TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 177, March 1973

POSITIVE APPROXIMANTS

BY

RICHARD BOULDIN

ABSTRACT. Let $T = B + iC$ with $B = B^*$, $C = C^*$ and let $\delta(T)$ denote the the distance of T to the set of nonnegative operators. We find upper and lower bounds for $\delta(T)$. We prove that if P is any best approximation for T among nonnegative operators then $P \leq B + ((\delta(T))^2 - C^2)^{1/2}$. Provided $B \geq 0$ or T is normal we characterize those T which have a unique best approximation among the nonnegative operators. If T is normal we characterize its best approximating nonnegative operators which commute with it. We characterize those T for which the zero operator is the best approximating nonnegative operator.

1. **Introduction.** This paper continues the research initiated by P. R. Halmos in [3]. In that paper Halmos suggested a number of research problems related to approximating a given operator with some nonnegative operator. He asked for a characterization of when a given operator has a unique nonnegative operator closest to it (in the sense of the operator norm) and he asked for a formula (if such exists) for the distance between a given operator and the set of nonnegative operators. He also indicated the desirability of describing all of the nonnegative operators which have the minimum distance to a given operator—especially when the given operator is normal.

In this paper we characterize when a normal operator has a unique closest nonnegative operator and we characterize when any operator with nonnegative real part has a unique closest nonnegative operator. We characterize those operators for which the zero operator is the closest nonnegative operator. Furthermore, for a normal operator we describe all of the nonnegative operators having the minimum distance to the given normal operator and commuting with it. More generally, we prove a theorem which replaces the problem of finding the nonnegative operators closest to a given operator with a seemingly easier problem. Then we give partial solutions to the easier problem for operators with nonnegative real parts and for normal operators.

The author is grateful to P. R. Halmos for a preprint of the paper [3] and for the subsequent stimulating correspondence.

Presented to the Society, January 25, 1973; received by the editors April 4, 1972.
AMS (MOS) subject classifications (1970). Primary 47A55; Secondary 47B15.

2. **Preliminaries.** By operator we shall mean a bounded linear operator on a complex Hilbert space. Let T be an operator on H. If S is a subspace of H then we denote the restriction of T to S by T/S; the spectrum of T is $\sigma(T)$; we simply write c when we mean cI with I the identity transformation. To say that T is normaloid means that the numerical radius of T, denoted $w(T)$, is equal to the norm of T, denoted $\|T\|$.

We shall follow Halmos' convention of using "positive operator" as synonymous with "nonnegative operator"; thus a positive operator is not necessarily invertible. We also use Halmos' phrase "positive approximant" to mean the closest positive operator or the best approximation among the positive operators. For the reader's convenience we restate the following results proved by Halmos in [3].

2.1. **Theorem.** *If $B + iC$ is the usual Cartesian representation for the operator T then*

$$\inf\{\|T - P\|: P \geq 0\} = \inf\{r: r \geq \|C\|,\ B + (r^2 - C^2)^{1/2} \geq 0\}.$$

The first infimum shall henceforth be denoted $\delta(T)$.

2.2. **Theorem.** *If $B + iC$ is the usual Cartesian representation for the operator T and if $P_0 = B + ((\delta(T))^2 - C^2)^{1/2}$ then P_0 is a positive approximant for T.*

We shall henceforth denote the above positive approximant for any operator T by P_0 and we shall refer to it as the Halmos positive approximant.

2.3. **Theorem.** *Any operator T has a representation of the form $P + U\delta(T)$ where $P \geq 0$ and U is unitary with negative real part. If T is not a positive operator then the above representation is unique.*

We shall use throughout this paper the existence and uniqueness of the Cartesian representation. For any operator T there are two unique selfadjoint operators B and C such that $T = B + iC$. The positive part of a selfadjoint operator such as B, which is denoted by B_+, can be defined in several equivalent ways. If we take only positive square roots then $2B_+ = B + |B^2|^{1/2}$. If $E(\cdot)$ denotes the spectral measure of B then $B_+ = BE([0, \infty)) = BE((0, \infty))$; this is a quick consequence of the usual operational calculus. The analogous statements for the negative part of B, denoted B_-, are clearly true. We shall use the equivalence of these different definitions without further explanation.

3. **The distance to the positive operators.** In this section we shall collect some basic results on the parameter $\delta(T)$. Although the conclusion of the first lemma seems to be well known, the author is unable to find a suitable reference and thus a brief proof is given.

3.1. **Lemma.** *If A and B are positive operators such that $B \geq A$ then their usual positive square roots have the same relation—i.e. $B^{1/2} \geq A^{1/2}$.*

Proof. Using a result of [2] we deduce this lemma as a consequence of the main results of Bendat and Sherman in [1]. Theorem 2.3 and Theorem 2.5 of the second paper imply that $x^{1/2}$ is a monotone operator function on any interval $(0, M)$ with M positive. Here an appropriate definition of square root is that $z^{1/2} = |z|^{1/2} \exp((i/2) \arg z)$ where $\arg z \in (-\pi, \pi]$; thus $\operatorname{Im} z^{1/2} > 0$ whenever $\operatorname{Im} z > 0$.

Thus if $B \geq A \geq \epsilon I > 0$ we can deduce that $B^{1/2} \geq A^{1/2}$. By Corollary 2 of [2], $(A + \epsilon)^{1/2}$ and $(B + \epsilon)^{1/2}$ converge to $A^{1/2}$ and $B^{1/2}$ as ϵ goes to 0 from the right. Consequently we deduce that $B^{1/2} \geq A^{1/2}$.

More generally for $\alpha \in (0, 1)$ the function x^{α} is a monotone operator function. Regretfully x^2 is not a monotone operator function. If we take any positive operator A and we let P be the orthogonal projection onto a unit vector which is not an eigenvector of A then $(A + P)^2 - A^2$ is not a positive operator.

An example due to Halmos shows that strict inequality can occur in the following inequality.

3.2. **Theorem.** *If $T = B + iC$ (with $B = B^*$, $C = C^*$) then $\|B_-^2 + C^2\|^{1/2} \geq \delta(T)$.*

Proof. By 2.1 it suffices to prove that

$$\inf\{r: r \geq \|C\|,\ B + (r^2 - C^2)^{1/2} \geq 0\} \leq \|B_-^2 + C^2\|^{1/2}.$$

Take r such that $r^2 \geq \|B_-^2 + C^2\|$ and note that

$$B_-^2 \leq r^2 - C^2,$$

$$B_- = (B_-^2)^{1/2} \leq (r^2 - C^2)^{1/2},$$

$$0 \leq -B_- + (r^2 - C^2)^{1/2},$$

$$0 \leq B_+ - B_- + (r^2 - C^2)^{1/2} = B + (r^2 - C^2)^{1/2}.$$

This shows that any lower bound for the set $\{r: r \geq \|C\|,\ B + (r^2 - C^2)^{1/2} \geq 0\}$ is not greater than $\|B_-^2 + C^2\|^{1/2}$.

3.3. **Corollary.** *If $T = B + iC$ (with $B = B^*$, $C = C^*$) and B is positive then $\delta(T) = \|C\|$.*

Proof. The inequality $\delta(T) \geq \|C\|$ is immediate from 2.1 and the opposite inequality is immediate from 3.2 since $B_- = 0$.

For T a normal operator, Halmos proved a useful formula for $\delta(T)$. We shall give an alternative proof of that formula; our proof exploits the observation that if A and B are commuting positive operators such that $B \geq A$ then $B^2 \geq A^2$.

3.4. **Theorem.** *If $N = B + iC$ (with $B = B^*$, $C = C^*$) is normal then $\delta(N) = \|-B_- + iC\|$.*

Proof. Because $-B_- + iC$ is normal we have $\|-B_- + iC\|^2 = \|B_-^2 + C^2\|$, and in view of 3.2 we need only prove that $\|B_-^2 + C^2\|^{1/2} \leq \delta(T)$. In order to prove this we show that $\|B_-^2 + C^2\|^{1/2}$ is a lower bound for the set $\{r: r \geq \|C\|, B + (r^2 - C^2)^{1/2} \geq 0\}$ and so we take r to be an element of that set. For any vector f we get

$$\langle [r^2 - C^2 - B_-^2]f, f\rangle = \langle [(r^2 - C^2)^{1/2} - B_-][(r^2 - C^2)^{1/2} + B_-]f, f\rangle$$

$$= \langle [(r^2 - C^2]^{1/2} - B_-][(r^2 - C^2)^{1/2} + B_-]^{1/2} f, [(r^2 - C^2)^{1/2} + B_-]^{1/2} f\rangle$$

and consequently $[r^2 - C^2 - B_-^2]$ is positive provided that $[(r^2 - C^2)^{1/2} - B_-]$ is positive. If $E(\cdot)$ is the spectral measure for B then $H_- = E((-\infty, 0))H$ reduces $B + (r^2 - C^2)^{1/2}$ to $[-B_- + (r^2 - C^2)^{1/2}]/H_-$ and so the latter operator is positive; the orthogonal complement of H_- reduces $[-B_- + (r^2 - C^2)^{1/2}]$ to the positive operator $(r^2 - C^2)^{1/2}$. Since $[(r^2 - C^2)^{1/2} - B_-]$ is positive we see that $[r^2 - C^2 - B_-^2]$ is positive and since $B_-^2 + C^2$ is selfadjoint this obviously implies that $r^2 \geq \|B_-^2 + C^2\|$. This completes the proof.

We obtain another result using the observation that $-B + P \geq -B \geq 0$ and $BP = PB$ imply that $(P - B)^2 \geq B^2$.

3.5. **Theorem.** *Let $T = B + iC$ with $B \leq 0$ and $C = C^*$. If T has a positive approximant which commutes with B then $\delta(T) = \|B^2 + C^2\|^{1/2}$.*

Proof. By 3.2 we have $\delta(T) \leq \|B^2 + C^2\|^{1/2}$. In order to use a basic inequality noted by Halmos we set $S = B - P + iC$ and we note that $(B - P)^2 + C^2$ is $\frac{1}{2}(S^*S + SS^*)$ and consequently $\|(B - P)^2 + C^2\| \leq \|S^*\|\,\|S\| = \|S\|^2$. Thus if P is a positive approximant for T which commutes with B we have

$$\delta(T)^2 = \|B + iC - P\|^2 \geq \|(P - B)^2 + C^2\|.$$

Since $-B \geq 0$ and $P(-B) = (-B)P$ we see that $-B + P \geq -B \geq 0$ and $(P - B)^2 \geq B^2$; thus $(P - B)^2 + C^2 \geq B^2 + C^2$ and $\|(P - B)^2 + C^2\| \geq \|B^2 + C^2\|$. Hence $\delta(T) \geq \|B^2 + C^2\|^{1/2}$, as desired.

The results of §4 suggest that generally there are many positive approximants which commute with the imaginary part of T but there are few, or none, which commute with the real part of T.

Now we obtain a lower bound for $\delta(T)$. It is easy to see that strict inequality can occur.

3.6. **Theorem.** *If $T = B + iC$ (with $B = B^*$, $C = C^*$) then*

$$\delta(T) \geq \sup\{|\langle(-B_- + iC)f, f\rangle|: f \in \ker B_+, \|f\| = 1\}.$$

Proof. The function $g(t) = -(r^2 - t^2)^{1/2}$ for $t \in [-r, r]$ and $g(t) = 0$ for $t \notin [-r, r]$ is clearly convex on the interval $(-\infty, \infty)$. Let $F(\cdot)$ be the spectral measure for C and note that for any unit vector f the positive measure $\langle F(\cdot)f, f\rangle$ is concentrated on $\sigma(C)$ with total mass 1. If we apply Jensen's inequality (see [5]) to this measure and the function $g(t)$ then we get

$$(*) \qquad (r^2 - \langle Cf, f\rangle^2)^{1/2} \geq \langle (r^2 - C^2)^{1/2} f, f\rangle$$

provided $r \geq \|C\|$.

We shall actually prove that $\sup\{|\langle(-B_- + iC)f, f\rangle|: f \in \ker B_+, \|f\| = 1\}$ is a lower bound for the set $\{r: r \geq \|C\|, B + (r^2 - C^2)^{1/2} \geq 0\}$. Thus we take r in the latter set. It is routine to see that $\ker B_+ = E((-\infty, 0])H = H_-$; we choose a unit vector $f \in H_-$. Since on H_-, $B + (r^2 - C^2)^{1/2}$ is just $-B_- + (r^2 - C^2)^{1/2}$ we see that $[-\langle B_- f, f\rangle + \langle (r^2 - C^2)^{1/2} f, f\rangle]$ is nonnegative for $f \in H_-$ and in view of $(*)$ above this implies that

$$(r^2 - \langle Cf, f\rangle^2)^{1/2} \geq \langle B_- f, f\rangle \geq 0,$$

$$r^2 - \langle Cf, f\rangle^2 \geq \langle B_- f, f\rangle^2,$$

$$r^2 \geq |-\langle B_- f, f\rangle + i\langle Cf, f\rangle|^2.$$

This proves the theorem.

3.7. **Corollary.** *If $T = B + iC$ (with $B = B^*$, $C = C^*$) and $B \leq 0$ then $\delta(T)$ is greater than or equal to the numerical radius of T, denoted $w(T)$.*

3.8. **Corollary.** *If $T = B + iC$ (with $B = B^*$, $C = C^*$) is normaloid and $B \leq 0$ then $\delta(T) = \|T\|$.*

Proof. Since $B = B_-$ and $w(T) = \|T\|$ we have $\|T\|^2 \geq \|B^2 + C^2\| = \|B_-^2 + C^2\| \geq \delta(T)^2 \geq w(T)^2 = \|T\|^2$.

4. **Positive approximants for an arbitrary operator.** The central result of this section will show that the Halmos positive approximant is absolutely maximal. The following lemma is the key to proving that theorem.

4.1. **Lemma.** *Let C be a selfadjoint contraction and set $U = -(1 - C^2)^{1/2} + iC$. If A is a selfadjoint operator such that $\|U + A\| \leq 1$ then A is positive.*

Proof. Let $E(\cdot)$ be the spectral measure for A and take a sequence of unit vectors $\{f_n: n = 1, 2, \cdots\}$ from $E((-\infty, 0))H$ such that $\{\|(-A - \|A_-\|)f_n\|: n = 1, 2, \cdots\}$ converges to 0; this is possible since $A_- = -AE((-\infty, 0))$ and the boundary of $\sigma(A_-)$ is contained in the approximate point spectrum of A_-. If we set $g_n = (-A - \|A_-\|)f_n$ then $Af_n = -\|A_-\|f_n - g_n$ and $\{g_n: n = 1, 2, \cdots\}$ converges to 0. Thus we have

$$1 \geq \|U + A\|^2 \geq \|(U + A)f_n\|^2 = \langle (I + AU + U^*A + A^2)f_n, f_n\rangle$$

$$= 1 + \|Af_n\|^2 + 2\,\mathrm{Re}\langle Uf_n, Af_n\rangle$$

or

$$-2\,\mathrm{Re}\langle Uf_n, Af_n\rangle \geq \|Af_n\|^2$$

or

$$-2\,\mathrm{Re}(-\|A_-\|\langle Uf_n, f_n\rangle - \langle Uf_n, g_n\rangle) \geq \|Af_n\|^2.$$

Thus we have

$$(*) \qquad 2\|A_-\|\,\mathrm{Re}\langle Uf_n, f_n\rangle + 2\|g_n\| \geq \|Af_n\|^2.$$

Suppose that $\{f_n : n = 1, 2, \cdots\}$ has a subsequence $\{w_n : n = 1, 2, \cdots\}$ such that $|\mathrm{Re}\langle Uw_n, w_n\rangle| \geq \delta > 0$ for $n = 1, 2, \cdots$. Because $\mathrm{Re}\langle Uf, f\rangle \leq 0$ for all f, the inequality $(*)$ implies that $\|A_-\| = 0$. The only other possibility is that $\{\mathrm{Re}\langle Uf_n, f_n\rangle : n = 1, 2, \cdots\}$ converges to 0 and $\{\langle (1 - C^2)^{1/2}f_n, f_n\rangle = \langle (-\mathrm{Re}\, U)f_n, f_n\rangle : n = 1, 2, \cdots\}$ converges to 0. This implies that $\{\|(1 - C^2)^{1/2}f_n\| : n = 1, 2, \cdots\}$ converges to 0 and since $\|(1 - C^2)^{1/2}f_n\|^2 = \langle (1 - C^2)f_n, f_n\rangle = 1 - \|Cf_n\|^2$, we conclude that $\{\|Cf_n\| : n = 1, 2, \cdots\}$ converges to 1. If we apply this fact in the elementary inequality $\|T\|^2 \geq \|(\mathrm{Re}\, T)^2 + (\mathrm{Im}\, T)^2\|$ we get

$$1 \geq \|U + A\|^2 \geq \|(\mathrm{Re}\, U + A)^2 + C^2\|$$

$$\geq \langle ((\mathrm{Re}\, U + A)^2 + C^2)f_n, f_n\rangle = \|(\mathrm{Re}\, U + A)f_n\|^2 + \|Cf_n\|^2.$$

Since the last sum converges to $\|A_-\|^2 + 1$, we conclude that $\|A_-\| = 0$.

The next theorem makes it clear that the Halmos positive approximant is significant in any scheme for constructing positive approximants.

4.2. **Theorem.** *If P is any positive approximant for $T = B + iC$ (with $B = B^*$, $C = C^*$) then $0 \leq P \leq P_0$ where $P_0 = B + (\delta(T)^2 - C^2)^{1/2}$.*

Proof. If T is positive then $\delta(T) = 0$, $C = 0$, and P_0 is the unique positive approximant of T; certainly this theorem is true in that case. If T is not positive then we may replace T with $T(1/\delta(T))$; equivalently we shall assume that $\delta(T) = 1$. Thus for any positive approximant P we have

$$1 = \|T - P\| = \|T - P_0 + P_0 - P\| = \|-(1 - C^2)^{1/2} + iC + (P_0 - P)\|$$

and we apply 4.1 to conclude that $P_0 - P \geq 0$ or $P_0 \geq P$.

The next theorem replaces the difficult task of finding the positive approximants of a nonnormal operator T with a very similar problem for a normal operator with a special form.

4.3. **Theorem.** *There is a one-to-one correspondence between the positive approximants of the operator* $T = B + iC$ *(with* $B = B^*$, $C = C^*$*) and the positive operators* A *such that*

(i) $0 \leq A \leq P_0 = B + (\delta(T)^2 - C^2)^{1/2}$ *and*

(ii) $\|-(\delta(T)^2 - C^2)^{1/2} + iC + A\| \leq \delta(T)$.

If A *satisfies the two conditions then* $(P_0 - A)$ *is the corresponding positive approximant.*

Proof. The theorem is a straightforward consequence of 4.2 and the equation $\|T - (P_0 - A)\| = \|-(\delta(T)^2 - C^2)^{1/2} + iC + A\|$.

The above result allows us to derive considerable information about the positive approximants for an operator with positive real part. If $E(\cdot)$ is the spectral measure for C in the next theorem then for any Borel set Δ we may take A to be $E(\Delta)(\delta(T)^2 - C^2)^{1/2}$ and then A has properties (i) and (ii). Consequently T has an abundance of positive approximants.

4.4. **Theorem.** *If* $T = B + iC$ *(with* $B = B^*$, $C = C^*$*) and* $B \geq 0$ *then* $P_0 - A$, *with* $P_0 = B + (\delta(T)^2 - C^2)^{1/2}$, *is a positive approximant of* T *for each* A *satisfying the two conditions*:

(i) $0 \leq A \leq (\delta(T)^2 - C^2)^{1/2}$, *and*

(ii) $AC = CA$.

Proof. For simplicity we assume $\delta(T) = 1$. For any unit vector f it follows from (i) that

$$0 \leq \langle((1 - C^2)^{1/2} - A)f, f\rangle \leq \langle(1 - C^2)^{1/2} f, f\rangle.$$

In the above inequality we replace each term with its square and we add $(\langle Cf, f\rangle)^2$ throughout; then by taking square roots we get

$$(*) \qquad \begin{aligned} |\langle(A - (1 - C^2)^{1/2} + iC)f, f\rangle| &\leq |\langle((1 - C^2)^{1/2} + iC)f, f\rangle| \\ &\leq \|(1 - C^2)^{1/2} + iC\| = 1. \end{aligned}$$

Because of (ii) the operator $(A - (1 - C^2)^{1/2} + iC)$ is normal and its norm is equal to its numerical radius; consequently $(*)$ implies that $\|A - (1 - C^2)^{1/2} + iC\| \leq 1$. This shows that (ii) of 4.3 is satisfied. Condition (i) above and the fact that $B \geq 0$ imply that (i) of 4.3 is satisfied and so 4.4 follows from 4.3.

4.5. **Lemma.** *Let* $T = B + iC$ *(with* $B = B^*$, $C = C^*$*) have positive real part. If* $C^2 \neq \|C\|^2 I$ *then* T *does not have a unique positive approximant.*

Proof. According to 3.3, $\delta(T) = \|C\|$ and since $\|T - B\| = \|iC\|$ we see that B is a positive approximant. Of course, $P_0 = B + (\|C\|^2 - C^2)^{1/2}$ is a positive approximant. If T had a unique positive approximant then it would follow that $(\|C\|^2 - C^2)^{1/2} = 0$ and so $C^2 = \|C\|^2 I$.

4.6. **Theorem.** *Let $T = B + iC$ (with $B = B^*$, $C = C^*$) be given with $B \geq 0$. The operator T has a unique positive approximant if and only if $C^2 = \|C\|^2 I$.*

Proof. It is immediate from 4.5 that T has a unique positive approximant only if $C^2 = \|C\|^2 I$. In order to prove the converse we assume that $C^2 = \|C\|^2 I$ and we set $Q_+ = F(\{\|C\|\})$, $Q_- = F(\{-\|C\|\})$ where $F(\cdot)$ is the spectral measure for C. If P is a positive approximant then 3.3 implies that

$$\|C\|^2 = \|B + iC - P\|^2 \geq \|Q_+(B + iC - P)Q_+\|^2$$

$$\geq \|(Q_+(B - P)Q_+)^2 + (Q_+CQ_+)^2\| \geq \langle (Q_+BQ_+ - Q_+PQ_+)^2 f, f\rangle + \|C\|^2$$

whenever $\|f\| = 1$. It follows that $(Q_+BQ_+ - Q_+PQ_+)^2 = 0$ or $Q_+BQ_+ = Q_+PQ_+$. Similarly we can show that $Q_-BQ_- = Q_-PQ_-$. If we represent B, P and C with respect to the decomposition $Q_+H \oplus Q_-H = H$ then $B_1 = P_1$ and $B_4 = P_4$ in the following:

$$B - P + iC = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} - \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} + i \begin{pmatrix} \|C\|I & 0 \\ 0 & -\|C\|I \end{pmatrix}$$

$$= \begin{pmatrix} i\|C\|I & B_2 - P_2 \\ B_3 - P_3 & -i\|C\|I \end{pmatrix}.$$

For any unit vector $f \in Q_+H$ we have

$$\|C\|^2 \geq \|(B - P + iC)f\|^2 = \|C\|^2 + \|(B_3 - P_3)f\|^2$$

and so $B_3 = P_3$. Similarly we can show that $B_2 = P_2$. Thus we have $P = B$, as desired.

The last theorem of this section describes the set of all positive approximants of a normal operator N which commute with N. Although a more concrete description is desirable, the theorem shows the nature of positive approximants.

4.7. **Theorem.** *Let $N = B + iC$ (with $B = B^*$, $C = C^*$) be a normal operator. An operator A which commutes with N satisfies conditions* (i), (ii) *of* 4.3 *if and only if there is a selfadjoint operator T and there are Borel measurable functions $b(t)$, $c(t)$, $a(t)$ such that*

(a) $A = a(T)$, $B = b(T)$, $C = C(T)$,

(b) $0 \leq a(t) \leq b(t) + (\delta(N)^2 - C(t)^2)^{1/2}$ *on* $\sigma(T)$,

(c) $|(a(t) - (\delta(N)^2 - C(t)^2)^{1/2} + iC(t))| \leq \delta(N)$ *on* $\sigma(T)$.

Proof. By the usual operational calculus for a selfadjoint operator applied to T we see that (i), (ii) of 4.3 follow from (a), (b), (c). The converse follows from the Putnam-Fuglede theorem, a theorem of Von Neumann (see [4, p. 358]), and the operational calculus for a selfadjoint operator.

5. **Normal operators with unique positive approximants.** The normal operators with unique positive approximants can be characterized by a simple geometric condition on the spectrum. The collection of all the points in the complex plane which have a distance d to the positive real numbers is a curve consisting of a semicircle with radius d in the left half-plane and in the right half-plane two straight lines parallel to the positive real numbers and passing through id and $-id$. This curve coincides with the nonnegative real numbers in the case that d is zero. A normal operator has a unique positive approximant if and only if its spectrum is a subset of the above curve for some choice of d.

5.1. **Lemma.** *Let $N = B + iC$ (with $B = B^*$, $C = C^*$) be a normal operator with spectral measure $E(\cdot)$ and let $F(\cdot)$ be the spectral measure for B. If Δ is a Borel subset of the reals then $F(\Delta) = E(S)$ where $S = \{z\colon \operatorname{Re} z \in \Delta\}$. In particular the positive and negative parts of B, denoted B_+ and B_-, are $BE(\{z\colon \operatorname{Re} z \geq 0\})$ and $BE(\{z\colon \operatorname{Re} z < 0\})$, respectively.*

Proof. If $h(t)$ is any Borel measurable function on $\operatorname{Re} \sigma(N)$ and f, g are any vectors then

$$\int_{\operatorname{Re}\sigma(N)} h(t)\,d\langle F(t)f, g\rangle = \int_{\sigma(B)} h(t)\,d\langle F(t)f, g\rangle$$
$$= \langle h(B)f, g\rangle = \int_{\sigma(N)} h(\operatorname{Re} z)\,d\langle E(z)f, g\rangle.$$

The proof of the next theorem requires a rather elaborate construction with spectral measures and the operational calculus for a normal operator. Consequently we have broken the proof into parts. Note that 3.4 implies that B_+ is a positive approximant for the normal operator $N = B + iC$ (with $B = B^*$, $C = C^*$).

5.2. **Theorem.** *If $N = B + iC$ (with $B = B^*$, $C = C^*$) is normal and N has a unique positive approximant then for some nonnegative number d we have*

$$\sigma(N) \subset \{z\colon \operatorname{Re} z \leq 0,\ |z| = d\} \cup \{z\colon 0 \leq \operatorname{Re} z \leq \|N\|,\ \operatorname{Im} z = \pm d\}.$$

Step 1. We assume the notation of 5.1 and we define C' to be $CE(\{z\colon \operatorname{Re} z \geq 0\})$. If Δ is a closed subset of $\{z\colon \operatorname{Re} z \geq 0\}$ such that $\Delta \cap \{z\colon \operatorname{Im} z = \pm \|C'\|\} = \varnothing$ then $E(\Delta) = 0$.

Proof of Step 1. For simplicity set $Q_+ = E(\{z\colon \operatorname{Re} z \geq 0\})$. Assume $E(\Delta)H \neq \{0\}$ and note that

$$(-B_- + iC)/E(\Delta)H = (-B_- + iC)E(\Delta)/E(\Delta)H = (-B_- + iC)Q_+E(\Delta)/E(\Delta)H$$
$$= (-B_-Q_+ + iCQ_+)E(\Delta)/E(\Delta)H = \|iC'/E(\Delta)E\|.$$

Thus we have $\|(-B_- + iC)/E(\Delta)H\| = \|iC'/E(\Delta)H\|$.

By well-known results (see [6]) we have

$$\sigma(iC'/E(\Delta)H) = \sigma(iC/E(\Delta)H) = \sigma(i \operatorname{Im} N/E(\Delta)H)$$
$$= i \operatorname{Im} \sigma(N/E(\Delta)H) \subset i[-\|C'\| + \delta,\ \|C'\| - \delta]$$

for some positive δ. Since $C'/E(\Delta)H$ is normaloid, for some positive ϵ we have

$$\|(-B_- + iC)/E(\Delta)H\| = \|C'/E(\Delta)H\| = \|C'\| - \epsilon.$$

On $E(\Delta)H$ define a nonzero positive operator P' such that $\|P'\| \leq \epsilon$ and note that if $P = P' \oplus 0$ on $E(\Delta)H \oplus (E(\Delta)H^{\perp}$ then we have $\|-B_- + iC - P\| = \|-B_- + iC\|$ since the norm of a direct sum operator is clearly the maximum of the norms of the direct summands.

By 3.4 it follows that $B_+ + P$ is a positive approximant. The contradiction proves that $E(\Delta)H = \{0\}$.

Step 2. We let Q_- denote $E(\{z\colon \operatorname{Re} z < 0\})$ and we define C'' to be CQ_-. If Δ is a closed subset of $\{z\colon \operatorname{Re} z < 0\}$ such that

$$\Delta \cap \{z\colon |z| = \|-B_- + iC''\|\} = \varnothing$$

then $E(\Delta) = 0$.

Proof of Step 2. The proof is analogous to the proof of Step 1.

Step 3. If $N = B + iC$ is normal operator with a unique positive approximant then either the quantities $\|-B_- + iC''\|$ and $\|C'\|$ are equal or the subspace Q_-H or Q_+H corresponding to the smaller quantity is trivial.

Proof of Step 3. Note that on $H = Q_-H \oplus Q_+H$ we have $-B_- + iC = (-B_- + iC'') \oplus iC'$ and so

$$\|-B_- + iC\| = \max\{\|-B_- + iC''\|, \|iC'\|\}.$$

For simplicity, assume $\|-B_- + iC''\| = \|C'\| + \delta$ with δ a positive number and $Q_+H \neq \{0\}$; we shall obtain a contradiction. Then we construct a nonzero positive operator P' on Q_+H such that $\|P'\| \leq \delta$ and we set $P = 0 \oplus P'$ on $Q_-H \oplus Q_+H$. It is routine to see that $\|-B_- + iC - P\| = \|-B_- + iC\|$ and so $B_+ + P$ is a positive approximant different from B_+. This contradiction proves that if $\|-B_- + iC''\| > \|C'\|$ then $Q_+H = \{0\}$. An analogous argument for the case $\|-B_- + iC''\| < \|C'\|$ completes the proof of Step 3.

Step 4. Theorem 5.2 now follows.

Proof of Step 4. If $Q_-H \neq \{0\}$ then we set $d_1 = \|-B_- + iC''\|$ and we let $S = \{z\colon \operatorname{Re} z \leq 0, |z| = d_1\}$. For each $n = 1, 2, \cdots$ set $S_n = \{z\colon \operatorname{Re} z < 0, \operatorname{dist}(z, S) \geq 1/n\}$ and note that Step 2 implies that $E(S_n) = 0$. Thus

$$Q_-H = E\left(\{z\colon \operatorname{Re} z < 0\} - \bigcup_{n=1}^{\infty} S_n\right) H = E(S)H$$

and by the usual operational calculus we have

$$\sigma(N/Q_-H) = \sigma(N/E(S)H) \subset S.$$

If $Q_+H \neq \{0\}$ then we set $d_2 = \|iC'\|$ and we let $L = \{z\colon 0 \leq \operatorname{Re} z \leq \|N\|, \operatorname{Im} z = \pm d_2\}$. For each $n = 1, 2, \cdots$ set $L_n = \{z\colon \operatorname{Re} z \geq 0, \operatorname{dist}(z, L) \geq 1/n\}$ and note

that Step 1 implies that $E(L_n) = 0$. Thus

$$Q_+H = E\left(\{z: \operatorname{Re} z \geq 0\} - \bigcup_{n=1}^{\infty} L_n\right)H = E(L)H$$

and $\sigma(N/Q_+H) = \sigma(N/E(L)H) \subset L$.

By Step 3 if both of the subspaces Q_-H and Q_+H are nontrivial then $d_1 = d_2$ and 5.2 follows from the fact that $\sigma(N) = \sigma(N/Q_-H) \cup \sigma(N/Q_+H)$. If one of these subspaces is trivial then $\sigma(N)$ is just the spectrum of N restricted to the nontrivial subspace and 5.2 follows from either the first or second paragraph above, depending on which subspace is nontrivial.

In general, a normal operator has such an abundance of positive approximants that describing all of them is somewhat analogous to describing all of the square roots of a normal operator. The following theorem is an obvious consequence of the method used to prove 5.2 and this theorem clearly suggests the difficulties in describing all positive approximants for a fixed normal operator.

5.3. **Theorem.** *Let $N = B + iC$ (with $B = B^*$, $C = C^*$) be a normal operator, let B_+ denote the positive part of B, and let $F(\cdot)$ denote the spectral measure of $N - B_+$. If P' is any positive operator on $F(\{z: |z| \leq \delta(N) - \epsilon\})H$ such that $\|P'\| \leq \epsilon$ then $(B_+ + P)$ is a positive approximant for N where P agrees with P' on its domain and is zero on the orthogonal complement of that domain.*

One of the ingredients for proving the converse of 5.2 has already been given in the form of 4.6 which characterized the operators with positive real part and unique positive approximants. We shall prove that a normal operator with its spectrum contained in a semicircle centered at the origin and contained in the left half-plane has a unique positive approximant. That is proved in the next lemma. Then we shall use a matrix argument similar to the one used in the proof of 4.6 to show that the converse of 5.2 holds.

5.4. **Lemma.** *If $U = B + iC$ (with $B = B^*$, $C = C^*$) is a unitary operator and $B \leq 0$ then the zero operator is the unique positive approximant of U.*

Proof. It is possible to write out a proof which is a simplified version of the proof of 4.2. However, we offer a quicker proof based on 4.2 and 2.3. From 3.4 it is apparent that $\delta(U) = 1$ and so 0 is a positive approximant of U; by 2.3 we know that the operator U has a unique representation $P_0 + U_1$ where P_0 is the Halmos' positive approximant and U_1 is a unitary operator with $\operatorname{Re} U_1 \leq 0$. Since $0 + U$ is such a representation, it must be that $0 = P_0$. It is now immediate from 4.2 that the only positive approximant is the zero operator.

It is interesting to note that the converse of the above lemma is true. Note that we do not assume that T is normal.

5.5. **Theorem.** *The zero operator is the unique positive approximant of T if and only if there is a selfadjoint contraction C such that $T = \delta(T)[-(1-C^2)^{1/2} + iC]$.*

Proof. If T is positive then it is a positive approximant for itself and $\delta(T) = 0$. Thus it is clear that $T = 0$ and the desired formula holds for any selfadjoint contraction C.

Provided T is not positive, $\delta(T)$ is not zero and we may replace T with $(1/\delta(T))T$. Thus we may assume that $\delta(T) = 1$. By hypothesis the Halmos' positive approximant is zero—that is, $B + (1-C^2)^{1/2} = 0$ or $B = -(1-C^2)^{1/2}$. Thus $T = -(1-C^2)^{1/2} + iC$, as desired.

Next we prove the converse of 5.2. This will establish the characterization of the normal operators with unique positive approximants as asserted in the first paragraph of this section.

5.6. **Theorem.** *Let $N = B + iC$ (with $B = B^*$, $C = C^*$) be a normal operator such that $\sigma(N)$ is contained in the set $\{z\colon \operatorname{Re} z \le 0,\ |z| = d\} \cup \{z\colon 0 \le \operatorname{Re} z \le \|N\|,\ \operatorname{Im} z = \pm d\}$ for some nonnegative number d. Then the positive part of B, denoted B_+, is the unique positive approximant of N.*

Proof. If $d = 0$ then N is positive and the above assertion is clear. Assuming $d > 0$ we replace N with $(1/d)N$;thus we may assume henceforth that $d = 1$. Let $E(\cdot)$ denote the spectral measure of B and define H_-, H_+ to be $E((-\infty, 0))H$, $E([0, \infty))H$, respectively. It is routine to see that

$$\sigma(N/H_-) = \sigma(N) \cap \{z\colon Re\ z \le 0\} \subset \{z\colon \operatorname{Re} z \le 0,\ |z| = 1\},$$
$$\sigma(N/H_+) = \sigma(N) \cap \{z\colon \operatorname{Re} z \ge 0\} \subset \{z\colon 0 \le \operatorname{Re} z \le \|N\|,\ \operatorname{Im} z = \pm 1\}.$$

If either H_- or H_+ is trivial then the desired conclusion follows from 4.6 or 5.4 depending on the case. Thus we may assume that both H_- and H_+ are nontrivial.

If P is a positive approximant for N and $Q_- = E((-\infty, 0))$ then the following shows that Q_-PQ_-/H_- is a positive approximant for N/H_-:

$$\begin{aligned}\|(-B_- + iQ_-CQ_- - Q_-PQ_-)/H_-\| &= \|Q_-(-B_- + iC - P)Q_-/H_-\| \\ &\le \|B + iC - P\| = \|-B_- + iC\| = \max\{\|-B_- + iCQ_-\|, \|iCQ_+\|\} = 1 \\ &= \|(-B_- + iCQ_-)/H_-\|\end{aligned}$$

where $Q_+ = E([0, \infty))$ and we are using that both B and C commute with the projections Q_+, Q_-. By 5.4 we deduce that $Q_-PQ_- = 0$ and using 4.6 we similarly deduce that $Q_+PQ_+ = B_+$.

We write P as a matrix relative to the decomposition $Q_-H \oplus Q_+H$: so

$$P = \begin{pmatrix} 0 & P_1 \\ P_2 & B_+ \end{pmatrix} \quad \text{and} \quad N - P = \begin{pmatrix} -B_- + iCQ_- & -P_1 \\ -P_2 & iCQ_+ \end{pmatrix}.$$

For each $f \in Q_- H$ we get

$$\|(-B_- + iCQ_-)f\|^2 \geq \|(N - P)f\|^2 = \|(-B_- + iCQ_-)f\|^2 + \|-P_2 f\|^2.$$

Thus $P_2 = 0$. Similarly $P_1 = 0$ and so

$$P = \begin{pmatrix} 0 & 0 \\ 0 & B_+ \end{pmatrix}$$

as desired.

6. **Questions and conjectures.** A very interesting problem that this paper leaves open is the characterization of those operators which have unique positive approximants. Using normal operators as a guide, one would try to handle the special cases where $\operatorname{Re} T \geq 0$ and $\operatorname{Re} T \leq 0$; then a matrix argument might handle the general case. The first case is covered by our 4.6 but the second case seems more difficult. Motivated by the case of normal operators, we conjecture that if T has a unique positive approximant and $\operatorname{Re} T \leq 0$ then for any unit vector f the norm of T restricted to the smallest T-invariant subspace containing f is $\|T\|$.

For any unit vector f we have $(\langle B_- f, f\rangle)^2 \leq \|B_- f\|^2$ and $(\langle Cf, f\rangle)^2 \leq \|Cf\|^2$ and consequently $|\langle -B_- + iC)f, f\rangle|$ is not greater than $(\|B_- f\|^2 + \|Cf\|^2)^{1/2} = (\langle (B_-^2 + C^2)f, f\rangle)^{1/2}$. By taking supremums we get that $\|B_-^2 + C^2\|^{1/2} \geq w(-B_- + iC)$. In our third section we showed that the larger of these two operator parameters is an upper bound for $\delta(B + iC)$ and we showed that the numerical radius of the compression of $-B_- + iC$ to $\ker B_+$ is a lower bound for $\delta(B + iC)$. What precisely is $\delta(B + iC)$? If $T = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ then $\delta(T) > W(-B_- + iC)$; what is an operator theoretic parameter between $\|B_-^2 + C^2\|^{1/2}$ and $w(-B_- + iC)$?

For other questions related to noncommutative approximation theory see the final section of [3].

BIBLIOGRAPHY

1. J. S. Bendat and S. Sherman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. 79 (1955), 58–71. MR 18, 588.

2. R. H. Bouldin, *Norm continuity properties of square roots*, SIAM J. Math. Anal. 3 (1972), 206–210.

3. P. R. Halmos, *Positive approximants of operators*, Indiana Univ. Math. J. (to appear).

4. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Akad. Kiadó, Budapest, 1953; English transl., Ungar, New York, 1955. MR 15, 132; MR 17, 175.

5. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966. MR 35 #1420.

6. R. Whitley, *The spectral theorem for a normal operator*, Amer. Math. Monthly 75 (1968), 856–861. MR 38 #5040.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30601