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PRIMARY IDEALS IN RINGS OF ANALYTIC FUNCTIONS[1]

BY

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ABSTRACT. Let A be the ring of all analytic functions on a connected, noncompact Riemann surface. We use the valuation theory of the ring A as developed by N. L. Alling to analyze the structure of the primary ideals of A. We characterize the upper and lower primary ideals of A and prove that every nonprime primary ideal of A is either an upper or a lower primary ideal. In addition we give some necessary and sufficient conditions for certain ideals of A to be intersections of primary ideals.

Introduction. Let A be the ring of all analytic functions on a connected, noncompact Riemann surface. This paper is an analysis of the primary ideals of A and an investigation of the ideals of A that are intersections of prime or of primary ideals.

In [$\mathbf{H}_1$] and [$\mathbf{H}_2$] the maximal ideals and the prime ideals of the ring of entire functions were analyzed by M. Henriksen. In [$\mathbf{A}_1$] N. L. Alling used H. Florack's generalization [**F**] of the Weierstrass and Mittag-Leffler theorems to extend Henriksen's analysis of maximal and prime ideals to the ring A of all functions analytic on an arbitrary connected, noncompact Riemann surface. In §2 we analyze the structure of the primary ideals of A using the valuation theory developed by Alling in [$\mathbf{A}_1$] and [$\mathbf{A}_2$]. We find that the structure of the primary ideals of A is strikingly similar to the structure of the primary ideals of the ring C of all real valued continuous functions on a completely regular topological space. (See [**K**] for an analysis of the primary ideals of C.) In particular, we show in Theorem 2.2 that every nonprime primary ideal of A is either an upper or a lower primary ideal.

In §3 we give some results concerning intersections of prime and primary ideals of A. This section attacks the general problem of finding conditions (necessary, sufficient, and where possible both necessary and sufficient) under

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which ideals of A will be intersections of prime or primary ideals. These results parallel the results of [W] concerning intersections of prime and primary ideals in the ring C. We focus our attention on ideals which we call "deep": an ideal I of A is *deep* if for every maximal ideal M containing I there exists a nonmaximal prime ideal P such that $I \subset P \subset M$. We show that if I is a deep ideal of A, then I is an intersection of prime ideals iff $I = I^2$ (Theorem 3.1). Concerning intersections of primary ideals, we find that the ideals $I \cdot I^{1/2}$ and $I : I^{1/2}$ are always intersections of primary ideals (Theorem 3.6) and that the condition $I^2 = I \cdot (I : I^{1/2})$ comes very close to being a necessary and sufficient condition for I to be an intersection of primary ideals (Theorem 3.12).

1. **Background.** Our terminology and notation will, with only a few exceptions, be that of [GJ]. The symbol $\subset$ will denote set inclusion, while $<$ will denote proper inclusion. The term "ring" unmodified will mean a commutative ring with identity. If I is an ideal of a ring, we will denote the radical of I by $I^{1/2}$.

Let A be the ring of all analytic functions on a connected, noncompact Riemann surface X. Before beginning our analysis of the primary ideals of A, we describe the maximal ideals of A, outline the valuation theory developed by Alling in [A_1] and [A_2], and discuss the "contracted ideals" and the prime ideals of A. The material of this section is, for the most part, either explicit or implicit in the papers of Henriksen [H_1], [H_2] and Alling [A_1], [A_2].

Suppose that M is a maximal ideal of A. Let $f \in M - \{0\}$, and set $D = Z(f)$. D is a nonempty, closed, discrete subset of X, either finite or countably infinite. Let $\mu = \{Z \cap D : Z \in Z[M]\}$. μ is an ultrafilter on the set D, and $M = \{f \in A : Z(f) \cap D \in \mu\}$. If μ is a fixed ultrafilter on D, then there exists $x \in D$ such that $\mu = \{S \subset D : x \in S\}$. In this case $M = \{f \in A : f(x) = 0\}$; M is called a *fixed maximal ideal* and is denoted by M_x. If μ is a free ultrafilter on D, then M is called a *free maximal ideal.*

Now we outline the valuation theory that we will use to analyze the primary ideals of A. (We do not include here a treatment of elementary valuation theory; for this we refer the reader to [A_2, §3], [S, Chapter 1], or [ZS].) Let F be the quotient field of A, i.e. F is the field of meromorphic functions on X. Let M be a maximal ideal of A, and set $A_M = \{f/g \in F : g \in A - M\}$. A_M is a proper subring of F called the "localization of A at M". In [A_1] Alling showed that A_M is a valuation ring of F and analyzed the value group G. The totally ordered group G can be described as follows: let $D \in Z[M] - \{X\}$, and let μ be the ultrafilter on D defined above. Denote the set of integers by Z, and let $H = \{s \in Z^D : s|U = 0$ for some $U \in \mu\}$. Note that H is a subgroup of the additive group Z^D. The value group G is the factor group, $G = Z^D/H$; and for $H(s) \in G$, $H(s) \geq 0$ iff there exists $t \in Z^D$ such that $H(t) = H(s)$ and $t(x) \geq 0$ for all $x \in D$. Observe that G has a smallest positive element and therefore a smallest nonzero convex subgroup, which we will identify with Z. If M is fixed, then $Z = G$; if M is free, then $Z < G$.

We can now describe the valuation map $V_M: F-\{0\} \to G$ associated with A_M. For $f \in F-\{0\}$, denote the order of the zero of f at $x \in X$ by $V_x(f)$. Let $d: F-\{0\} \to Z^D$ be defined by $d(f)(x) = V_x(f)$ for all $x \in D$, and let $\rho: Z^D \to Z^D/H$ be the canonical homomorphism. Then for $f \in F-\{0\}$, $V_M(f) = \rho(d(f))$. Thus $V_M(f) \geq 0$ iff there exists $U \in \mu$ such that $V_x(f) \geq 0$ for all $x \in U$. Note that for a fixed maximal ideal M_x, $V_{M_x} = V_x$.

In addition to this valuation theory, we will also make use of the "contracted ideals" of A. Let I be an ideal of A. If M is a maximal ideal of A that contains I, we set

$$\begin{aligned} I_M^c &= (I \cdot A_M) \cap A \\ &= \{f \in A: fg \in I \text{ for some } g \in A - M\} \\ &= \{f \in A: V_M(f) \geq V_M(h) \text{ for some } h \in I\}. \end{aligned}$$

I_M^c is an ideal of A that contains I, and $(I_M^c)_M^c = I_M^c$. A nonzero ideal I of A is called a *contracted ideal* if $I = I_M^c$ for some maximal ideal M that contains I. Thus I is a contracted ideal iff, for some maximal ideal M containing I, I satisfies the following condition: if $V_M(f) \geq V_M(h)$ for some $h \in I$, then $f \in I$. We will need two facts about contracted ideals:

(a) A nonzero ideal I of A is contained in a unique maximal ideal M iff $I = I_M^c$, i.e. the contracted ideals are the ideals of A that are contained in unique maximal ideals.

(b) The contracted ideals contained in a given maximal ideal form a chain (i.e. are totally ordered under set inclusion).

(We should note that the contracted ideals of A are called "primary ideals" in [A_2]. As we shall see, contracted ideals are not necessarily primary in the algebraic sense, so to avoid confusion we will not use this terminology.)

Finally, we describe the prime ideals of A. Every nonzero prime ideal is clearly a contracted ideal, so every nonzero prime ideal is contained in a unique maximal ideal; and the prime ideals contained in a given maximal ideal form a chain. Let M be a maximal ideal of A. The ideal

$$P_M^* = \bigcap_{n \in N} M^n = \{f \in A: V_M(f) > n \text{ for all } n \in N\}$$

is prime and is the largest nonmaximal prime ideal contained in M. Note that when M is fixed, $P_M^* = \{0\}$; when M is free, $\{0\} < P_M^*$. Let $g \in M-\{0\}$; there is a smallest prime ideal P_M^g contained in M that contains g and a largest prime ideal P_{Mg} contained in M that does not contain g:

$$P^g_M = \{f \in A: V_M(f^n) \geq V_M(g) \text{ for some } n \in N\},$$

$$P_{Mg} = \{f \in A: V_M(f) > V_M(g^n) \text{ for all } n \in N\}.$$

Note that when $V_M(g) \in N$ (i.e. when $g \in M - P^*_M$), then $P^g_M = M$ and $P_{Mg} = P^*_M$.

1.1 **Definition.** In any ring an *upper prime ideal* (resp. *upper primary ideal*) is a prime (resp. primary) ideal that has an immediate predecessor in the set of all prime (resp. primary) ideals partially ordered by inclusion. A *lower prime ideal* (resp. *lower primary ideal*) is a prime (resp. primary) ideal that has an immediate successor in the set of all prime (resp. primary) ideals partially ordered by inclusion.

The ideals of the form P^g_M are the upper prime ideals of A, and the ideals of the form P_{Mg} are the lower prime ideals of A. There are prime ideals of A that are neither upper nor lower prime ideals.

2. **The primary ideals of** A. Every nonzero primary ideal of A is clearly a contracted ideal, so every nonzero primary ideal is contained in a unique maximal ideal; and the primary ideals contained in a given maximal ideal form a chain. In our first theorem we demonstrate the existence of upper and lower primary ideals in the ring A.

2.1. **Theorem.** *Let M be a maximal ideal of A, and let $g \in M - \{0\}$. Set*

$$P_M|^g = \{f \in A: V_M(f) + V_M(h) \geq V_M(g) \text{ for some } h \in A - P^g_M\},$$

and

$$P_M|_g = \{f \in A: V_M(f) > V_M(g) + V_M(h) \text{ for all } h \in A - P^g_M\}.$$

Then

(a) $P_M|^g$ *is the smallest primary ideal contained in M that contains g.*

(b) $P_M|_g$ *is the largest primary ideal contained in M that does not contain g.*

Note. We will drop the subscript M from the valuation map V_M, the prime ideals P^*_M, P^g_M, P_{Mg}, and the primary ideals $P_M|^g$, $P_M|_g$ in this and succeeding proofs when this will lead to no confusion.

Proof. (a) It is a routine matter to verify that $P|^g$ is an ideal of A. We claim that $(P|^g)^{1/2} = P^g$. It is sufficient to show that $P_g < P|^g \subset P^g$. Clearly $P_g \subset P|^g$, and since $g \in P|^g - P_g$, $P_g < P|^g$. Let $f \in P|^g$. Then $V(fh) \geq V(g)$ for some $h \in A - P^g$, so $fh \in P^g$. Since $h \notin P^g$, $f \in P^g$, and we have $P|^g \subset P^g$.

Now suppose that $fr \in P|^g$ and $r \notin (P|^g)^{1/2} = P^g$. Since $fr \in P|^g$, $V(fr) + V(s) = V(f) + V(rs) \geq V(g)$ for some $s \in A - P^g$. Since $r \notin P^g$ and $s \notin P^g$, $rs \notin P^g$, so $f \in P|^g$, and $P|^g$ is primary.

Finally, let J be any primary ideal contained in M that contains g. If $P^g \subset J$, then $P|^g \subset J$ and we are done, so assume that $J \subset P^g$. Let $f \in P|^g$. Then $V(fh) \geq V(g)$ for some $h \in A - P^g$. Since J is a contracted ideal, $fh \in J$. And since $h \notin P^g$, $h \notin J^{1/2} \subset P^g$, so $f \in J$, and we have $P|^g \subset J$.

(b) It is a routine matter to verify that $P|_g$ is an ideal of A. We claim that $(P|_g)^{1/2} = P^g$. It is sufficient to show that $P_g < P|_g < P^g$. Clearly $P|_g \subset P^g$, and since $g \in P^g - P|_g$, $P|_g < P^g$. Let $f \in P_g$. For all $h \in A - P^g$, $V(h) < V(g)$, so $V(f) > 2V(g) > V(g) + V(h)$ for all $h \in A - P^g$, and therefore $f \in P|_g$. Since $g^2 \in P|_g - P_g$, we have $P_g < P|_g$.

Now suppose that $fr \in P|_g$ and $r \notin (P|_g)^{1/2} = P^g$. Let $s \in A - P^g$. Then $rs \notin P^g$, so since $fr \in P|_g$, $V(fr) > V(g) + V(rs)$. This implies that $V(f) > V(g) + V(s)$. Since s is arbitrary, $f \in P|_g$, and $P|_g$ is primary.

Finally, let I be any primary ideal contained in M that properly contains $P|_g$. If $P^g \subset I$, then $g \in I$ and we are done, so assume that $I \subset P^g$. Let $f \in I - P|_g$. Then $V(f) \leq V(g) + V(h) = V(gh)$ for some $h \in A - P^g$. Since I is a contracted ideal, $gh \in I$. And since $h \notin P^g$, $h \notin I^{1/2} \subset P^g$, so $g \in I$. Q.E.D.

As we observed in the proof of Theorem 2.1, we always have $P_g < P|_g < P|^g \subset P^g$. If $V(g) = 1$, then $P|^g = P^g = M$. But if $V(g) > 1$, then $P|^g < P^g$: for, choose $f \in A$ such that

$$Z(f) = \{x \in X: V_x(g) \geq 2\} \in Z[M]$$

and

$$V_x(f) = [V_x(g)/2] \quad \text{for all } x \in Z(f),$$

where $[V_x(g)/2]$ denotes the largest integer $\leq V_x(g)/2$. Then $3V(f) \geq V(g)$, so $f \in P^g$. But $f \notin P|^g$, for if we have $V(f) + V(h) \geq V(g)$, then $2V(h) \geq V(g)$, so that $h \in P^g$.

The ideals of the form $P|^g$ and $P|_g$ are respectively upper and lower primary ideals of A, and these are the only upper and lower primary ideals of A. In particular, then, no nonmaximal prime ideal can be an upper or a lower primary ideal. If $V(g) \in N$, then

$$P|^g = \{f \in A: V(f) \geq V(g)\} = M^{V(g)}$$

and

$$P|_g = \{f \in A: V(f) > V(g)\} = M^{V(g)+1}.$$

Thus for $n \in N$, $n \geq 2$, the ideals M^n are both upper and lower primary ideals of A. We will see in Corollary 2.5 that these are the only ideals of A that are both upper and lower primary ideals.

We have mentioned that a contracted ideal of A need not be primary. We show now that if M is a free maximal ideal and $g \in M-\{0\}$ satisfies $V_M(g) > N$, then $P_M|_g < (g)^c_M < P_M|^g$, so that the contracted ideal $(g)^c_M$ is not primary. It is clear that $P_M|_g < (g)^c_M \subset P_M|^g$. Choose $f \in A$ such that $Z(f) = \{x \in X: V_x(g) \geq 2\}$ and $V_x(f) = V_x(g)-1$ for all $x \in Z(f)$. Then $V_M(f)+1 = V_M(g)$. Since $V_M(g) > N$, $1 \notin V_M(P^g)$, and it follows that $f \in P_M|^g$. But $f \notin (g)^c_M$. For assume, on the contrary, that $f \in (g)^c_M$. Then $hf = rg$ for some $r \in A$ and $h \in A-M$. For all $x \in Z(g)-Z(h)$, which is nonempty, we have $V_x(f) = V_x(fh) = V_x(rg) \geq V_x(g)$, a contradiction. Therefore $f \in P_M|^g - (g)^c_M$.

In our next theorem we show that, just as in the ring C, every nonprime primary ideal of A is either an upper or a lower primary ideal.

2.2. **Theorem.** *Let M be a maximal ideal of A, and let J be a primary ideal contained in M. Then J is either prime or has the form $P_M|^{g^*}$ or $P_M|_{g^*}$ for some $g^* \in M-\{0\}$.*

2.3. **Lemma.** [A_1, Theorem 2.6]. *Let M be a maximal ideal of A. The value group G of A_M satisfies the following property: if L and U are nonempty countable subsets of G such that $a \leq b$ for all $a \in L$ and $b \in U$, then there exists $f \in A$ such that $a \leq V(f) \leq b$ for all $a \in L$ and $b \in U$.*

Proof of Theorem 2.2. Assume that J is not prime. Then since J cannot be the union of a chain of prime ideals, $P^g \not\subset J$ for some $g \in J-\{0\}$. For each $n \in N$ choose $g_{(n)} \in A$ so that $Z(g_{(n)}) = Z(g)$ and $V_x(g_{(n)}) = \max\{1, [V_x(g)/n]\}$ for all $x \in Z(g)$. (The existence of an element of A that satisfies these conditions is guaranteed by the generalized Weierstrass (product) theorem [F]. We pick one function from among the functions satisfying these conditions and denote it by $g_{(n)}$, agreeing to always take $g_{(1)} = g$. We will denote the mth power of $g_{(n)}$ by $g^m_{(n)}$.) Note that since $P^g \not\subset J$, $g_{(k)} \notin J$ for some $k \in N$. Let

$$L = \{V(g^m_{(n)}):\ g^m_{(n)} \notin J,\ (n, m) \in N \times N\},$$

and

$$U = \{V(g^m_{(n)}):\ g^m_{(n)} \in J,\ (n, m) \in N \times N\}.$$

L and U are countable, nonempty ($V(g_{(k)}) \in L$, $V(g) \in U$) subsets of the value group, and $L < U$ since J is a contracted ideal. By Lemma 2.3 there exists $g^* \in A$ such that $L \leq V(g^*) \leq U$. We claim

(a) if $g^* \in J$, then $J = P|^{g^*}$;

(b) if $g^* \notin J$, then $J = P|_{g^*}$.

(a) Clearly $P|^{g^*} \subset J$. Let $f \in J$. We want to show that $V(f) + V(h) \geq V(g^*)$ for some $h \in A - P^{g^*}$. This is clear if $V(f) \geq V(g^*)$, so assume that $V(f) < V(g^*)$. Then there exists $f' \in A$ such that $V(f') = V(f)$ and $g^*/f' \in A$. We have $V(f) + V(g^*/f') = V(g^*)$, so the proof of (a) will be complete if we show that $g^*/f' \notin P^{g^*}$. First, note that

$$(*) \qquad \text{for all } k \in N, \quad V(g_{(k)}) + V(f) > V(g^*).$$

This is clear if $g_{(k)} \in J$. Suppose that $g_{(k)} \notin J$, and let m be the smallest positive integer such that $g_{(k)}^m \in J$. Then $g_{(k)}^{m-1} \notin J$, so $V(f) > V(g_{(k)}^{m-1})$. Therefore $V(g_{(k)}) + V(f) > V(g_{(k)}^m) \geq V(g^*)$, and we have $(*)$. Now by $(*)$, $V(g^*/f') < V(g_{(k)})$ for all $k \in N$, and therefore $g^*/f' \notin P^g$. Finally, since obviously $P^g = P^{g^*}$, we have $g^*/f' \notin P^{g^*}$.

(b) Clearly $J \subset P|_{g^*}$. Let $f \in A - J$. We want to show that $V(f) \leq V(g^*) + V(h)$ for some $h \in A - P^{g^*}$. This is clear if $V(f) \leq V(g^*)$, so assume that $V(f) > V(g^*)$. Then there exists $f'' \in A$ such that $V(f'') = V(f)$ and $f''/g^* \in A$. We have $V(f) = V(g^*) + V(f''/g^*)$, so the proof of (b) will be complete if we show that $f''/g^* \notin P^{g^*}$. First, note that

$$(**) \qquad \text{for all } k \in N, \quad V(f) < V(g^*) + V(g_{(k)}).$$

This if clear if $g_{(k)} \in J$ since then $V(g_{(k)}) > V(f)$. Suppose that $g_{(k)} \notin J$, and let m be the smallest positive integer such that $g_{(k)}^m \in J$. Then

$$V(f) < V(g_{(k)}^m) = V(g_{(k)}^{m-1}) + V(g_{(k)}) \leq V(g^*) + V(g_{(k)})$$

since $g_{(k)}^{m-1} \notin J$, and we have $(**)$. Now by $(**)$, $V(f''/g^*) < V(g_{(k)})$ for all $k \in N$. As in the proof of (a), this implies that $f''/g^* \notin P^{g^*}$. Q.E.D.

Now we show that the upper and the lower prime ideals of A and "most" of the upper and lower primary ideals of A have expressions analogous to the expressions for the upper and lower prime and primary ideals of C. (See **[GJ**, Chapter 14] and **[K]** for these latter expressions.)

2.4. Theorem. *Let M be a maximal ideal of A, and let $g \in M - \{0\}$. For $n \in N$ define $g_{(n)}$ as in the proof of Theorem 2.2. Then*

(a) $P_M^g = \{f \in A: V_M(f) \geq V_M(g_{(n)})$ *for some* $n \in N\}$,

(b) $P_{M^g} = \{f \in A: V_M(f) > V_M(g^n)$ *for all* $n \in N\}$.

If we assume, in addition, that $V_M(g) > N$ (i.e. that $g \in P_M^*$), then

(c) $P_M|^g = \{f \in A: V_M(f) > V_M(g_{(n)}^{n-1})$ *for all* $n \in N\}$,

(d) $P_M|_g = \{f \in A: V_M(f) \geq V_M(g_{(n)}^{n+1})$ *for some* $n \in N\}$.

Proof. (a) $\subset$ is clear and was used several times in the proof of Theorem 2.2. For the reverse inclusion, suppose that $V(f) \geq V(g_{(n)})$. Then

$$(2n-1)V(f) \geq (2n-1)V(g_{(n)}) = nV(g_{(n)}) + (n-1)V(g_{(n)})$$
$$\geq nV(g_{(n)}) + n - 1 \geq V(g).$$

Therefore $f \in P^g$.

(b) This has been noted previously and is restated here only for emphasis.

(c) Suppose that $V(f) > V(g^{n-1}_{(n)})$ for all $n \in N$. If $V(f) \geq V(g)$, then $f \in P|^g$; so assume that $V(f) < V(g)$. Choose $h \in A$ such that $Z(h) = \{x \in X: V_x(f) < V_x(g)\}$ and $V_x(h) = V_x(g) - V_x(f)$ for all $x \in Z(h)$. Then $V(f) + V(h) = V(g)$, and we claim that $h \notin P^g$. For assume, on the contrary, that $h \in P^g$. Then for some $k \in N$, $V(h) \geq V(g_{(k)})$, and so $V(f) \leq V(g) - V(g_{(k)})$. Now since

$$(2k-1)V(g_{(2k)}) + V(g_{(k)}) \geq (2k-1)V(g_{(2k)}) + 2V(g_{(2k)}) = 2kV(g_{(2k)}) + V(g_{(2k)})$$
$$> 2kV(g_{(2k)}) + 2k \qquad (\text{since } g_{(2k)} \in P^*_M)$$
$$> V(g),$$

we have $V(f) \leq V(g) - V(g_{(k)}) < V(g^{2k-1}_{(2k)})$, a contradiction.

For the reverse inclusion, let $f \in P|^g$, and let $n \in N$. We want to show that $V(f) > V(g^{n-1}_{(n)})$. Since $f \in P|^g$, $V(f) + V(h) \geq V(g)$ for some $h \in A - P^g$. Since $h \notin P^g$, $V(f) \geq V(g) - V(h) > V(g) - V(g_{(k)})$ for all $k \in N$, so $V(f) > V(g) - V(g_{(2n)}) > (n-1)\, V(g_{(n)})$.

(d) Suppose that $V(f) \geq V(g^{n+1}_{(n)})$ for some $n \in N$. Note that

$$(n+1)V(g_{(n)}) - V(g_{(2n)}) \geq 2(n+1)V(g_{(2n)}) - V(g_{(2n)}) = 2nV(g_{(2n)}) + V(g_{(2n)})$$
$$> 2nV(g_{(2n)}) + 2n > V(g).$$

Therefore

$$V(f) \geq (n+1)V(g_{(n)}) > V(g) + V(g_{(2n)})$$
$$> V(g) + V(h) \qquad \text{for all } h \in A - P^g,$$

and so $f \in P|_g$.

For the reverse inclusion, suppose that $V(f) < V(g^{n+1}_{(n)})$ for all $n \in N$. If $V(f) \leq V(g)$, then $f \notin P|_g$; so assume that $V(f) > V(g)$. Let $h \in A$ such that $Z(h) = \{x \in X: V_x(f) > V_x(g)\}$ and $V_x(h) = V_x(f) - V_x(g)$ for all $x \in Z(h)$. Then $V(f) = V(g) + V(h)$, so to complete the proof it is sufficient to show that $h \notin P^g$. Assume, on the contrary, that $h \in P^g$. Then $V(h) \geq V(g_{(k)})$ for some $k \in N$, and we have $V(f) \geq V(g) + V(g_{(k)}) \geq V(g^{k+1}_{(k)})$, a contradiction. Q.E.D.

It is easy to see that if $V(g) \in N$, the equalities of (c) and (d) do not necessarily hold. For example, if $V(g) = 3$, then

$$\{f \in A: V(f) > V(g^{n-1}_{(n)}) \text{ for all } n \in N\} = P^*, \quad \text{but } P|^g = M^3;$$

and

$$\{f \in A: V(f) \geq V(g^{n+1}_{(n)}) \text{ for some } n \in N\} = M^3, \quad \text{but } P|_g = M^4.$$

2.5. **Corollary.** (a) *No prime ideal of A can be both an upper and a lower prime ideal.*

(b) *No primary ideal of A contained in P^*_M can be both an upper and a lower primary ideal.*

Proof. (a) Suppose that $P^g \subset P_h$ for some $g, h \in M-\{0\}$. Then $V(g_{(n)}) > V(h^m)$ for all $(n, m) \in N \times N$. By Lemma 2.3 there exists $f \in A$ such that $V(g_{(n)}) \geq V(f) \geq V(h^m)$ for all $(n, m) \in N \times N$. Now since $P_h < M$, $P_h \subset P^*$, and so $g \in P^*$. Therefore $V(g_{(k)}) > V(g_{(k+1)})$ for all $k \in N$, so $V(g_{(n)}) > V(f)$ for all $n \in N$, and we have $f \notin P^g$. Since clearly $f \in P_h$, $P^g < P_h$.

(b) Suppose that $P|_r \subset P|^s \subset P^*$, for some $r, s \in M-\{0\}$. Then $V(r_{(n)}^{n+1}) > V(s_{(m)}^{m-1})$ for all $(n, m) \in N \times N$. By Lemma 2.3 there exists $f \in A$ such that $V(r_{(n)}^{n+1}) \geq V(f) \geq V(s_{(m)}^{m-1})$ for all $(n, m) \in N \times N$. Clearly $f \in P|^s - P|_r$.

3. **Intersections of primary ideals.** In this section we investigate intersections of prime and primary ideals of A. These results are aimed at the problem of characterizing the ideals of A that are intersections of prime or of primary ideals. In our first theorem we use a result of Alling to characterize the ideals that are intersections of nonmaximal prime ideals.

3.1. **Theorem.** *An ideal I of A is an intersection of nonmaximal prime ideals iff $I = I^2$ (i.e. I is idempotent).*

3.2. **Lemma** [A_2, **Lemma** 5.2]. *If I is any ideal of A and we denote the set of maximal ideals containing I by $v(I)$, then $I = \bigcap_{M \in v(I)} I^c_M$.*

Proof of Theorem 3.1. Suppose that $I = \bigcap_\alpha P_\alpha$, where the P_α are nonmaximal prime ideals. If g is a nonzero element of I, then for each α, $g_{(3)} \in P_\alpha$; and if V is the valuation map associated with the maximal ideal containing P_α, then $V(g/g_{(3)}) \geq V(g_{(3)})$, so $g/g_{(3)} \in P_\alpha$. Hence $g = g_{(3)} \cdot g/g_{(3)} \in (\bigcap_\alpha P_\alpha) \cdot (\bigcap_\alpha P_\alpha) = I^2$.

Suppose, conversely, that $I = I^2$ and let $f \in A - I$. We want to find a nonmaximal prime ideal that contains I but not f. By Lemma 3.2 there exists $M \in v(I)$ such that $f \notin I^c_M$. Since $I = I^2$ and $I \subset M$, $I \subset \bigcap_{n \in N} M^n = P^*_M$. If $f \notin P^*_M$, we are done; so assume that $f \in P^*_M$. Since $f \in P_M|^f - I^c_M$ and the contracted ideals contained in M form a chain, $I^c_M \subset P_M|^f$. Therefore $I \subset P_M|^f$. We complete the proof by showing that $I \subset P_{Mf}$. Let $g \in I$, and let $n \in N$. Since $g \in I^{3n}$, there exist $k \in N$ and functions $g_{ji} \in I$, $1 \leq j \leq 3n$, $1 \leq i \leq k$, such that

$$g = \sum_{i=1}^{k} g_{1i} \cdots g_{3ni}.$$

We have

$$V_M(g) \geq \min_{1 \leq i \leq k} V_M(g_{1i} \cdots g_{3ni}) \geq 3nV_M(f_{(2)}) \quad \text{by Theorem 2.4(c)}$$
$$> V_M(f^n).$$

Since n is arbitrary, $g \in P_{Mf}$. Since g is arbitrary, we have $I \subset P_{Mf}$.

3.3. **Corollary.** *Every idempotent contracted ideal of A is a nonmaximal prime ideal.*

3.4. **Definition.** An ideal I of A is called *deep* iff, for all $M \in v(I)$, $I \subset P^*_M$.

We take a moment to justify our terminology. Let R be any ring, and let J be an ideal of R. The "depth" of J [N, p. 25] is defined to be the largest nonnegative integer n such that there exist prime ideals $P_0, \cdots, P_n$ in R with $J \subset P_0 < P_1 < \cdots < P_n$. (If there is no such n, J is said to be of "infinite depth".) In Definition 3.4 we have called an ideal I of A deep if for every $M \in v(I)$ the ideal I^c_M is of positive depth.

From now on we restrict ourselves to the consideration of deep ideals of A. We do so for two reasons. First of all we are looking for results that parallel the results of [W] concerning intersections of prime and primary ideals in the ring C. We have seen in Theorem 2.4 that the structure of the primary ideals of A that are contained in P^*_M is similar to the structure of the primary ideals of C. Hence it is natural to consider ideals I of A whose contracted components I^c_M are contained in P^*_M. Secondly, the deep ideals of A are in many respects the most interesting ideals of A. Among the class of contracted ideals, for example, the only ideals that are not deep are those that are powers of maximal ideals. Note that in Theorem 3.1 we have already implicitly restricted ourselves to deep ideals since idempotent ideals are necessarily deep. Theorem 3.1 can be rephrased as follows: a deep ideal I of A is an intersection of prime ideals iff $I = I^2$. We turn now to the study of intersections of primary ideals.

3.5. **Definitions.** (a) If I and J are ideals of a ring R, the *quotient* $I:J$ is defined as follows: $I : J = \{r \in R : rJ \subset I\}$. (It is clear that $I : J = R$ iff $J \subset I$; and if $J \not\subset I$, then $I:J$ is an ideal of R which contains I.)

(b) If I is an ideal of a ring R and K is a primary ideal of R which contains I, then K is called a *minimal primary ideal of I* if there does not exist a primary ideal J of R such that $I \subset J < K$.

If I is a deep ideal of A, we will denote by I^* the intersection of all the primary ideals containing I. The next theorem shows that the ideals $I \cdot I^{1/2}$ and $I:I^{1/2}$ are always intersections of primary ideals and provides useful descriptions of the ideals I, I^*, and $I^{1/2}$. The theorem involves collections of upper and lower prime and primary ideals; we abbreviate our notation by denoting $P^{f\alpha}_{M\alpha}$ by $P^{f\alpha}_{\alpha}$, $P_{M\beta}|^{f\beta}$ by $P_{\beta}|^{f\beta}$, etc. The proof of this theorem and of

Theorem 3.12 are omitted since the proofs are straightforward modifications of the proofs given in [W] for the corresponding results in the ring C.

3.6. **Theorem**. *Let I be a deep ideal of A. Let $\{Q_\alpha: \alpha \in A\}$ be the set of minimal primary ideals of I that are prime, $\{P_\beta|^{f_\beta}: \beta \in B\}$ be the set of minimal primary ideals of I that are upper primary ideals, and $\{P_\gamma|_{f_\gamma}: \gamma \in \Gamma\}$ be the set of minimal primary ideals of I that are lower primary ideals. Then*

(a) $I \cdot I^{1/2} = (\bigcap_{\alpha \in A} Q_\alpha) \cap (\bigcap_{\beta \in B} P_\beta|_{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|_{f_\gamma})$,

(b) $I = (\bigcap_{\alpha \in A} Q_\alpha) \cap (\bigcap_{\beta \in B} I^c_\beta) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|_{f_\gamma})$,

(c) $I^* = (\bigcap_{\alpha \in A} Q_\alpha) \cap (\bigcap_{\beta \in B} P_\beta|^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|_{f_\gamma})$,

(d) $I^{1/2} = (\bigcap_{\alpha \in A} Q_\alpha) \cap (\bigcap_{\beta \in B} P_\beta^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma^{f_\gamma})$,

(e) $I : I^{1/2} = (\bigcap_{\beta \in B} P_\beta|^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|^{f_\gamma})$.

Theorem 3.6 gives us two obvious sufficient conditions for a deep ideal to be an intersection of primary ideals.

3.7. **Corollary**. *Let I be a deep ideal of A. If $I = I \cdot I^{1/2}$ or $I = I : I^{1/2}$, then I is an intersection of primary ideals.*

Further, for the class of deep contracted ideals we have

3.8. **Corollary**. *If I is a deep contracted ideal of A that is not prime, then $I : I^{1/2}$ is an upper primary ideal, and $I \cdot I^{1/2}$ is the corresponding lower primary ideal.*

3.9. **Corollary**. *Let I be a deep contracted ideal of A. I is primary iff either $I = I \cdot I^{1/2}$ or $I = I : I^{1/2}$.*

It is natural to ask if Corollary 3.9 is true for all deep ideals of A, i.e. is the union of the two sufficient conditions $I = I \cdot I^{1/2}$ and $I = I : I^{1/2}$ a necessary condition for an arbitrary deep ideal I to be an intersection of primary ideals? We show now that the answer is no.

3.10. **Example**. We construct a deep ideal I of A such that I is an intersection of primary ideals, but $I \cdot I^{1/2} < I < I : I^{1/2}$. Let $D = \{x_n: n \in N\}$ be a countably infinite closed discrete subset of X. Let $\mathfrak{U}_M$ and $\mathfrak{U}_N$ be free ultrafilters on D such that $\{x_n: n \text{ odd}\} \in \mathfrak{U}_M$ and $\{x_n: n \text{ even}\} \in \mathfrak{U}_N$. Set $M = \{f \in A: Z(f) \cap D \in \mathfrak{U}_M\}$ and $N = \{f \in A: Z(f) \cap D \in \mathfrak{U}_N\}$; M and N are maximal ideals of A. Let $i \in A$ such that $Z(i) = D$, and $V_{x_n}(i) = n$ for all $n \in N$. We set $I = P_M|_i \cap P_N|^i$. Then I is an intersection of primary ideals, and since $v(I) = \{M, N\}$, it is clear that I is deep. It is easy to see that $\{P_M|_i, P_N|^i\}$ is the set of minimal primary ideals of I, so by Theorem 3.6, $I : I^{1/2} = P_M|^i \cap P_N|^i$ and $I \cdot I^{1/2} = P_M|_i \cap P_N|_i$. We have then $i \in I : I^{1/2} - I$. Let $f \in A$ such that $Z(f) = D$, and $V_{x_n}(f) = n$

for n even, $V_{x_n}(f) = n^n$ for n odd. Then $V_M(f) > V_M(i^k)$ for all $k \in N$, so $f \in P_{Mi} \subset P_M|_i$; also $V_N(f) = V_N(i)$, so $f \in P_N|^i$, and we have $f \in P_M|_i \cap P_N|^i = I$. But since $V_N(f) = V_N(i)$, $f \notin P_N|_i$, and so $f \notin I \cdot I^{1/2}$.

3.11. **Definition.** Let $\{J_\alpha : \alpha \in A\}$ be a collection of ideals of a ring. We shall say that the intersection of the collection $\{J_\alpha : \alpha \in A\}$ is *irredundant* if for all $\beta \in A$ we have $\bigcap_{\alpha \in A,\, \alpha \neq \beta} J_\alpha \not\subset J_\beta$.

In our final theorem we give a necessary condition for a deep ideal I to be an intersection of primary ideals. If the intersection of all the minimal primary ideals of I is irredundant, the condition is also a sufficient condition for I to be an intersection of primary ideals.

3.12. **Theorem.** *Let I be a deep ideal of A. If I is an intersection of primary ideals, then $I^2 = I \cdot (I : I^{1/2})$. Conversely, if the intersection of all the minimal primary ideals of I is irredundant, then the condition $I^2 = I \cdot (I : I^{1/2})$ implies that I is an intersection of primary ideals.*

3.13. **Corollary.** *Let I be a deep ideal of A such that I^* can be written as a finite intersection of primary ideals. Then I is an intersection of primary ideals iff $I^2 = I \cdot (I : I^{1/2})$.*

Proof. We can write I^* as a finite irredundant intersection of minimal primary ideals of I, $I^* = J_1 \cap \cdots \cap J_n$. A routine argument then shows that every minimal primary ideal of I appears in this expression, and by Theorem 3.12 the corollary follows.

We have characterized the deep contracted ideals of A that are primary in Corollary 3.9. We can now further characterize such ideals as follows:

3.14. **Corollary.** *Let I be a deep contracted ideal of A. I is primary iff $I^2 = I \cdot (I : I^{1/2})$.*

It may seem reasonable to conjecture that the condition $I^2 = I \cdot (I : I^{1/2})$ is a sufficient condition for an arbitrary deep ideal to be an intersection of primary ideals. We close by giving a counterexample to this conjecture.

3.15. **Example.** We will construct a deep ideal I of A such that $I^2 = I \cdot (I : I^{1/2})$, but $I \neq I^*$. Let N be a countably infinite, closed discrete subset of X; as our notation indicates, we think of N as an embedded copy of the space of positive integers. In our construction we will use the points of $\beta N - N$, where βN is the Stone-Čech compactification of N, to index the set of all free ultrafilters on N: if $\alpha \in \beta N - N$, the unique free ultrafilter on N that converges to α in the topology of βN will be denoted by $\mathfrak{U}_\alpha$, i.e., $\mathfrak{U}_\alpha = \{S \subset N : \alpha \in \mathrm{cl}_{\beta N} S\}$. For each free ultrafilter $\mathfrak{U}_\alpha$ on N, M_α will denote the free maximal ideal $M_\alpha = \{f \in A : Z(f) \cap N \in \mathfrak{U}_\alpha\}$.

Now we construct our ideal I. We define a sequence U_n of subsets of N as follows: $U_1 = N$, $U_2 = \{n \in N : n \text{ is odd}\}$, $U_3 = \{1, 5, 9, \cdots\}$, and in general $U_k = \{2^{k-1}(n-1)+1 : n \in N\}$. By [GJ, 6W.3] there is a point $\gamma \in \beta N - N$ such that $\gamma \in \bigcap_n \mathrm{cl}_{\beta N} U_n - \mathrm{int}_{\beta N - N}[\bigcap_n \mathrm{cl}_{\beta N} U_n]$. Let $\Lambda = (\beta N - N) \cap [\bigcup_n \mathrm{cl}_{\beta N}(U_n - U_{n+1})]$. Choose $i \in A$ such that $Z(i) = N$, and $V_n(i) = n$ for all $n \in N$. For each positive integer k choose $f_k \in A$ such that $Z(f_k) = N$, and $V_x(f_k) = V_x(i_{(nk)}^{nk+1})$ for all $x \in U_n - U_{n+1}$. We will denote by L the ideal of A generated by the set $\{f_k : k \in N\}$. Finally we set $I = L^c_{M_\gamma} \cap (\bigcap_{\lambda \in \Lambda} P_{M_\lambda}|^i)$. We leave to the reader the verification that I is deep, that $i \in I^* - I$, and that $I^2 = I \cdot (I : I^{1/2})$ (see [W, Example 2.21]).

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