k-CONGRUENCE ORDERS FOR \( E_k \)

BY

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ABSTRACT. This paper generalizes the notion of congruence order for metric spaces to k-metric (k-dimensional metric) spaces. The k-congruence order of \( E_k \) with respect to the class of oriented semi k-metric spaces is determined. An example shows that this result is sharp.

Introduction. The congruence order and 'best' congruence indices of classes of semi-metric spaces is well known [1, pp. 93–118]. This paper deals with the analogous problem for oriented semi k-metric spaces with respect to \( E_k \) (considered as an oriented k-metric space).

A k-metric space consists of a set \( S \) together with a real-valued function \( d_k \) defined for \( k + 1 \)-tuples of \( S \) and satisfying

1. if \( a_1, a_2 \) are 2 distinct points of \( S \), there exist points \( a_3, \ldots, a_{k+1} \) such that \( d_k(a_1, \ldots, a_k, a_{k+1}) \neq 0 \), and

2. for each \( k + 2 \) points of \( S \) there exist \( k + 2 \) points of \( E_{k+1} \), the \( k + 1 \)-dimensional Euclidean space, and a 1-1 mapping between the two \( k + 2 \)-tuples such that the values of \( d_k \) and the \( k \)-dimensional hypervolume (unsigned) are the same for corresponding \( k + 1 \)-tuples.

This reduces to a metric space for \( k = 1 \), \( k = 2 \) gives a generalized area and other values give generalized 'volume' spaces. The Euclidean space \( E_n \) is a k-metric space for every \( k \leq n \) if we take the k-dimensional volume for the k-metric.

A 1-1 onto mapping between \( S \) and \( S' \), subsets of k-metric spaces, is called a k-congruence if it preserves the k-metric. \( S \) and \( S' \) are then said to be k-congruent and we write \( S \cong S' \). With this definition condition (2) above could be changed to: each \( k + 2 \) points are k-congruent with \( k + 2 \) points of \( E_{k+1} \).

If condition (2) is reduced to the requirement that every \( k + 1 \)-tuple be k-congruent with a \( k + 1 \)-tuple of \( E_k \), the resulting space is called a semi k-metric space. The spaces are said to be oriented if each ordered \( k + 1 \)-tuple is attached a sign according to some rule. The usual orientation for \( E_k \) is given by the sign of the determinant which gives the hypervolume of each \( k + 1 \)-tuple. If a
$k$-congruence between two sets in oriented $k$-metric spaces either preserves all orientations or reverses all orientations, it is said to be positive (denoted by $k_+\text{-congruence and } \equiv$).

The results mentioned earlier for metric spaces depended heavily on the fact that an independent $n+1$-tuple in $E_n$ is a complete metric basis for $E_n$ (i.e., any point of $E_n$ can be uniquely determined by giving its distances from points in an independent $n+1$-tuple and congruences between subsets of $E_n$ can be extended to motions if the subsets contain independent $n+1$-tuples). Considering $E_k$ as a $k$-metric space the corresponding statement is true for an independent $k+1$-tuple only if the $k$-congruences are all positive. If $a, b, c$ and $d$ are the vertices of a rectangle in $E_2$, then $a, b, c, d$ are $2$-congruent with $a, b, d, c$, but the correspondence cannot be extended to all of $E_2$ (where would the point of intersection of the diagonals go?). The difficulty in this example lies in the fact that the $2$-congruence between $bcd$ and $bdc$ is not positive, while that between $abc$ and $abd$ is. The correspondence between $abc$ and $abd$ can be extended positively to $abcd$ by mapping $d$ into the point which is the reflection of $c$ through $d$. The latter $2$-congruence can be extended to a $2$-motion of $E_2$ (a $k$-motion of a $k$-metric space $M$ is a $k$-congruence of $M$ with itself).

**Bases in $E_k$.**

**Definition.** If each point of a $k$-metric ($k_+$-metric) space $M$ is uniquely determined when the values of the $k$-metric ($k_+$-metric) for each ordered $k+1$-tuple containing that point and $k$ points of some subset $B$ are given, then $B$ is called a $k$-metric ($k_+$-metric) basis for $M$.

**Definition.** A $k$-metric ($k_+$-metric) basis $B$ is said to be complete if, whenever $B \approx B^*$ ($B \equiv B^*$) and $B^* \subseteq S^* \subseteq M$, there exists a subset $S$ of $M$ containing $B$ and $k$-congruent ($k_+$-congruent) with $S^*$ and this correspondence is an extension of that between $B$ and $B^*$.

**Definition.** A $k+1$-tuple in $E_k$ is called independent if it is not contained in a lower dimensional subspace.

If $(x_{i_1}, \ldots, x_{i_k})$, $i = 0, 1, \ldots, k$, are the rectangular representations of $k+1$ points of $E_k$, then $V_k$, the $k_+$-metric (signed $k$-dimensional volume) of the ordered $k+1$-tuple, is given by

\[
(k!) V_k(x_0, \ldots, x_k) = \begin{vmatrix}
  x_{01} & x_{02} & \cdots & x_{0k} & 1 \\
  x_{11} & & & & 1 \\
  \vdots & & & & \vdots \\
  x_{k1} & x_{k2} & \cdots & x_{kk} & 1
\end{vmatrix}
\]

In order to facilitate working with these determinants, we will adopt the convention that if $x$ is a point in $E_k$, $x'$ is the point in $E_{k+1}$ whose first $k$ coordinates
are the same as \( x \) and whose \( k + 1 \)st coordinate is 1. The value of the \( k + 1 \)-
metric above would then be \( (1/k!) \det(x'_0, x'_1, \ldots, x'_k) \).

**Theorem 1.** If \( p_0, p_1, \ldots, p_k \) and \( q_0, q_1, \ldots, q_k \) are two independent \( k + 1 \)-tuples in \( E_k \) with the same \( k \)-dimensional volume and \( x \) is a point of \( E_k \), then there is one and only one \( y \) in \( E_k \) such that \( p_0, p_1, \ldots, p_k, x = q_0, q_1, \ldots, q_k, y \).

**Proof.** Without loss of generality we may assume the \( p \)'s to be in an \( E_k \) such that \( p_0 = (0, 0, \ldots, 0) \), \( p_i = (a_{1i}, a_{2i}, \ldots, a_{ii}, 0, \ldots, 0) \) and the \( q \)'s in an \( E_k \) with \( q_0 = (0, \ldots, 0) \), \( q_i = (b_{1i}, \ldots, b_{ii}, 0, \ldots, 0) \). For definiteness we assume that \( p_0, \ldots, p_k \) and \( q_0, \ldots, q_k \) have the same orientation.

Let \( x \) and \( y \) be given by \( (x'_0, x'_1, \ldots, x'_k) \) and \( (y_0, y_1, \ldots, y_k) \). The \( k + 1 \)-congruence requires that the \( k + 1 \) equations

\[
det(p'_0, p'_1, \ldots, p'_{i-1}, x', p'_{i+1}, \ldots, p'_k) = det(q'_0, q'_1, \ldots, q'_{i-1}, y', q'_{i+1}, \ldots, q'_k), \quad i = 0, \ldots, k,
\]

be satisfied. The theorem will be proved by showing that there is a unique solution to the system of equations obtained by taking \( i \geq 1 \) and that it satisfies the equation for \( i = 0 \). By expanding the determinants for \( i \geq 1 \) we get \( k \) linear equations in the \( k \) unknowns \( y_0, \ldots, y_k \). The \( i \)th equation so obtained is of the form

\[
\sum_{j=i}^k C_{ij} y_j = \sum_{j=i}^k D_{ij} x_j, \quad \text{where } C_{ij} = \frac{\prod_{j=1}^k b_{jj}}{b_{ii}}
\]

and is, therefore, not zero. The coefficient vectors in the system are then in echelon form. It follows that there is a unique solution \( y \). In order to show that the solution obtained is ‘compatible’ with the equation for \( i = 0 \), we examine the determinants a little more closely. From

\[
det(p'_0, \ldots, p'_{i-1}, x', \ldots, p'_k) = det(q'_0, \ldots, q'_{i-1}, y', \ldots, q'_k)
\]

follows

\[
\begin{array}{cccc|cccc}
   a_{11} & 0 & \cdots & 0 & b_{11} & 0 & \cdots & 0 \\
   \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
   \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
   a_{i-11} & 0 & \cdots & 0 & b_{i-11} & \cdots & 0 & \\
   x_1 & \cdots & x_k & y_1 & \cdots & y_k \\
   \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
   a_{kk} & \cdots & a_{kk} & b_{kk} & \cdots & b_{kk} \\
\end{array}
\]
If the $x$'s and $y$'s are moved to the top row and the preceding rows down one, the determinants are the same as the minors of the elements in the $i + 1$st row and last column of the determinants for $i = 0$, $\det(x', p'_1, \ldots, p'_k)$ and $\det(y', p'_1, \ldots, p'_k)$. The minors of the elements in the 1st row and last column are equal because they are the volumes of the $p$'s and the $q$'s. Pairwise equality of these minors implies equality of the determinants when we recall that the elements in the last column are all 1's.

Corollary 1.1. An independent $k + 1$-tuple in $E_k$ is a $k_+$-metric basis for $E_k$.

Proof. Let $p'_i = q'_i$ for $i = 0, \ldots, k$.

Corollary 1.2. There is one and only one $k_+$-motion of $E_k$ that takes $p_0, \ldots, p_k$ onto $q_0, \ldots, q_k$, where $p_i$ and $q_i$ are the points in Theorem 1.

Proof. If there is a motion it must take each point onto that point determined in the proof of the theorem, so it is unique. We must show that the mapping which takes each $x$ onto the $y$ such that $p_0, \ldots, p_k, x \equiv q_0, \ldots, q_k, y$ is a $k_+$-motion. That the mapping is onto $E_k$ follows from the fact that the row space in the system of equations is of dimension $k$. It remains to show that the $k_+$-metric is preserved. Let $x^0, x^1, \ldots, x^k$ be $k + 1$ points of $E_k$ that map onto $y^0, y^1, \ldots, y^k$, respectively. From (4) we see that $y'_i = \Sigma^k_{j=1} B_{ji} x'_j$ and these are the elements in the $i$th column of $\det(y^0', \ldots, y^k')$. By elementary column operations the elements in the $i$th column may be reduced to $B_{i} x'_i$. It follows that $\det(y^0', \ldots, y^k') = A \det(x^0', \ldots, x^k')$ for some constant $A$. From $\det(p'_0, \ldots, p'_k) = \det(q'_0, \ldots, q'_k)$, we have $A = 1$ and the mapping is a $k_+$-motion.

Corollary 1.3. An independent $k + 1$-tuple in $E_k$ is a complete $k_+$-metric basis for $E_k$.

Proof. Let $l$ and $l^*$ be independent $k + 1$-tuples in $E_k$, $l^* \subseteq S^* \subseteq E_k$ for some $S^*$ and $l \equiv l^*$. Since $l \equiv l^*$, there is just one $k_+$-motion $\phi$ which will take $l$ onto its image $l^*$. The proof is completed by letting $S$ be the set of pre-images of $S^*$ under $\phi$.

The need for orientation can be seen by noting that without it, equations (3) would have to be $\det(p'_0, \ldots) = \pm \det(q'_0, \ldots)$ and there would not be unique solutions for all $x$.

The analogue in an arbitrary $k$-metric space of an independent $k + 1$-tuple in $E_k$ is a $k + 1$-tuple on which the $k$-metric function is not zero. Such a $k + 1$-tuple is called nontrivial.
k-congruence order of $E_k$.

Definition. A semi $k$-metric space is said to be $k$-congruently imbeddable ($k_+$-congruently imbeddable) in a semi $k$-metric space $M$ if $S$ is $k$-congruent ($k_+$-congruent) with a subset of $M$.

Definition. A semi $k$-metric space $M$ has $k$-congruence indices ($k_+$-congruence indices) $(n, q)$ with respect to a class $\{S\}$ of spaces provided any space $S$ of $\{S\}$, containing more than $n + q$ pairwise distinct points, is $k$-congruently imbeddable ($k_+$-congruently imbeddable) in $M$ whenever each $n$ of its points has that property.

Definition. If a space $M$ has $k$-congruence indices ($k_+$-congruence indices) $(n, 0)$ with respect to a class $\{S\}$ of spaces, $M$ is said to have $k$-congruence order ($k_+$-congruence order) $n$ with respect to that class.

Two $k + 1$-tuples in a semi $k$-metric space are said to $k$-touch if they have $k$ points in common.

Definition. Two nontrivial $k + 1$-tuples, $P$ and $Q$, of points in a semi $k$-metric space are chain connected if there exists a finite sequence $X_i$ ($i = 0, \ldots, n$) of nontrivial $k + 1$-tuples of points of the semi $k$-metric space with each $X_i$ $k$-touching $X_{i+1}$, $X_0 = P$ and $X_n = Q$.

A semi $k$-metric space has property $C_k$ if each pair of its nontrivial $k + 1$-tuples is chain connected.

Lemma 1. $E_n$ has property $C_k$ for each $k \leq n$.

Proof. The theorem is obvious for $k = 1$. We assume it is true for $k - 1$ and proceed by induction. It is sufficient to show that if $x_0, \ldots, x_i, y_0, \ldots, y_j$ ($i + j = k$) are $k + 2$ points in $E_n$, such that the first $k + 1$ of them are independent and that the $y$'s are independent, then for a suitable ordering of the $x$'s, $x_1, \ldots, x_i, y_0, \ldots, y_j$ is an independent $k + 1$-tuple.

Assume $x_1, \ldots, x_i, y_0, \ldots, y_j$ is not an independent $k + 1$-tuple. The points satisfy the hypothesis of the lemma for $k - 1$, so there is an $x$ which may be deleted to leave an independent $k$-tuple. Let it be $x_1$. Then $x_2, \ldots, x_i, y_0, \ldots, y_j$ generate an $E'_{k-1}$. Since adding $x_1$ does not yield an independent $k + 1$-tuple, $x_1$ must be in $E'_{k-1}$. If $x_0, x_2, \ldots, x_i, y_0, \ldots, y_j$ is independent, the theorem is proved; if not, then $x_0$ is in $E'_{k-1}$. But $x_0$ and $x_1$ both in $E'_{k-1}$ contradicts the independence of $x_0, \ldots, x_i, y_0, \ldots, y_{j-1}$.

Let $\mathcal{K}$ be the class of oriented semi $k$-metric spaces which have property $C_k$. We may now state the following:

Theorem 2. $E_k$ has $k_+$-congruence order $k + 3$ with respect to the class $\mathcal{K}$.

Proof. Let $p_0, p_1, \ldots, p_k$ be a nontrivial $k + 1$-tuple of a semi $k_+$-metric space $M$ in $\mathcal{K}$, every $k + 3$ points of which are $k_+$-congruent with $k + 3$ points.
of $E_k$. Let $p'_0, p'_1, \ldots, p'_k$ be a $k + 1$-tuple of $E_k$ with $p'_0, p'_1, \ldots, p'_k = p'_0$, $p'_1, \ldots, p'_k$. If $x$ is a point in $M$, then there exist points $p''_0, p''_1, \ldots, p''_k, x$ in $E_k$ such that $p'_0, p'_1, \ldots, p'_k, x = p''_0, p''_1, \ldots, p''_k, x$. Since $V_k(p''_0, \ldots, p''_k) = V_k(p'_0, \ldots, p'_k)$, we know that there is a $k_+$-motion of $E_k$, call it $g$, with $g(p''_i) = p'_i$ ($i = 0, 1, \ldots, k$). Let $x'$ denote $g(x'')$. By repeating the procedure for each $x$ in $M$ we define a mapping of $M$ into a subset of $E_k$. In order to show that the mapping is a $k_+$-congruence we let $X_0, X_1, \ldots, X_n$ be a chain connecting $P = [p_0, \ldots, p_k]$ and the nontrivial $k + 1$-tuple $[x_0, \ldots, x_k]$, then show that the mapping may be carried across this chain. Let $X_1 = [u, p_0, \ldots, p_k]$. For each $x$ in $M$, there exist points $u^*, p_0^*, \ldots, p_k^*, x^*$ in $E_k$ such that $u, p_0, \ldots, p_k, x = u^*, p_0^*, \ldots, p_k^*, x^*$. Since $V_k(u^*, p_0^*, \ldots, p_k^*) = V_k(u', p_1', \ldots, p_k')$, there exist $p_0^{**}$ and $x^{**}$ with $u', p_1', \ldots, p_k', p_0^{**}, x^{**} = u^*, p_1^*, \ldots, p_k^*, p_0^*$, by Corollary 1.3. From this follows $u, p_1, \ldots, p_k, p_0 = u', p_1', \ldots, p_k', p_0^{**}$, which, together with $u, p_1, \ldots, p_k, p_0 = u', p_1', \ldots, p_k', p_0^{**}$, and transitivity of $k_+$-congruence, yields $p_0 = p_0^{**}$ by Theorem 1. That $x = x^{**}$ will then follow from $p'_0, \ldots, p'_k, x' = p''_0, \ldots, p''_k, x^{**}$ in a similar manner. Thus the mapping determined by the nontrivial $k + 1$-tuple $u, p_1, \ldots, p_k$ coincides with a mapping determined by $p_0, \ldots, p_k$. The same argument may be employed to establish that if the mapping determined by $X_{i-1}$ takes each $x$ onto $x'$, then the one determined by $X_i$ may be made to do that also. By induction, the mapping can be carried across to $X_n$, from which we have $V_k(x_0, \ldots, x_k) = d_k(x_0, \ldots, x_k)$, the value of the $k_+$-metric on $X_n$.

We still need to show that the mapping is a $k_+$-congruence for trivial $k + 1$-tuples. We assume that $x_0^*, \ldots, x_k^*$ is nontrivial and $x_0^*, \ldots, x_k^*$ is trivial, for otherwise the theorem follows.

By Lemma 1 there is a chain connecting $p_0', \ldots, p_k'$ and $x_0', \ldots, x_k'$. We note that this chain contains only points from the two $k + 1$-tuples. This chain and the proof above may be used to construct a chain in $M$ because the $k_+$-congruence of each $X_i$ with its counterpart $X_i'$ in $E_k$ was established as soon as the mapping was extended to $X_{i-1}$. Consequently, the nontriviality of $X_i'$ establishes that of the $X_i$ and the chain in $M$ is just the 'parallel' to the one in $E_k$.

Since $X_i = X_i'$ for each $X_i'$ in the chain and $V_k(x_0, \ldots, x_k) \neq 0$, then $d_k(x_0, \ldots, x_k) \neq 0$, a contradiction.

In the last part of the proof of the theorem we did not need the fact that $M$ was chain connected. It is natural to ask if the theorem is true for a wider class of spaces than $K$. It turns out that, if $k$ is less than 4, then we need not assume that $M$ is chain connected, and may simply drop that requirement. However, for $k > 3$, we must assume that more points are $k_+$-congruently contained in $E_k$ in order to 'bridge' nontrivial $k + 1$-tuples, if we do not have chain connectedness. The following theorem, together with the fact that $k + 3 \geq 2k$ if $k \leq 3$, indicates the reason for this behavior.
Theorem 3. If every $2k$ points of a $k$-metric space $M$ are $k$-congruently contained in $E_k$, then $M$ is chain connected.

Proof. Let $x_0, \ldots, x_k$ and $p_0, \ldots, p_k$ be two nontrivial $k + 1$-tuples with no points in common. The imbeddability of the $k + 2$-tuple $p_0, \ldots, p_k, x_0$ in $E_{k+1}$ implies that, for some $j$, $p_0, \ldots, p_{j-1}, x_0, p_{j+1}, \ldots, p_k$ is nontrivial. We suppose, without loss of generality, that the $k + 1$-tuple $p_0, \ldots, p_{k-1}, x_0$ is nontrivial. If there exist $i, j, i > 0$ and $j < k$, such that $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k, p_j$ is nontrivial, then the union of the $k + 1$-tuples $x_0, \ldots, x_k, p_j$ and $p_0, \ldots, p_{k-1}, x_0$ contains $2k$ points. Since the $2k$ points are $k$-congruently contained in $E_k$, an application of Lemma 1 completes the proof in that case. If no such $i$ and $j$ exist, then $d_{2k}(p_j, x_1, \ldots, x_k) = d_{2k}(x_0, \ldots, x_k)$ for every $j < k$ so the $k + 1$-tuple $p_j, x_1, \ldots, x_k$ is nontrivial. One of the $k + 1$-tuples $p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k, x_j$ is nontrivial for $j < k$. We may combine that $k + 1$-tuple with $p_n, x_1, \ldots, x_k$ ($n \neq j$) to get $2k$ points and then proceed as above.

An easy corollary is

Corollary 3.1. The $k_+$-congruence order of $E_k$ with respect to the class $S$ of all oriented semi $k$-metric spaces is $\max \{k + 3, 2k\}$.

We note that for $k = 1, 2$ or $3$, Corollary 3.1 is a sharper result than Theorem 2.

Examples. The following examples show that the previous results are sharp.

Example 1 is a 4-metric space, every 7 points of which are 4-congruently contained in $E_4$, but which is not chain connected. Therefore $2k$ may not be replaced by $2k - 1$ in Theorem 3.

$k_+$-congruence order $m$ is the same as $k_+$-congruence indices $(m, 0)$. $E_1$ has congruence order 4, but also has congruence indices $(3, 1)$, see [1, p. 118]. Example 2 shows that, for $k \geq 2$, $E_k$ does not have $k_+$-congruence indices $(k + 2, n)$ for any positive integer $n$. Therefore, for $k \geq 2$, Theorem 2 and Corollary 3.1 are sharp whether they are stated in terms of $k_+$-congruence order or $k_+$-congruence indices.

Example 1. Let $S = \{x_0, x_1, x_2, x_3, x_4, p_0, p_1, p_2, p_3, p_4\}$ and define $d_4(X) = 0$ if $X$ is any 5-tuple containing 3 $p$'s and 2 $x$'s or 3 $x$'s and 2 $p$'s; $d_4(X) = 1$ for all other 5-tuples of distinct points of $S$. Let $Y$ be a subset of $S$ containing 7 points. Since $d_4$ is symmetric with respect to $p$'s and $x$'s and independent of indices, we may assume $Y = \{p_0, p_1, p_2, p_3, x_0, x_1, x_2\}$ or $\{p_0, p_1, p_2, p_3, p_4, x_0, x_1\}$. In the first case let $f(p_0) = (0, 0, 0, 0), f(p_1) = (1, 0, 0, 0), f(p_2) = (0, 2, 0, 0), f(p_3) = (0, 0, 3, 0), f(x_0) = (0, 0, 0, 4) = f(x_1) = f(x_2)$. The second case is the same, except $f(p_4) = (0, 0, 0, 4)$ and $f(x_0) = (1, -2, 3, -4) = f(x_1)$. In both cases $f$ is a 4-congruence.
Example 2. We construct the example first for $k$ even.
Let $k$ be an even positive integer. Let $M_k = (S, d_k)$, where $S = \{a_i\}$ is any countable set and $d_k$ is defined as follows:

$$d_k(a_{i_0}, a_{i_1}, \ldots, a_{i_k}) = \frac{1}{k!}$$ if $\{i_j\}$ is increasing,

$$d_k(\pi(a_{i_0}, \ldots, a_{i_k})) = -d_k(a_{i_0}, \ldots, a_{i_k})$$ if $\pi$ is a transposition, and

$$d_k(a_{i_0}, a_{i_0}, a_{i_1}, \ldots) = 0.$$

For any increasing indices $i_0, i_1, \ldots, i_{k+1}$ define $f(a_{i_j}) = P_j$, where the $P_j$'s are points in $E_k$ with $P_0 = (0, 0, \ldots, 0)$, $P_1 = (1, 0, \ldots, 0)$, $P_2 = (0, 1, 0, \ldots, 0)$, $P_k = (0, 0, \ldots, 0, 1)$ and $P_{k+1} = (1, -1, 1, -1, \ldots, 1, -1, 1, 1)$. Verification that $f$ is a $k$-congruence mapping the $k+2$-tuple $\{a_{i_j}\}$ onto $E_k$ is straightforward and may be done by induction.

Let $M_{k+1} = (S^*, d_{k+1})$ where $S^* = S \cup \{x\}$ ($x \neq a_i$ for any $i$) and

$$d_{k+1}(x, a_{i_0}, a_{i_1}, \ldots, a_{i_k}) = d_k(a_{i_0}, \ldots, a_{i_k}),$$

$$d_{k+1}(a_{i_0}, a_{i_1}, \ldots, a_{i_{k+1}}) = 0.$$

Each $k+3$-tuple of $M_{k+1}$ may be imbedded in $E_{k+1}$ by mapping the $a_i$'s into a $k$-flat in the manner described above and by mapping $x$ to an appropriate point outside of the $k$-flat.

That the spaces $M_k$ and $M_{k+1}$ are not congruently contained in $E_k$ and $E_{k+1}$ follows from the fact that each contains an independent 'tuple' which would uniquely determine the image of every other point. In fact, all other points would have to map onto the same point in $E_k$ or $E_{k+1}$.

BIBLIOGRAPHY


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