ON THE ZEROS OF POWER SERIES WITH HADAMARD GAPS–DISTRIBUTION IN SECTORS

BY

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ABSTRACT. We give a sufficient condition for a power series with Hadamard gaps to assume every complex value infinitely often in every sector of the unit disk.

I. Introduction. Let

\[ f(z) = c_0 + \sum_{k=1}^{\infty} c_k z^{n_k} \]

be a power series convergent in \(|z| < 1\), with Hadamard gaps, \(n_{k+1}/n_k > q > 1\), \(k \geq 1\). Given a complex number \(c\), we are interested in the distribution of the zeros of \(f(z) - c\). We shall discuss the problem in term of the zeros of \(f\), replacing the constant term \(c_0\) of (1) by \(c_0 - c\) if necessary.

It has been shown that

(i) \(f\) has infinitely many zeros in the unit disk if \(\sum_{k=0}^{\infty} |c_k| = \infty\) and \(q \geq q_0\), where \(q_0\) is about 100 [5].

(ii) \(f\) has infinitely many zeros in any sector \(\theta_2 < \arg z < \theta_1\), \(|z| < 1\), if \(\lim_{k \to \infty} |c_k| > 0\) [2].

It remains undetermined whether \(f\) has zeros in the unit disk, or perhaps in any sector, if \(\sum_{k=0}^{\infty} |c_k| = \infty\), \(\lim_{k \to \infty} c_k = 0\), and \(1 < q < q_0\). We prove

Theorem 1. Let \(f(z) = c_0 + \sum_{k=1}^{\infty} c_k z^{n_k}\) be a power series convergent in \(|z| < 1\), with

(i) \(n_{k+1}/n_k > q > 1\) \((k \geq 1)\),

(ii) \(\lim_{k \to \infty} c_k = 0\),

(iii) \(\sum_{k=0}^{\infty} |c_k|^{2+\epsilon} = \infty\) for some positive \(\epsilon\).

Then \(f\) has infinitely many zeros in any sector \(\theta_2 < \arg z < \theta_1\), \(|z| < 1\).

II. A formula. Basic to the proof of Theorem 1 is a formula for functions

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meromorphic in sectors. The basic idea of this formula goes back to V. P. Petrenko [3]. The following lemma can be found in [2].

**Lemma 1.** Suppose \( f(z) \) is meromorphic in the sector \( |\arg z| < \pi/\nu \ (\nu > 1), |z| \leq R \). Let \( z = t \ (0 < t < R) \) be a regular point of \( f \) on the real axis, where \( f(t) \neq 0 \). For \( z \neq t \), \( R^2/t \), define

\[
a(z) = a(R, z, t) = \log \left| \frac{(R^2 - tz)/R(z - t)} \right|
\]

and

\[
\Lambda(R, z, t) = a(z) - a(-|z|).
\]

If we write

\[
I_1(R, t, \nu) = \int_0^R \left\{ \int_{-\pi/\nu}^{\pi/\nu} \log |(re^{i\theta})| \, d\theta \right\} \xi_1(R, r, t, \nu) \, dr,
\]

\[
I_2(R, t, \nu) = \int_{-\pi/\nu}^{\pi/\nu} \log |(Re^{i\theta})| \xi_2(R, \theta, t, \nu) \, d\theta,
\]

where

\[
\xi_1(R, r, t, \nu) = \frac{\nu^2}{2\pi} \frac{r^{\nu-1}R^{2\nu} - t^{2\nu}}{(r^{\nu} + t^{2\nu})(r^{2\nu} + t^{2\nu})^2},
\]

\[
\xi_2(R, \theta, t, \nu) = \frac{\nu}{\pi} \frac{R^{\nu}t^{\nu}(R^2 - t^2)(1 + \cos \nu\theta)}{(R^{\nu} + t^{2\nu})(R^{2\nu} + t^{2\nu} - 2R^\nu t^\nu \cos \nu\theta)},
\]

then

\[
\log |f(t)| = I_1(R, t, \nu) + I_2(R, t, \nu) + \sum_{i} A(R^\nu, t^\nu, b_i^\nu)
\]

\[
- \sum_{i} A(R^\nu, t^\nu, a_i^\nu)
\]

where the summation is taken over the zeros \( \{a_i\} \) and the poles \( \{b_i\} \) of \( f \) which lie in the interior of the sector.

Without loss of generality, we may assume that \( f(0) = 1 \) (consider \( f(z)/c\rho z^p \) if necessary). Suppose now that \( f \) has no zero in some sector, which we may assume to be the sector \( |\arg z| < \pi/\nu_0, |z| < 1 \), where \( \nu_0 > 1 \). We shall show that this leads to the conclusion

\[
\lim \sup_{R \to 1} [I_1(R, 2\nu_0) + I_2(R, 2\nu_0)] = \infty
\]

whereas (2) now reduces to the contradictory result

\[
I_1(R, 2\nu_0) + I_2(R, 2\nu_0) = \log |f(t)|.
\]

In the next section, we derive estimates which will be used to establish (3) in §IV.
III. Lower bounds for \(|f(z)|\). Transform the domain of \(f\) to the right half-plane with the change of variable \(z = e^{-w}\), and write (1) as

\[ F(w) = f(e^{-w}) = c_0 + \sum_{k=1}^{\infty} c_k e^{-nkw} \]

Lemma 2. There exist a subsequence \(\{c_{k(i)}\}\) of the coefficients \(\{c_k\}\) of (4) and positive constants \(U_0(q), u_0(q), p_0(q)\) such that the derivatives of \(F\) satisfy

\[ F^{(p)}(w) = (-n_{k(i)})^p c_{k(i)} e^{-nk(i)w} + R_i(w), \]

\[ |R_i(w)| \leq \frac{1}{2} |c_{k(i)}|^p c_{k(i)} e^{-nk(i)\Re(w)} \]

whenever \(p \geq p_0(q)\), and \(\Re(w)\) is in the range

\[ u_0(q)/n_{k(i)} < \Re(w) < u_0(q)/n_{k(i)} \]

Proof. Consider the sequence \(\{d_k\}\), where

\[ d_0 = \max \{|c_0|, |c_1|, |c_2|, \ldots\}, \]

\[ d_k = \max \{|d_{k-1}|, |c_k|, |c_{k+1}|, \ldots\} \quad (k \geq 1), \]

one verifies readily that

(a) \(d_k > 0\) for all \(k\),

(b) \(1/2 \leq d_{k+1}/d_k \leq 1\), and also

(c) \(d_k \geq |c_k|\), with equality occurring infinitely often.

If in (c), equality occurred finitely often, then \(d_{k+1} = d_k\) for \(k \geq k_0\). In this case

\[ \sum_{k=k_0}^{\infty} |c_k| \leq \sum_{k=k_0}^{\infty} d_k = d_{k_0} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j < \infty, \]

contradicting the assumptions that \(\sum_{k=0}^{\infty} |c_k|^{2^k} = \infty\) and \(\lim_{k \to \infty} c_k = 0\).

Let \(\{c_{k(i)}\}\) be the subsequence of \(\{c_k\}\) satisfying \(d_{k(i)} = |c_{k(i)}|, i = 1, 2, \ldots\). Differentiating \(F(w)\) \(p\) times, (4) becomes

\[ F^{(p)}(w) = \sum_{k=1}^{\infty} \delta_k a_k(w) \]

where \(\delta_k = (-1)^p c_k/d_k\), and \(a_k(w) = (n_{k(i)}^p d_k e^{-nk(i)w})\). We can find, for each \(k(i)\), and for sufficiently large \(p\), a set of \(w\) such that

(5) \[ |a_{k+1}(w)|/|a_k(w)| > 5 \quad \text{for } k < k(i), \]

(6) \[ |a_{k+1}(w)|/|a_k(w)| < 1/5 \quad \text{for } k \geq k(i). \]
For, (5) holds if
\[ \operatorname{Re}(n_{k+1}w) < \left( p \log t_k - \log 10 \right) / (1 - 1/t_k) \]

where \( t_k = n_{k+1}/n_k > q \). For sufficiently large \( p \),
\[ f(t) = \left( p \log t - \log 10 \right) / (1 - 1/t) \]
is a positive increasing function of \( t \) in \( t \geq q \). Therefore, (5) holds if \( p > p_0 \), and if
\[ \operatorname{Re}(w) < \left( 1/n_{k(i)} \right) \left( p \log q - \log 10 \right) / (1 - 1/q). \]

Similarly (6) holds, if
\[ \operatorname{Re}(n_kw) > \left( p \log t_k + \log 5 \right) / (t_k - 1). \]
The right-hand side of this inequality is bounded above by \( (p \log q + \log 5) / (q - 1) \), so that (6) holds if
\[ \operatorname{Re}(w) > \left( 1/n_{k(i)} \right) \left( p \log q + \log 5 \right) / (q - 1). \]

We note that if \( u = (p \log q + \log 5) / (q - 1) \), and \( U = (p \log q - \log 10) / (1 - 1/q) \), then for large \( p \), \( U/u = q (1 + O(1/p)) > c > 1 \). Thus (5) and (6) hold simultaneously, if \( p > p_1 \), and \( \operatorname{Re}(w) \) satisfies
\[ u/n_{k(i)} < \operatorname{Re}(w) < U/n_{k(i)}. \]

If \( w \) is in the range of (7), we have then
\[ F(p)(w) = \delta_{k(i)}a_{k(i)}(w) + \sum_{k \neq k(i)} \delta_k a_k(w) \]
\[ = \delta_{k(i)}a_{k(i)}(w) + R_i(w) \]
where \( \left| \delta_{k(i)} \right| = 1 \), and
\[ \left| R_i(w) \right| \leq \sum_{1 \leq k < k(i)} \left| a_{k(i)}(w) \right| (5)^{k-k(i)} + \sum_{k > k(i)} \left| a_{k(i)}(w) \right| (5)^{k(i)-k} \]
\[ \leq 2 \left| a_{k(i)}(w) \right| \sum_{j=1}^{\infty} (5)^{-j} = \frac{\left| a_{k(i)}(w) \right|}{2}. \]

**Lemma 3.** Let \( F(w) \) be holomorphic in \( |w - w_0| < R \). If for some \( p \),
\[ |F(p)(w)| \geq m > 0 \text{ and } \sup_{|w - w_0| < R} |F(p)(w)| = M, \text{ then the image of } |w - w_0| < R \text{ under } F \text{ covers the disk} \]
\[ \{ z : |z - F(w_0)| < K_p R_p m^{p+1} M^{-p} \} \]
where \( K_p \) is a positive constant depending on \( p \) only [1].
We infer from Lemma 2 and Lemma 3 the following

**Lemma 4.** If the function $f$ of Theorem 1 has no zero in the sector $|\arg z| \leq \pi/v_0$, $|z| < 1$, then there exist positive constants $U_1, u_1$, and $L$, depending on $q$ only, such that $|f(z)| > L |c_{k(i)}|$ in

$$S_i : \exp \left( - U_1/n_{k(i)} \right) < |z| < \exp \left( - u_1/n_{k(i)} \right),$$

$$|\arg z| < \pi/v \quad (v = 2v_0).$$

Here $\{k(i)\}$ is the sequence defined by $|c_{k(i)}| = d_{k(i)}$.

We next estimate the size of the set of points where

$$|f(z)| \left( \frac{1}{2} \left| c_0 \right|^2 + \sum_{k=1}^{\infty} \left| c_k \right|^2 |z|^{2n_k} \right)^{-\frac{1}{2}}$$

is bounded away from zero. The following result is due to R. Salem and A. Zygmund. The basic idea of the proof can be found in [4]. Define

$$\Lambda(r) = \left( \frac{1}{2} \left| c_0 \right|^2 + \sum_{k=1}^{\infty} \left| c_k \right|^2 |z|^{2n_k} \right)^{\frac{1}{2}}.$$

**Lemma 5.** If $f$ satisfies the conditions of Theorem 1, then, in any measurable subset $E \subset [0, 2\pi]$, the linear measure

$$m(\theta \in E \mid |f(re^{i\theta})| \Lambda(r)^{-1} \leq \gamma)$$

tends to $(m(E)/2\pi) \int_0^{2\pi} \int_0^\gamma e^{-r^2/2} dr = m(E)(1 - e^{-\gamma^2/2})$ as $r \to 1$.

**Lemma 6.** For any measurable subset $E \subset [0, 2\pi]$, and any positive $\delta < 1$, there is $r_0$ such that whenever $r \geq r_0$,

$$m(\theta \in E \mid |f(re^{i\theta})|\Lambda^{-1}(r) > \delta) \geq m(E)(1 - \delta).$$

**Proof.** By Lemma 5, for $r < 1$,

$$m(\theta \in E \mid |f(re^{i\theta})|\Lambda^{-1}(r) \leq \gamma) = m(E) - m(\theta \in E \mid |f(re^{i\theta})|\Lambda^{-1}(r) > \gamma).$$

$$\to m(E)e^{-\gamma^2/2} \quad (r \to 1).$$

Set $\gamma = \delta$. Since $\exp(-\delta^2/2) > 1 - \delta^2/2 > 1 - \delta$, (8) is proved.

**IV. Lower bounds for $I_1(R) + I_2(R)$**. In the following derivations, we shall use letters $K_1, K_2, K_3, \ldots$ for positive constants which depend on $f, t$ and $v$, but not on $R$.

With the notations of Lemma 1,

$$I_2(R, t, \nu) \geq \int_{-\pi/2\nu}^{\pi/2\nu} \log^+ |f(RE^{i\theta})| \xi_2 d\theta - \int_{-\pi/\nu}^{\pi/\nu} \log^+ |1/(RE^{i\theta})| \xi_2 d\theta.$$
In the first integral of the right-hand side, \( \xi_2 \geq K_1 \) for all \( R \) sufficiently close to 1. Choose \( \delta \) in the interval \( 0 < \delta < 1/2 \). By Lemma 6, if \( R \in S \), \( (i \geq i_0) \), \( \log^+ |f| \geq \log A(R) + \log \delta \) in a subset of measure \( > \pi/2 \nu \) of \( (-\pi/2 \nu, \pi/2 \nu) \).

In the second integral \( 0 \leq \xi_2 \leq K_2^2 \). By Lemma 6, \( \log^+ |1/(Re^{i\theta})| = 0 \) outside a set of \( \theta \) of measure \( < K_3 \delta \). In this set, by Lemma 4, \( \log^+ |1/(Re^{i\theta})| < - \log (L|c_k(i)|) \).

Therefore, for all large \( i \) and \( R \in S \),

\[
I_2 \geq K_4 \log A(R) + K_4 \log \delta + K_5 \delta \log |c_k(i)| - K_6
\]

(9)

Next we find a lower bound for \( I_1(R, t, \nu) \). From Lemma 1,

\[
I_1 = 2\pi \int_0^R \xi_1(R, r, t, \nu) \left\{ \frac{1}{2\pi} \int_{-\pi/\nu}^{\pi/\nu} \log |f(r e^{i\theta})| d\theta \right\} dr
\]

and we see that \( \xi_1 \) satisfies \( 0 \leq \xi_1 \leq K_8(R - r) \). By the first fundamental theorem of Nevanlinna,

\[
\frac{1}{2\pi} \int_{-\pi/\nu}^{\pi/\nu} \log^+ |1/(r e^{i\theta})| d\theta \leq T(r, f) = m(r, f).
\]

By the inequality of the arithmetic and geometric mean

\[
m(r, f) \leq K_9 \log A(r) \leq K_9 \log A(R) \quad (r \leq R).
\]

Therefore, if \( 0 < s < R \),

\[
I_1 \geq 2\pi \int_0^s \xi_1 \left\{ \frac{1}{2\pi} \int_{-\pi/\nu}^{\pi/\nu} \log |f(r e^{i\theta})| d\theta \right\} dr
\]

\[
- 2\pi \int_s^R \xi_1 \left\{ \frac{1}{2\pi} \int_{-\pi/\nu}^{\pi/\nu} \log^+ |1/(r e^{i\theta})| d\theta \right\} dr
\]

(10)

\[\geq D(s) - K_{10} \int_s^R (R - r)A(R) dr \geq D(s) - K_{11}(R - s)^2A(R).\]

By choosing \( s \) sufficiently close to 1, we can make

\[K_7 - K_{11}(R - s)^2 > K_7 - K_{11}(1 - s)^2 > \frac{1}{2} K_7.\]

Combining (9) and (10)

\[I_1 + I_2 \geq D(s_0) + \frac{1}{2} K_7 \log A(R) + K_5 \delta \log |c_k(i)| + K_4 \log \delta \quad (s_0 < R).\]

Since \( R \in S \), \( A(R) \geq K_{12} \sum_{k=0}^{k(i)} |c_k|^2 \), and thus

\[I_1(R) + I_2(R) \geq K_{13} \left\{ \log \left| c_k(i) \right| \delta \sum_{k=0}^{k(i)} |c_k|^2 \right\} + \log \delta \].

To show that
(11) \[ \limsup_{R \to 1} \left( l_1(R) + l_2(R) \right) = \infty \]

it is therefore enough to show that for some \( \delta \),

(12) \[ \limsup_{i \to \infty} \left| c_{k(i)} \left( \sum_{k=0}^{i} |c_k|^2 \right)^{\delta} \right| = \infty. \]

We prove first that if \( 0 < \delta < \epsilon/2 \) where \( \epsilon \) is the exponent of condition (iii) of Theorem 1, then

\[ W(\delta) = \limsup_{p \to \infty} \left| c_p \right| \left( \sum_{k=0}^{p} |c_k|^2 \right)^{\delta} \]

is infinite.

Suppose \( W(\delta) < \infty \), then for some \( K > 0 \), and all \( c_p \) with \( |c_p| < 1 \),

(13) \[ |c_p|^{2+\epsilon} \leq |c_p|^{2+2\delta} \leq K |c_p|^2 \left( \sum_{k=0}^{p} |c_k|^2 \right)^{2}. \]

Summing (13) over \( p \),

\[ \sum_{p=0}^{\infty} |c_p|^{2+\epsilon} \leq K \left( \sum_{p=0}^{\infty} \left( \sum_{k=0}^{p} |c_k|^2 \right)^{2} \right) \]

The left-hand side of the inequality is infinite by assumption. The right-hand side is finite by a well-known theorem on divergent series, stating that if \( a_n > 0 \), and \( \sum_{n=0}^{\infty} a_n = \infty \), then for any positive \( \rho \),

\[ \sum_{p=0}^{\infty} \left( \sum_{n=0}^{p} a_n \right)^{1+\rho} < \infty. \]

\( W(\delta) \) must therefore be infinite.

Let \( S = |c_p|^\delta \left( \sum_{k=p}^{\infty} |c_k|^2 \right) \). We now prove (12) by showing that for at least one of the members of the sequence \( \{k(i)\} \) which are closest to \( p \),

\[ |c_{k(i)}|^{\delta} \left( \sum_{k=0}^{k(i)} |c_k|^2 \right) > \frac{25}{3}. \]

The case \( p \in \{k(i)\} \) is trivial. Suppose that \( K < p \), and \( K' > p \) are the two member of \( \{k(i)\} \) which are closest to \( p \). If, for some \( k \) in \( K < k \leq p \), \( d_k = |c_k| \) \((k > k)\), then \( l \in \{k(i)\} \), and by the definition of \( K \) and \( K' \), we must have \( l = K' \) and \( |c_p| < |c_{K'}| \), so that
The only other possibility is that $d_k = \frac{1}{2} d_{k-1}$ ($K < k < p$) and so $|c_k| \leq 2^{-k+K} d_k$

$$|c_K|^\delta \left( \sum_{k=0}^{K} |c_k|^2 \right) > |c_p|^\delta \left( \sum_{k=0}^{p} |c_k|^2 \right) = S.$$

$$\sum_{k=0}^{K} |c_k|^2 \geq \sum_{k=0}^{p} |c_k|^2 \left( 1 - \frac{1}{3} \frac{|c_K|^2}{\sum_{k=0}^{p} |c_k|^2} \right) \geq \frac{2}{3} \sum_{k=0}^{p} |c_k|^2$$

if $p$ is so large that $|c_K| < 1$, $\sum_{k=0}^{p} |c_k|^2 > 1$. We have now

$$|c_K|^\delta \left( \sum_{k=0}^{K} |c_k|^2 \right) \geq \frac{2}{3} |c_p|^\delta \left( \sum_{p=0}^{p} |c_k|^2 \right) = \frac{25}{3}.$$ This proves (11) and completes the proof of Theorem 1.

REFERENCES


