(x) \geq 0$, $b^{(n)}(x) \geq 0$ on $[\alpha, \beta]$ and $u_0^{(n)}(a) > 0$, $b^{(n)}(a) > 0$, we have that $u_0^{(n-1)}(x)$ and $b^{(n-1)}(x)$ are positive on $(\alpha, \beta]$. Moreover, since both of these functions are continuous on $[\delta, \beta]$, they have a positive maximum and minimum on this interval. Let $P = \max u_0^{(n-1)}(x)$ on $[\delta, \beta]$ and $p = \min b^{(n-1)}(x)$ on $[\delta, \beta]$. We can choose ϵ_2 such that $\epsilon_2 P \leq p$, and consequently we have that $\epsilon_2 u_0^{(n-1)}(x) \leq b^{(n-1)}(x)$ on $[\delta, \beta]$. If we define $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ then it follows that $\epsilon u_0^{(n-1)}(x) \leq b^{(n-1)}(x)$ on $[\alpha, \beta]$. This together with the boundary conditions (3.6) implies that $\epsilon u_0(x) \leq b(x)$ on $[\alpha, \beta]$; that is, $\epsilon \int_a^\beta G_a(x, s)c(s)ds \leq \int_I G_a(x, s)c(s)ds$.

Now let $u(x)$ be a nonzero element of the cone \mathcal{P}_1 . Then a closed interval I can be found such that $u(x) \geq \alpha_0 > 0$ for $x \in I$. Consequently,

$$\begin{aligned} M[u] &= \int_a^\beta G_a(x, s)c(s)u(s)ds \geq \alpha_0 \int_I G_a(x, s)c(s)ds \\ &\geq \alpha_0 \epsilon \int_a^\beta G_a(x, s)c(s)ds = \alpha_0 \epsilon u_0(x). \end{aligned}$$

This proves that the operator M is u_0 -bounded below. The u_0 -boundedness above is evident:

$$M[u] = \int_a^\beta G_a(x, s)c(s)u(s)ds \leq \max_{x \in [\alpha, \beta]} |u(x)|u_0(x).$$

This completes the proof of the lemma.

One might now suspect that Lemma 3.3 is true under the weaker assumption $\int_x^\beta c(x)dx > 0$ on $[\alpha, \beta]$. However, this is not the case. The precise reason for this will become apparent later. At this point, it will suffice to say that the cone \mathcal{P}_1 contains "too many" elements. In an attempt to circumvent this problem we define the smaller cone $\mathcal{P}_2 = \{u \in \mathcal{B} \mid u'(x) \geq 0 \text{ on } [\alpha, \beta]\}$. We are now able to prove a lemma similar to Lemma 3.3, but with substantially weaker positivity assumptions on $c(x)$.

Lemma 3.4. *If $c(x) \neq 0$ and $\int_x^\beta c(s)ds \geq 0$ on $[\alpha, \beta]$ then the operator (3.4) is u_0 -positive with respect to the cone \mathcal{P}_2 .*

Proof. We shall show that the operator (3.4) is u_0 -positive where

$$u_0(x) = \int_a^\beta G_a(x, s)c(s)ds.$$

Let $u(x)$ be a nonzero element of the cone \mathcal{P}_2 and define

$$b(x) = \int_a^\beta G_a(x, s)c(s)u(s)ds.$$

Observe that

$$\begin{aligned} u_0^{(n)}(\alpha) &= \frac{1}{a(\alpha)(n-1)!} \int_{\alpha}^{\beta} (s-\alpha)^{n-1} c(s) ds \\ &\geq \frac{1}{a(\alpha)n!} \int_{\alpha}^{\beta} (s-\alpha)^{n-2} \int_s^{\beta} c(r) dr ds > 0, \end{aligned}$$

and that

$$\begin{aligned} b^{(n)}(\alpha) &= \frac{1}{a(\alpha)(n-1)!} \int_{\alpha}^{\beta} (s-\alpha)^{n-1} c(s) u(s) ds \\ &\geq \frac{1}{a(\alpha)(n-1)!} \int_{\alpha}^{\beta} [(n-1)(s-\alpha)^{n-2} u(s) + (s-\alpha)^{n-1} u'(s)] \int_s^{\beta} c(r) dr ds > 0, \end{aligned}$$

since $u \in \mathcal{P}_2$. Thus there exist positive real numbers r_1 and r_2 and a real number δ , $\alpha < \delta \leq \beta$, such that $r_1 u_0^{(n)}(x) \leq b^{(n)}(x) \leq r_2 u_0^{(n)}(x)$ on $[\alpha, \delta]$. Using reasoning similar to that in Lemma 3.3 we can conclude that there exist positive real numbers k_1 and k_2 such that $k_1 u_0^{(n-1)}(x) \leq b^{(n-1)}(x) \leq k_2 u_0^{(n-1)}(x)$ on $[\alpha, \beta]$. Integrating and using boundary conditions, we obtain

$$k_1 u_0'(x) \leq \frac{d}{dx} \left[\int_{\alpha}^{\beta} G_a(x, s) c(s) u(s) ds \right] \leq k_2 u_0'(x).$$

Consequently operator (3.4) is u_0 -positive with respect to \mathcal{P}_2 .

Although we shall not make explicit use of it, one can now extend an observation made by St. Mary [8] for the special second order case ($n = 1$). The proof given by St. Mary does not readily generalize to the higher order case.

Corollary 3.1. *Let $y_0(x)$ be the eigenvector corresponding to the smallest positive eigenvalue of problem (1.1). If $\int_{\alpha}^{\beta} c(s) ds \geq 0$ on $[\alpha, \beta]$ then $y_0'(x) \geq 0$ on $[\alpha, \beta]$.*

Proof. We first need to observe that the eigenvalues of problem (1.1) are simple. By this it is meant that to each eigenvalue λ there belongs exactly one eigenfunction (except for a multiplicative constant). The proof is similar to that given for the case $n = 2$ by Leighton and Nehari [6, p. 338] and will not be repeated here. Now to prove the corollary, notice that the eigenvalue problem (1.1) is equivalent to the integral equation $M[u] = u/\lambda$, where M is defined by (3.4). Since M is a compact linear operator which is u_0 -positive with respect to the cone \mathcal{P}_2 , we know that M has a unique (normalized) eigenvector in the cone \mathcal{P}_2 and the corresponding eigenvalue is positive and larger than the absolute value of any other eigenvalue. Since $1/\lambda_0$ is the largest positive eigenvalue of M and this eigenvalue is simple, the eigenvector corresponding to it must be in the cone \mathcal{P}_2 . Therefore $y_0'(x) \geq 0$ on $[\alpha, \beta]$.

It is known [3] that if $c(x)$ changes sign in $[\alpha, \beta]$ problem (1.1) has an

infinite set of real eigenvalues which have the limit points $+\infty$ and $-\infty$. For the case $n = 1$, it has been further shown [3] that if y_m^+ and y_m^- are the eigenvectors corresponding to the m th positive and negative eigenvalues, respectively, then y_m^+ and y_m^- have exactly m zeros in the interval (α, β) . Thus the eigenvectors corresponding to the smallest positive and largest negative eigenvalues are both positive in (α, β) and consequently are in the cone \mathcal{P}_1 . This is precisely the reason that the cone \mathcal{P}_1 is inadequate in the case where $c(x)$ is allowed to change sign in $[\alpha, \beta]$. However, if $\int_x^\beta c(s)ds \geq 0$ we can show that only the eigenvector corresponding to the smallest positive eigenvalue is in the cone \mathcal{P}_2 .

Corollary 3.2. *Let $y_0^-(x)$ be the eigenvector corresponding to the largest negative eigenvalue of problem (1.1) (assuming it exists). If $\int_x^\beta c(s)ds \geq 0$ on $[\alpha, \beta]$ then $dy_0^-(x)/dx$ changes sign on $[\alpha, \beta]$.*

Proof. We may assume that $dy_0^-(x)/dx$ is nonnegative near α . If it is nonnegative throughout $[\alpha, \beta]$ then $y_0^-(x)$ is in the positive cone \mathcal{P}_2 . However, we know from Corollary 3.1 that $y_0^+(x)$ is also in the cone \mathcal{P}_2 . This contradicts Theorem 2.1 which states that the compact u_0 -positive operator $M[u]$ has only one eigenvector in the cone \mathcal{P}_2 . Thus $dy_0^-(x)/dx$ changes sign in $[\alpha, \beta]$.

We shall assume that $M[u]$ and $N[u]$ are defined by (3.4) and (3.5), and define

$$(3.7) \quad R[u] = \int_\alpha^\beta G_A(x, s)c(s)u(s)ds.$$

Lemma 3.5. *If $\int_\alpha^x (1/a(s))ds \leq \int_\alpha^x (1/A(s))ds$ on $[\alpha, \beta]$ and $c(x) \geq 0$, then $M \leq R$ with respect to the cone \mathcal{P}_1 .*

Proof. The proof is a direct consequence of the definitions and will be omitted.

Lemma 3.6. *If $\int_x^\beta c(s)ds \leq \int_x^\beta C(s)ds$ on $[\alpha, \beta]$ then $R \leq N$ with respect to the cone \mathcal{P}_2 .*

Proof. We must show that $(N - R)u \in \mathcal{P}_2$ for all $u \in \mathcal{P}_2$. It is clear that $(N - R)u$ satisfies the boundary conditions $[(N - R)u](\alpha) = [(N - R)u]'(\alpha) = \dots = [(N - R)u]^{(n-1)}(\alpha) = 0$. The case $n = 1$ is similar to the one given below and will be omitted. If $n \geq 2$ then

$$\begin{aligned}
 R^{(n)}[u](x) &= \frac{1}{A(x)(n-1)!} \int_x^\beta (s-x)^{n-1} c(s) u(s) ds \\
 &= \frac{1}{A(x)(n-1)!} \int_x^\beta ((s-x)^{n-1} u(s))' \int_s^\beta c(r) dr ds \\
 &\leq \frac{1}{A(x)(n-1)!} \int_x^\beta ((s-x)^{n-1} u(s))' \int_s^\beta C(r) dr ds \\
 &= N^{(n)}[u](x).
 \end{aligned}$$

Integrating over (α, x) and using the boundary conditions we obtain $R'[u](x) \leq N'[u](x)$ on $[\alpha, \beta]$ for all $u \in \mathcal{P}_2$.

Lemma 3.7. *If $0 < A(x) \leq a(x)$ and $\int_x^\beta c(s) ds \leq \int_x^\beta C(s) ds$ then $M \leq N$ with respect to the cone \mathcal{P}_2 .*

Proof. The proof is similar to the above and will be omitted.

IV. **Comparison theorems for eigenvalues.** Z. Nehari [7] was the first to give an integral criterion for the comparison of λ_0 and Λ_0 . For $n = 1$ and $a(x) \equiv A(x) \equiv 1$, he shows that if $c(x)$ is nonnegative and

$$(4.1) \quad \int_x^\beta c(s) ds \leq \int_x^\beta C(s) ds \quad \text{on } [\alpha, \beta]$$

then $\Lambda_0 \leq \lambda_0$, with equality if and only if $c(x) \equiv C(x)$. Theorem 4.3 below removes the assumption that $c(x)$ be nonnegative and replaces (4.1) with $0 \leq \int_x^\beta c(s) ds \leq \int_x^\beta C(s) ds$.

H. Howard [2, p. 310] shows that Nehari's result is true when (4.1) is replaced by the apparently more general condition that there exists a positive, nonincreasing function $f(x)$ of class $C^1[\alpha, \beta]$ such that

$$(4.2) \quad \int_x^\beta c(s) f(s) ds \leq \int_x^\beta C(s) f(s) ds \quad \text{on } [\alpha, \beta].$$

In fact, conditions (4.1) and (4.2) are equivalent. That (4.1) implies (4.2) is evident. To see the converse, assume that condition (4.2) holds and notice that

$$\begin{aligned}
 \int_x^\beta c(s) ds &= - \int_x^\beta \left[\int_s^\beta c(r) f(r) dr \right]' \frac{1}{f(s)} ds \\
 &= \frac{1}{f(x)} \int_x^\beta c(r) f(r) dr + \int_x^\beta - \frac{f'(s)}{f^2(s)} \int_s^\beta c(r) f(r) dr ds \\
 &\leq \frac{1}{f(x)} \int_x^\beta C(r) f(r) dr + \int_x^\beta - \frac{f'(s)}{f^2(s)} \int_x^\beta C(r) f(r) dr ds \\
 &= \int_x^\beta C(s) ds.
 \end{aligned}$$

In the same paper [2, p. 311] Howard considers the case $n = 1$ and $c(x) \equiv C(x) > 0$. He shows that if

$$(4.3) \quad \int_{\alpha}^x A(s)f(s) ds \leq \int_{\alpha}^x a(s)f(s) ds,$$

where $f(x)$ is a positive function of class $C^1[\alpha, \beta]$ such that $a^2(x)f(x)$ is non-decreasing on $[\alpha, \beta]$, then $\Lambda_0 \leq \lambda_0$. In the eigenvalue comparison theorems following, it will be shown that condition (4.3) can be replaced by the much more natural condition

$$(4.4) \quad \int_{\alpha}^x \frac{1}{a(s)} ds \leq \int_{\alpha}^x \frac{1}{A(s)} ds \quad \text{on } [\alpha, \beta].$$

Howard [2] also extends the above two results to the case $n = 2$. However, he assumes that both $c(x)$ and $C(x)$ are positive on $[\alpha, \beta]$, a condition that we shall remove in Theorem 4.3 below.

In his recent book on oscillation theory [9, p. 147], C. A. Swanson states that it would be of interest to find comparison theorems for eigenvalues of second and fourth order equations such as (1.1) and (1.2) with integral conditions on both $a(x)$ and $c(x)$. This is accomplished in Theorem 1.1. We shall now prove this theorem, which we restate here for convenience.

Theorem 4.1. *Consider the eigenvalue problems (1.1) and (1.2). If*

$$(4.5) \quad 0 < c(x) \quad \text{on } [\alpha, \beta],$$

$$(4.6) \quad \int_{\alpha}^{\beta} c(s) ds \leq \int_{\alpha}^{\beta} C(s) ds \quad \text{on } [\alpha, \beta],$$

$$(4.7) \quad \int_{\alpha}^x \frac{1}{a(s)} ds \leq \int_{\alpha}^x \frac{1}{A(s)} ds \quad \text{on } [\alpha, \beta],$$

then $\Lambda_0 \leq \lambda_0$ with equality holding if and only if $a(x) \equiv A(x)$ and $c(x) \equiv C(x)$.

Proof. Consider the eigenvalue problem

$$(4.8) \quad [A(x)v^{(n)}]^{(n)} + \mu(-1)^{n+1}c(x)v = 0,$$

$$v(\alpha) = v'(\alpha) = \dots = v^{(n-1)}(\alpha) = v_1(\beta) = v_1'(\beta) = \dots = v_1^{(n-1)}(\beta) = 0.$$

Eigenvalue problems (1.1), (1.2), and (4.8) are equivalent, respectively, to the integral equations $M[u] = \lambda^{-1}u$, $N[u] = \Lambda^{-1}u$, and $R[u] = \mu^{-1}u$. Condition (4.5) and Lemma 3.3 imply that $M[u]$ is u_0 -positive with respect to \mathcal{P}_1 . From (4.7) and Lemma 3.5 we have that $M \leq R$ with respect to \mathcal{P}_1 . Therefore by Theorem 2.3, $\mu_0 \leq \lambda_0$ with equality holding if and only if the eigenvector corresponding to μ_0 is a multiple of the one corresponding to λ_0 . However this happens if and only if $a(x) \equiv A(x)$. Conditions (4.5), (4.6) and Lemmas 3.4 and 3.6 imply that $R[u]$

is u_0 -positive with respect to \mathcal{P}_2 and $R \leq M$ with respect to \mathcal{P}_2 . Thus $\Lambda_0 \leq \mu_0$ with equality holding if and only if $c(x) \equiv C(x)$. The theorem now follows.

The proof of the next two theorems is similar to the above and will be omitted.

Theorem 4.2. Consider equations (1.1) and (1.2). If $0 < C(x)$ on $[\alpha, \beta]$ and

$$0 \leq \int_x^\beta c(s) ds \leq \int_x^\beta C(s) ds \quad \text{on } [\alpha, \beta],$$

$$\int_\alpha^x \frac{1}{a(s)} ds \leq \int_\alpha^x \frac{1}{A(s)} ds \quad \text{on } [\alpha, \beta],$$

then $\Lambda_0 \leq \lambda_0$ with equality if and only if $a(x) \equiv A(x)$ and $c(x) \equiv C(x)$.

Theorem 4.3. Consider equations (1.1) and (1.2). If

$$0 \leq \int_x^\beta c(s) ds \leq \int_x^\beta C(s) ds \quad \text{on } [\alpha, \beta],$$

$$0 < A(x) \leq a(x) \quad \text{on } [\alpha, \beta],$$

then $\Lambda_0 \leq \lambda_0$ with equality if and only if $A(x) \equiv a(x)$ and $c(x) \equiv C(x)$.

As a final application of the theory that has been developed, we will improve a result by Z. Nehari [7]. Although we shall prove the result for the case $n = 1$, an appropriate generalization can be proven for arbitrary n .

Theorem 4.4. Consider equations (1.1) and (1.2) for the case $n = 1$. Suppose that (1.1) has a solution u_0 for which $u_0(\alpha) = u_0'(\beta) = 0$, and that (1.2) has a solution v_0 for which $v_0(\beta) > 0$ and $v_0'(\beta) \geq 0$. If (4.5) – (4.7) hold and either $c(x) \not\equiv C(x)$ or $a(x) \not\equiv A(x)$ then at least one of the functions v_0, v_0' changes sign in (α, β) .

Proof. Suppose neither v_0 nor v_0' changes sign in (α, β) . Since $v_0(\beta) > 0, v_0'(\beta) \geq 0$, we have that $v_0(x) \geq 0, v_0'(x) \geq 0$ in (α, β) . There is no loss of generality in assuming that $u_0(x)$ does not change sign in (α, β) , since otherwise α could be replaced by the first zero of $u(x)$ to the left of β . We may also assume that $u_0(x)$ is nonnegative in the interval. Now consider the integral equations:

$$M[u] = \int_\alpha^\beta G_a(x, s)c(s)u(s) ds, \quad R[u] = \int_\alpha^\beta G_A(x, s)c(s)u(s) ds,$$

$$S[u] = \int_\alpha^\beta G_A(x, s)C(s)u(s) ds + A(\beta)v_0'(\beta) \int_\alpha^x \frac{1}{A(s)} ds + v_0(\alpha).$$

Observe that $M[u_0] = u_0$ and $S[v_0] = v_0$. Since $u_0 \in \mathcal{P}_1$ and $0 \leq M \leq R$ with respect to \mathcal{P}_1 , we see that $1 \leq \mu_0$ where μ_0 is the largest eigenvalue of $R[u] = \mu u$. Also $1 = \mu_0$ if and only if $a(x) \equiv A(x)$. Now we notice that $0 \leq R \leq S$ with re-

spect to \mathcal{P}_2 and that $v_0 \in \mathcal{P}_2$. Therefore $\mu_0 \leq 1$ with equality if and only if $c(x) \equiv C(x)$. This leads to a contradiction and consequently either v_0 or v_0' changes sign in (α, β) .

V. **Remarks.** It should be noted that the above technique applies equally well to the eigenvalue problems

$$(5.1) \quad \begin{aligned} & [a(x)u^{(n)}]^{(n)} + \lambda(-1)^{n+1}c(x)u = 0, \\ & u_1(\alpha) = u_1'(\alpha) = \dots = u_1^{(n-1)}(\alpha) = u(\beta) = u'(\beta) = \dots = u^{(n-1)}(\beta) = 0; \end{aligned}$$

$$(5.2) \quad \begin{aligned} & [A(x)v^{(n)}]^{(n)} + \Lambda(-1)^{n+1}C(x)v = 0, \\ & v_1(\alpha) = v_1'(\alpha) = \dots = v_1^{(n-1)}(\alpha) = v(\beta) = v'(\beta) = \dots = v^{(n-1)}(\beta) = 0, \end{aligned}$$

where, as before, $a(x), A(x) > 0$ and $u_1(x) \equiv a(x)u^{(n)}(x)$, $v_1(x) \equiv A(x)v^{(n)}(x)$.

We confine ourselves here to stating the following result:

Theorem 5.1. *Consider equations (5.1) and (5.2). If $0 < c(x)$, $\int_{\alpha}^x c(s) ds \leq \int_{\alpha}^x C(s) ds$, $\int_x^{\beta} 1/a(s) ds \leq \int_x^{\beta} 1/A(s) ds$ then $\Lambda_0 \leq \lambda_0$ with equality if and only if $a(x) \equiv A(x)$ and $c(x) \equiv C(x)$.*

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