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k-PARAMETER SEMIGROUPS OF MEASURE-PRESERVING TRANSFORMATIONS

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ABSTRACT. An individual ergodic theorem is proved for semigroups of measure-preserving transformations depending on k real parameters, which generalizes N. Wiener's ergodic theorem.

In this paper we use the method introduced in ergodic theory by A. P. Calderón [1] combined with a covering lemma due to N. M. Rivière to obtain a pointwise ergodic theorem concerning k-parameter semigroups of measure-preserving transformations. Let X be a σ-finite measure space. We denote by T the set of all points $t = (t_1, \cdots, t_k)$ with nonnegative coordinates in k-dimensional euclidean space.

By a k-parameter semigroup of measure-preserving transformations we mean a system of mappings $(\theta_t, t \in T)$ of X into itself having the following properties:

(i) $\theta_t(\theta_s x) = \theta_{t+s} x$, $\theta_0 x = x$ for every t and s in T and every x in X.

(ii) For every measurable subset E of X the measure of $\theta_t^{-1}(E)$ equals the measure of E, for any t in T.

As usual, we shall assume that for any function f measurable on X, the function $f(\theta_t x)$ is measurable on the product space $T \times X$, where T is endowed with Lebesgue measure. In the next sections we give sufficient conditions for the almost everywhere convergence of the averages

$$A_\alpha f(x) = \frac{1}{|D_\alpha|} \int_{D_\alpha} f(\theta_t x)\, dt \quad \text{as } \alpha \to \infty,$$

where f is an arbitrary function in $L^1(X)$, D_α is an increasing family of regions in T depending on the positive real parameter α, and the vertical bars stand for Lebesgue measure.

For the definitions of sublinearity, strong and weak type properties of operators as used in the sequel, we refer to Zygmund [5, vol. 2, p. 111].

1. **A covering lemma of N. M. Rivière.** We will make use of the following

Lemma 1. *Let $(U_\alpha, \alpha > 0)$ be a one-parameter family of open sets in R^k, containing the origin and such that*

(1) $\alpha < \beta$ *implies* $U_\alpha \subset U_\beta$,

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(2) $|U_\alpha - U_\alpha + U_\alpha| \leq \text{const } |U_\alpha|$,

where $U_\alpha - U_\alpha + U_\alpha$ *denotes the set of all points which can be represented in the form* $u - v + w$ *with* u, v *and* w *in* U_α. *Under these conditions, if* $\alpha(x)$ *is a positive real-valued function defined on the compact set* K, *then there exists a finite set* $x_1, \cdots, x_n$ *of points in* K, *such that the sets* $x_i + U_{\alpha(x_i)}$ $(i = 1, \cdots, n)$ *are disjoint and*

$$|K| \leq C \sum_{i=1}^{n} |U_{\alpha(x_i)}|$$

where C *is the same constant that figures in condition* (2).

Proof. Since the sets $x + U_{\alpha(x)}$, $x \in K$, form an open covering of K, there exists a finite set $F \subset K$, such that the sets $y + U_{\alpha(y)}$, $y \in F$, also cover K. Choose x_1 in F so that $\alpha(x_1) = \max\{\alpha(y): y \in F\}$. If $x_1, \cdots, x_n$ have been chosen, we consider the set

$$A_n = K \cap \left(\bigcup_{i=1}^{n} (x_i + V_{\alpha(x_i)}) \right)^c$$

where $V_\alpha = U_\alpha - U_\alpha + U_\alpha$ and the upper c denotes complement. If $A_n = \emptyset$ we stop; otherwise we choose a point x_{n+1} in the set

$$B_n = \{y \in F: (y + U_{\alpha(y)}) \cap A_n \neq \emptyset\}$$

so that $\alpha(x_{n+1}) = \max\{\alpha(y): y \in B_n\}$. Obviously x_{n+1} is different from all the preceding points. Since the sequence of sets A_n is decreasing, so is the sequence of numbers $\alpha(x_n)$. The set F being finite, we must have $A_n = \emptyset$ for some n and the process stops there.

To prove the lemma, it will suffice to show that the sets $x_i + U_{\alpha(x_i)}$ $(i = 1, \cdots, n)$ are disjoint. Assume on the contrary that $(x_i + U_{\alpha(x_i)}) \cap (x_j + U_{\alpha(x_j)}) \neq \emptyset$, with $1 \leq i < j \leq n$, say. Then there are points u in $U_{\alpha(x_i)}$ and v in $U_{\alpha(x_j)}$ such that $x_i + u = x_j + v$. Since $x_j \in B_{j-1}$ there is a point w in $U_{\alpha(x_j)}$ such that the point $z = x_j + w$ is in A_{j-1}. Since $A_{j-1} \subset A_i$ and $U_{\alpha(x_j)} \subset U_{\alpha(x_i)}$ it follows that $z = x_i + u - v + w$ is in A_i and also in $x_i + V_{\alpha(x_i)}$, but this is a contradiction.

Corollary. *If the family of sets* $(U_\alpha, \alpha > 0)$ *satisfies the hypothesis of Lemma* 1 *and for every* $g(x)$ *in* $L^1(R^k)$ *we define the "maximal function"*

$$Sg(x) = \sup_{\alpha > 0} \frac{1}{|U_\alpha|} \int_{x + U_\alpha} |g(y)|\, dy,$$

then S *is of weak type* (1, 1).

In fact, let K be an arbitrary compact subset of $E = \{x: Sg(x) > \lambda\}$. For every x in K there is a positive number $\alpha(x)$ such that

$$\int_{x+U_{\alpha(x)}} |g(y)|\,dy > \lambda |U_{\alpha(x)}|.$$

By virtue of the preceding lemma there exists a finite sequence $x_1, \cdots, x_n$ of points in K such that the sets $x_i + U_{\alpha(x_i)}$ are disjoint and

$$|K| \le C \sum_{i=1}^{n} |U_{\alpha(x_i)}| \le \frac{C}{\lambda} \sum_{i=1}^{n} \int_{x_i + U_{\alpha(x_i)}} |g(y)|\,dy \le C\,\|g\|_1/\lambda.$$

Remarks. (i) Condition 2 of Lemma 1 is satisfied with the same constant C for all one-parameter families of open cells containing the origin (by a cell we mean the cartesian product of k linear intervals). As a consequence, the statement of the corollary remains true if the sets U_α are replaced by a one-parameter family $(P_\alpha, \alpha > 0)$ of closed cells containing the origin. To see this we construct for each α a decreasing sequence $U_\alpha^{(n)}$ $(n = 1, 2, \cdots)$ of open cells whose intersection is P_α. If we denote by $\tilde{S}$ the maximal operator associated with the cells P_α,

$$\tilde{S}g(x) = \sup_{\alpha > 0} \frac{1}{|P_\alpha|} \int_{x+P_\alpha} |g(y)|\,dy,$$

while S_n is the maximal operator associated with the family of open cells $U_\alpha^{(n)}$, $\alpha > 0$, then we have

$$\{x: \tilde{S}g(x) > \lambda\} \subset \liminf \{x: S_n g(x) > \lambda\}$$

from which we deduce

$$|\{x: \tilde{S}g(x) > \lambda\}| \le \liminf_{n\to\infty} |\{x: S_n g(x) > \lambda\}| \le C\,\|g\|_1/\lambda.$$

(ii) The statement of the corollary still remains true if the sets U_α are replaced by a one-parameter family of regions D_α containing the origin, provided that these regions satisfy the following hypothesis, which we shall assume to hold throughout the sequel.

(A) There exists a one-parameter family of closed cells P_α such that, for each α, $P_\alpha \supset D_\alpha$ and $|D_\alpha| \ge C\,|P_\alpha|$, where C is a constant. In fact, if S is the maximal operator associated with the family of regions $(D_\alpha, \alpha > 0)$, while $\tilde{S}$ is defined as in the previous remark, then $Sg(x) \le \text{const}\, \tilde{S}g(x)$.

2. **The maximal ergodic inequality.** Let $(D_\alpha, \alpha > 0)$ be an increasing family of regions in T, depending on the real parameter α and subject to hypothesis (A) of the preceding section. For each function $f(x)$ in $L^1(X)$ we define the maximal ergodic operator M by the formula

$$Mf(x) = \sup_{\alpha > 0} \frac{1}{|D_\alpha|} \int_{D_\alpha} |f(\theta_t x)|\, dt.$$

Following A. P. Calderón we prove

Theorem 1. *The maximal ergodic operator* M *is of weak type* (1, 1).

Proof. For any function $g(t)$ integrable over the parameter set T in R^k and for each positive integer N, we write

$$S_N g(t) = \sup_{\delta(D_\alpha) < N} \frac{1}{|D_\alpha|} \int_{D_\alpha} |g(t+s)|\, ds$$

if $|t| \leq N$; $S_N\, g(t) = 0$ otherwise, where $\delta(D_\alpha)$ denotes the diameter of D_α, while as before

$$Sg(t) = \sup_{\alpha > 0} \frac{1}{|D_\alpha|} \int_{D_\alpha} |g(t+s)|\, ds,$$

so that $S_N\, g(t) \leq Sg(t)$ and $\lim_{N\to\infty} S_N\, g(t) = Sg(t)$. From the preceding section we derive the inequalities

$$|\{t\colon S_N g(t) > \lambda\}| \leq |\{t\colon Sg(t) > \lambda\}| \leq \frac{C}{\lambda} \int_T |g(t)|\, dt.$$

Let us define the function $F(t, x) = f(\theta_t x)$ if $|t| \leq 2N$; $F(t, x) = 0$ otherwise. It follows from Fubini's theorem that $F(t, x)$ is an integrable function of t for almost all x. For a given $\lambda > 0$ consider the set E of all pairs (t, x) such that $S_N F(t, x) > \lambda$ and its sections $E_t = \{x\colon (t, x) \in E\}$, $E^x = \{t\colon (t, x) \in E\}$.

We observe that for $|t| \leq N$, $S_N F(t, x) = S_N F(0, \theta_t x)$, and therefore $E_t = \theta_t^{-1}(E_0)$ for $|t| \leq N$ while $E_t = \emptyset$ if $|t| > N$. If we denote by ρ the product of Lebesgue measure with the measure on X, then

$$\rho(E) = \int_T \operatorname{meas}(E_t)\, dt = \int_{|t| \leq N} \operatorname{meas}(E_t)\, dt = w_k N^k \operatorname{meas}(E_0),$$

where w_k is the measure of T intersected with the unit ball in R^k.

On the other hand,

$$\rho(E) = \int_X |E^x|\, dx \leq \int_X dx\, \frac{C}{\lambda} \int_{|t| \leq 2N} |f(\theta_t x)|\, dt$$

$$= \frac{C}{\lambda} \int_{|t| \leq 2N} dt \int_X |f(\theta_t x)|\, dx = \frac{C w_k (2N)^k}{\lambda} \|f\|_1.$$

Therefore

$$\operatorname{meas}(E_0) \leq 2^k C \, \|f\|_1/\lambda$$

and Theorem 1 follows from the last inequality by letting $N \to \infty$.

Corollary. *Since M does not increase the L^∞-norm of any function, it follows that M is of strong type (p, p) for any $p > 1$.*

3. **A pointwise ergodic theorem.** In this section we shall assume that in addition to hypothesis (A) of the first section, the family of sets D_α, $\alpha > 0$, and the semigroup $(\theta_t, t \in T)$ satisfy the following assumptions:

(B) For any t in R^k

$$\lim_{\alpha \to \infty} \frac{|(t + D_\alpha) \, \Delta \, D_\alpha|}{|D_\alpha|} = 0,$$

where Δ denotes the symmetric difference.

(C) If $B_{K,\alpha}$ denotes the set of all points t in R^k such that $t + D_\alpha$ intersects the compact set K without covering it, then

$$\lim_{\alpha \to \infty} \frac{|B_{K,\alpha}|}{|D_\alpha|} = 0.$$

(D) For any $f(x)$ in $L^p(X)$ and any $g(x)$ in $L^q(X)$, where $1 < p < \infty$ and $1/p + 1/q = 1$,

$$\int_X f(\theta_t x) g(x) \, dx$$

is a continuous function of t.

We can now state the following:

Theorem 2. *If the family of regions D_α and the semigroup θ_t satisfy the preceding conditions, then for any f in $L^1(X)$ the averages*

$$A_\alpha f(x) = \frac{1}{|D_\alpha|} \int_{D_\alpha} f(\theta_t x) \, dt$$

converge almost everywhere in X as $\alpha \to \infty$.

Proof. Let us consider the set of all functions $h(x)$ which can be represented in the form

$$h(x) = \int_T f(\theta_t x) \phi(t) \, dt \tag{1}$$

where f is a bounded function having support of finite measure and $\phi(t)$ is infinitely differentiable with compact support in T and vanishing integral. For any function h of this form we have

$$A_\alpha h(x) = \frac{1}{|D_\alpha|} \int_{D_\alpha} h(\theta_u x)\, du = \frac{1}{|D_\alpha|} \int_{D_\alpha} du \int_T f(\theta_{s+u} x)\, \phi(s)\, ds$$

$$= \int_T dt\, f(\theta_t x)\, \frac{1}{|D_\alpha|} \int_{D_\alpha} \phi(t-u)\, du = \int_T dt\, f(\theta_t x)\, \frac{1}{|D_\alpha|} \int_{t-D_\alpha} \phi(s)\, ds.$$

If we call B_α the set of points t such that $t - D_\alpha$ intersects the support of ϕ without covering it, we can estimate the L^1-norm of the expression depending on α in the last integral as follows:

$$\frac{1}{|D_\alpha|} \int_T dt \left| \int_{t-D_\alpha} \phi(s)\, ds \right| \leq \|\phi\|_1 \frac{|B_\alpha|}{|D_\alpha|},$$

which tends to zero as $\alpha \to \infty$ by virtue of (C). Since $f(\theta_t x)$ is a bounded function of t for almost all x, we see at once that $A_\alpha h(x)$ tends to zero for almost all x as $\alpha \to \infty$.

We will say that a function $l(x)$ in $L^p(X)$ is invariant if for every t, $l(\theta_t x) = l(x)$ for almost all x. If $l(x)$ is an invariant function, for almost all x we have $l(\theta_t x) = l(x)$ for almost all t. Therefore

$$\frac{1}{|D_\alpha|} \int_{D_\alpha} l(\theta_t x)\, dt = l(x)$$

for all α almost everywhere in X.

We conclude that the averages $A_\alpha f(x)$ converge almost everywhere if f is in the linear span of the functions h and l. Theorem 2 will follow by a standard argument if we prove that this linear span is dense in $L^p(X)$. In fact, it is enough to recall that M is of weak type (p, p) for $1 \leq p < \infty$. For this purpose, let us assume that a certain function $g_0(x)$ in the dual space $L^q(X)$, where $p > 1$ and $1/q = 1 - 1/p$, is orthogonal to all functions h and l. Therefore

$$0 = \int_X h(x) g_0(x)\, dx = \int_X dx\, g_0(x) \int_T f(\theta_t x)\, \phi(t)\, dt$$

$$= \int_T dt\, \phi(t) \int_X f(\theta_t x) g_0(x)\, dx,$$

for any infinitely differentiable function $\phi(t)$ with compact support in T and vanishing integral. This implies that the integral

$$\int_X f(\theta_t x) g_0(x)\, dx \tag{2}$$

is equal to a certain constant a for almost all t, in fact, for all t by virtue of (D). In order to prove that this constant a is actually equal to zero we consider the sequence

$$f_n(x) = \frac{1}{|D_n|} \int_{D_n} f(\theta_s x)\, ds.$$

This sequence is bounded in L^p and consequently, it contains a subsequence which converges weakly to a certain function $l(x)$. For simplicity of notation we assume that the whole sequence f_n converges weakly to l. Then for every t the sequence of functions $f_n(\theta_t x)$ converges weakly to $l(\theta_t x)$. It will follow that the limit function $l(x)$ is an invariant function if we show that the difference $f_n(\theta_t x) - f_n(x)$ converges weakly to zero.

Let $g(x)$ be any function in $L^q(X)$. Then

$$\left| \int_X (f_n(\theta_t x) - f_n(x))\, g(x)\, dx \right|$$

$$= \left| \frac{1}{|D_n|} \int_X dx\, g(x) \left(\int_{D_n} f(\theta_{t+s} x)\, ds - \int_{D_n} f(\theta_s x)\, ds \right) \right|$$

$$= \frac{1}{|D_n|} \left| \left(\int_{t+D_n} - \int_{D_n} \right) ds \int_X f(\theta_s x) g(x)\, dx \right|$$

$$\leq \|f\|_p \, \|g\|_q \, |(t + D_n) \,\Delta\, D_n| / |D_n|,$$

and the last expression tends to zero as $n \to \infty$, by virtue of (B). Since $g_0(x)$ is orthogonal to all invariant functions the sequence

$$\int_X f_n(x) g_0(x)\, dx \tag{3}$$

tends to zero as $n \to \infty$, A simple computation shows that each member of this sequence is equal to a, so that $a = 0$. Making $t = 0$ in (2) we see that $\int_X f(x)\, g_0(x) dx = 0$ for any bounded function $f(x)$ with support of finite measure. Then $g_0(x) = 0$ for almost all x, which proves the density in $L^p(X)$ of the linear span of the functions h and l and thus concludes the proof of Theorem 2.

One final remark is in order with regard to the assumptions we made on the family of regions D_α. Let us denote by K_α^u the set of all points in R^k whose distance to the complement of D_α is not less than the positive number u. We wish to show that the family of regions D_α satisfies condition (B) provided that the following holds.

(B′) $$\lim_{\alpha \to \infty} \frac{|K_\alpha^u|}{|D_\alpha|} = 1 \quad \text{for every positive number } u.$$

The proof is simple. Given t in R^k let us choose a number u which exceeds the length of the vector t. Then K^u_α is contained both in D_α and $t + D_\alpha$. If we denote by $A\setminus B$ the points in A not in B, we have

$$\frac{|D_\alpha \Delta (t + D_\alpha)|}{|D_\alpha|} = \frac{|D_\alpha \setminus (t + D_\alpha)| + |(t + D_\alpha)\setminus D_\alpha|}{|D_\alpha|}$$

$$\leq \frac{|D_\alpha \setminus K^u_\alpha| + |(t + D_\alpha)\setminus K^u_\alpha|}{|D_\alpha|} = 2\left(1 - \frac{|K^u_\alpha|}{|D_\alpha|}\right).$$

Condition (B$'$) is readily verified in the case of most of the familiar figures of geometry. In fact, Professor N. M. Rivière has proved that (B$'$) holds if the regions D_α are convex.

Let now D^u_α be the set of points in R^k whose distance to D_α does not exceed u. We claim that both hypotheses (B) and (C) can be replaced by a single one, namely

(S) $$\lim_{\alpha \to \infty} \frac{|K^u_\alpha|}{|D^u_\alpha|} = 1 \quad \text{for every positive number } u.$$

On the one hand (S) clearly implies (B$'$). Since $|B_{K,\alpha}| \leq |D^u_\alpha| - |K^u_\alpha|$ with $u =$ diameter of K, it also implies (C).

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