THE TRACE-CLASS OF A FULL HILBERT ALGEBRA

BY

MICHAEL R. W. KERVIN(1)

ABSTRACT. The trace-class of a full Hilbert algebra $A$ is the set $\tau(A) = \{xy | x \in A, y \in A\}$. This set is shown to be a *-ideal of $A$, and possesses a norm $\tau$ defined in terms of a positive hermitian linear functional on $\tau(A)$. The norm $\tau$ is in general both incomplete and not an algebra norm, and is also not comparable with the Hilbert space norm $\|\|$ on $\tau(A)$. However, a one-sided ideal of $\tau(A)$ is closed with respect to one norm if and only if it is closed with respect to the other. The topological dual of $\tau(A)$ with respect to the norm $\tau$ is isometrically isomorphic to the set of left centralizers on $A$.

Introduction. The methods of Schatten [10], employed by Saworotnow and Friedell [8] in the $H^*$-algebra setting, are used here in §§1 and 2, to show that the trace-class $\tau(A)$ of a full Hilbert algebra $A$ is a *-ideal of $A$ (Theorem 2.2), and to define a norm $\tau$ and a positive hermitian linear functional on $\tau(A)$. These enjoy many of the same properties as for $H^*$-algebras, with some exceptions (Theorem 2.5): the norm $\tau$ is generally incomplete and is not an algebra norm on $\tau(A)$, unless $A$ itself is complete, in which case $A$ is an $H^*$-algebra in a trivially equivalent norm.

§3 deals with two theorems concerning the trace-class (see [10, §IV.1], and also [9] for the $H^*$-algebra setting). Theorem 3.1 shows that the topological dual of $\tau(A)$ is isometrically isomorphic to the set of left centralizers on $A$, while Theorem 3.2 says that $\tau(A)$ is isometrically isomorphic to a subspace of $C^*(A)$, the $C^*$-algebra of $A$. An example is given (3.3) to show that this subspace may not even be dense in $C^*(A)$.

In §4 we examine the relation between the two norms $\tau$ and $\|\|$ on the trace-class. They are in general incomparable (Theorem 2.5 and 4.1). However, in Theorem 4.5 it is shown that a one-sided ideal of $\tau(A)$ is closed in one norm topology if and only if it is closed in the other. It is noted (Theorem 4.7) that $\tau(A)$ is an orthocomplemented Hilbert algebra. The final result is that the closed ideals of $\tau(A)$ are precisely the trace-classes of closed ideals of $A$. Many of these results

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are generalizations of the same occurrences for \(H^*\)-algebras, as discovered by Smith in [11].

1. Basic results. Let \(A\) denote a Hilbert algebra with inner product \((x, y)\), norm \(\|x\| = (x, x)^{1/2}\), and involution \(x \rightarrow x^*\). The definition and elementary properties of a Hilbert algebra can be found in [1] and [7]; see also [2], [6] and [12]. Our notation follows that of Yood in [14]. The Hilbert space which is the completion of \(A\) in the norm \(\|x\|\) is denoted \(H\), and \(A_{fe}\) is the fulfillment of \(A\). \(B(H)\) shall denote the space of all bounded linear operators on \(H\).

Put \(\Lambda(A) = \{L_x: x \in A\}'\) (the commutant taken in \(B(H)\)) and \(P(A) = \{L_x: x \in A\}'\). The commutation theorem (see [2] or [7]) states that \(\Lambda(A)' = P(A)\) and \(P(A)' = \Lambda(A)\). Moreover, \(\Lambda(A)\) (resp. \(P(A)\)) is the strong closure of \(\{L_x: x \in A\}\) (resp. of \(\{L_y: y \in A\}\), and \(\Lambda(A) = \Lambda(A_b)\), \(P(A) = P(A_b)\). Since \(A_b\) is invariant under \(\Lambda(A_b)\) \(\cup P(A_b)\), Proposition 1.6], \(\Lambda(A_b)\) may also be defined as the set of operators in \(B(H)\) satisfying \(T(xy) = T(x)y, \forall x, y \in A_b\). We call such operators left centralizers on \(A_b\) after B. E. Johnson [4]; operators in \(P(A_b)\) are called right centralizers on \(A_b\). Johnson's terminology differs (i.e. "left" in place of "right") from other notions of centralizer found, for example, in [13] and [5].

As in [8], a projection shall be a self-adjoint idempotent, and a projection base for \(A\) shall be a maximal family of nonzero mutually orthogonal projections of \(A\). A Hilbert algebra in general need not contain any nonzero projections (see [14, §4]), but projection bases exist in full Hilbert algebras (see [7, Theorem 2.3]). Thus we restrict our attention to the latter; in the remainder of the paper, \(A\) shall denote a full Hilbert algebra.

We need two results of Rieffel [7, Theorems 3.8 and 3.9], but enlarged to involve projection bases.

1.1 Lemma. Let \(T \in \Lambda(A)\) (resp. \(T \in P(A)\)). The following statements are equivalent:

1. \(T = L_x\) (resp. \(T = R_x\)) for some \(x\) in \(A\).
2. \(\sup \{\|T_e\|: e\ \text{is a projection of } A\} < +\infty\).
3. There is a projection base \(\{e_\gamma: \gamma \in \Gamma\}\) for \(A\) such that \(\sum_{\gamma \in \Gamma} \|T e_\gamma\|^2 < +\infty\).

1.2 Lemma. Let \(a \in A\). The following statements are equivalent:

1. \(a\) is positive and integrable.
2. \(a\) is positive and \(\sum_{\gamma \in \Gamma} (a, e_\gamma) < +\infty\) for some projection base \(\{e_\gamma: \gamma \in \Gamma\}\) for \(A\).
3. \(a = b^2\) for some unique positive \(b\) in \(A\).
4. \(a = xx^*\) for some \(x\) in \(A\).
Proof. Clearly (3) \implies (4) \implies (1) \implies (2). If (2) holds, and $T$ is the unique positive square root in $\Lambda(A)$ of the positive operator $\bar{L}_a$, then

$$\sum_{\gamma \in \Gamma} \|T e_\gamma\|^2 = \sum_{\gamma \in \Gamma} (T^2 e_\gamma, e_\gamma) = \sum_{\gamma \in \Gamma} (ae_\gamma, e_\gamma) < +\infty,$$

so $T = \bar{L}_b$ for some positive $b$ in $A$ by Lemma 1.1. Since the mapping $x \to \bar{L}_x$ is an algebra *-isomorphism on $A$, it follows that $b$ is the unique positive square root of $a$. Thus (2) \implies (3).

Using notation of Schatten (see [10] and [8]), we let $[x]$ denote the positive square root in $A$ of $x^* x$, for each $x$ in $A$. Note that $\|L_x\| = \|x\|$, and $\bar{L}_x = [\bar{L}_x]$, the positive square root of $\bar{L}_x^* x$.

It is interesting to observe that a complete analogue of the polar decomposition theorem for operators in $\mathcal{B}(H)$ (see [10]) obtains in the full Hilbert algebra setting. The partial isometry involved does not have as exact a description as for $\mathcal{B}(H)$ or any $H^*$-algebra (see [8]), however that is an unnecessary detail.

1.3 Theorem. For each $x$ in $A$, there is a partial isometry $W_x$ in $\Lambda(A)$ with initial set $[x]A^H$ (the closure in $H$ of $[x]A$) and final set $xA^H$ such that

1. $x = W_x([x])$;
2. $[x] = W_x^*[x]$;
3. $x^* = W_x^*[x^*]$;
4. $[x^*] = W_x(x^*)$.

Moreover, if $x = W(b)$ for some positive $b$ in $A$ and partial isometry $W$ in $\Lambda(A)$ with initial set $bA^H$, then $b = [x]$ and $W = W_x$.

Proof. Use the polar decomposition theorem in [10] to obtain a partial isometry $W_x$ in $\mathcal{B}(H)$ with initial set $[x]A^H$ and final set $xA^H$ such that $\bar{L}_x = W_x [\bar{L}_x] = W_x^* \bar{L}_x$, etc. For convenience let $\mathcal{N}$ denote the orthogonal complement in $H$ of $[x]A^H$. $\mathcal{N}$ is invariant under each $\bar{R}_y$ (y in $A$) so $\bar{R}_y W_x$ and $W_x \bar{R}_y$ agree on $H = [x]A^H \oplus \mathcal{N}$, for each $y$ in $A$. Thus $W_x$ is a left centralizer. (1)-(4) now follow using the semisimplicity of $A$.

If $x = W(b)$ as in the last sentence of the theorem, then

$$\bar{L}_x^* x = \bar{L}_x^* \bar{L}_x = \bar{L}_b W^* W \bar{L}_b = \bar{L}_b^2 = \bar{L}_b^2,$$

so $b = [x]$. If follows that $W$ and $W_x$ agree on $[x]A$, hence are equal.

As one might expect, there is a parallel result concerning right centralizers.

1.4 Theorem. For each $x$ in $A$, there is a partial isometry $V_x$ in $P(A)$ with initial set $\Lambda(x)^H$ and final set $\Lambda x^*H$, such that

1. $x^* = V_x([x])$;
2. $[x] = V_x^*[x^*]$;
(3) $x = V_x (x^*)$;
(4) $[x^*] = V_x (x)$.

If $x^* = V(b)$ for some positive $b$ in $A$ and partial isometry $V$ in $P(A)$ with initial set $Ab^H$, then $b = [x]$ and $V = V_x$.

Suppose for any $S$ in $B(H)$ we define an operator $S^\#$ on $H$ by $S^\#(\xi) = S(\xi^*)^*$, $\xi$ in $H$. The mapping $S \to S^\#$ is a conjugate-linear isometric automorphism of period 2 of $B(H)$ onto itself, with the following properties:

(a) $L_x^\# = \overline{R}_x$, for any $x$ in $A$ [2, Lemma 3];
(b) $\Lambda(A)^\# = P(A)$, $P(A)^\# = \Lambda(A)$;
(c) if $P_M$ is the projection of $H$ onto a closed subspace $M$, then $P_M^\# = P_{M^*}$;
(d) if $U$ is a partial isometry with initial set $M$ and final set $N$, then $U^H$ is a partial isometry with initial set $M^*$, final set $N^*$;
(e) $\Lambda^\#$ commutes with $\Lambda^\#$, the adjoint operation on $B(H)$.

From this it follows that $V_x = W_x^H$, $W_x^* = W^*_x$, and $V_x^* = V_x^*$ for each $x$ in $A$.

2. The trace-class. The trace-class of $A$ is the set $\tau(A) = \{xy: x \in A, y \in A\}$. This set is not obviously closed under addition. To show that this is so, we emulate the procedure in [10] and, more exactly, in [8]. To begin with, every element in the trace-class is integrable. It is not clear whether the converse obtains—however, it does for positive elements.

2.1 Lemma. For any $a$ in $A$, the following statements are equivalent:
(1) $a$ is in $\tau(A)$.
(2) $[a]$ is in $\tau(A)$.
(3) $[a]$ is integrable.
(4) There is a projection base \{e_\gamma: \gamma \in \Gamma\} for $A$ such that $\sum_{\gamma \in \Gamma}(\langle a, e_\gamma \rangle) < +\infty$.
(5) $[a]$ has a unique positive square root $[a]^{1/2}$ in $A$.

Proof. Use 1.2 and Theorem 1.3.

For any $x$ and $y$ in $A$ and any projection base $\{e_\gamma: \gamma \in \Gamma\}$ for $A$, the sum $\sum_{\gamma \in \Gamma}(xy, e_\gamma)$ converges absolutely to the number $(x, y^*)$, and is therefore independent of the choice of projection base. This number is called the trace of $xy$, $\text{tr}(xy)$: $\text{tr}(a) = \sum_{\gamma \in \Gamma}(a, e_\gamma)$ for any $a$ in $\tau(A)$ and projection base $\{e_\gamma: \gamma \in \Gamma\}$ for $A$.

2.2 Theorem. $\tau(A)$ is a dense $^*$-ideal of $A$ which is invariant under left or right centralizers. $\tau$ is a positive hermitian linear functional on $\tau(A)$ such that
(1) $\text{tr}(xy) = \text{tr}(yx) = (x, y^*)$,
(2) $\text{tr}(x^*x) = \|x\|^2$,
for any $x$ and $y$ in $A$. 

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Proof. It is clear that \( \tau(A) \) is invariant under left or right centralizers. The proof that \( \tau(A) \) is closed under addition is similar to Schatten’s [10, Lemma 3, p. 38]. The rest of the theorem now follows easily.

Now define \( \tau(a) = \text{tr}(a) \) for each \( a \) in \( \tau(A) \). Then \( \tau(a) = \sum_{\gamma \in \Gamma} \langle [a], e_{\gamma} \rangle \) for each projection base \( \{ e_{\gamma} : \gamma \in \Gamma \} \) for \( A \). Right away we see that

\[
\tau(a) = \tau([a]) = \| [a]^{1/2} \|^2
\]

and \( \tau(\lambda a) = \text{tr}(\lambda [a]) = |\lambda| \tau(a) \), for any \( a \) in \( \tau(A) \) and complex number \( \lambda \). \( \tau \) will be a norm on the trace-class once we show it is subadditive, and so we come to the next result (see [10, p. 39] as well as [8]).

2.3 Lemma. For any \( a \) in \( \tau(A) \) and operator \( T \) in \( \Lambda(A) \cup P(A) \),

\[
(1) |\text{tr}(Ta)| \leq \| T \| \tau(a),
(2) \tau(Ta) \leq \| T \| \tau(a).
\]

Proof. If \( T \) is a left centralizer, the proof is the same as in [8]. If \( T \) is a right centralizer, we proceed slightly differently:

\[
|\text{tr}(Ta)| = |\text{tr}(TW_a ([a]^{1/2} [a]^{1/2}))| = |\text{tr}(T(W_a [a]^{1/2})[a]^{1/2})|
= |\text{tr}(W_a [a]^{1/2})T([a]^{1/2})| = \| W_a [a]^{1/2}, (T[a]^{1/2})^* \|
\leq \| W_a [a]^{1/2} \| T[a]^{1/2} \| \leq \| [a]^{1/2} \| T \| [a]^{1/2} \| = \| T \| \tau(a).
\]

Since \( T \) commutes with operators in \( \Lambda(A) \), we have

\[
\tau(Ta) = \text{tr}([Ta]) = \text{tr}(W_{Ta}^* (Ta)) = \text{tr}(TW_{Ta}^* (a))
\leq \| T \| \tau(W_{Ta}^* (a)) \leq \| T \| \| W_{Ta}^* \| \tau(a) \leq \| T \| \tau(a).
\]

2.4 Theorem. \( \tau \) is a linear space norm on \( \tau(A) \) with the following properties:

1. multiplication in \( \tau(A) \) is separately \( \tau \)-continuous;
2. \( \tau(a^*) = \tau(a) \) for each \( a \) in \( \tau(A) \);
3. \( |\text{tr} \ a| \leq \tau(a) \) for each \( a \) in \( \tau(A) \);
4. \( \tau(xy) \leq \| x \| \| y \| \) for every \( x \) and \( y \) in \( A \);
5. \( \tau(T) = \| T \| \) for every \( T \) in \( \Lambda(A) \cup P(A) \).

Proof. \( \tau \) is subadditive: for any \( a \) and \( b \) in \( \tau(A) \), we have

\[
\tau(a + b) = \text{tr}([a + b]) = \text{tr}(W_{a+b}^* (a) + W_{a+b}^* (b))
\leq |\text{tr}(W_{a+b}^* (a))| + |\text{tr}(W_{a+b}^* (b))| \leq \tau(a) + \tau(b),
\]

using Theorem 1.3 and Lemma 2.3. Thus \( \tau \) is a linear space norm on the trace-class. (1) follows from Lemma 2.3 also, as does (3). To prove (2), we have, using Theorem 1.4 and Lemma 2.3,
for each $a$ in $\mathcal{A}$, so equality obtains. (4) is proven as in [8, Corollary 4]. If $T$ is a left or right centralizer on $A$, its restriction to the normed linear space $\mathcal{A}$ is continuous with respect to the norm $\tau$ by Lemma 2.3. The norm of the restricted operator is denoted $\tau(T)$:

$$
\tau(T) = \sup \{ \tau(Ta) : a \in \mathcal{A} \text{ and } \tau(a) \leq 1 \}.
$$

Now $\tau(T) \leq \|T\|$ by 2.3. We prove the reverse inequality for left centralizers (proof is similar for right centralizers): for any $x$ in $A$,

$$
\|Tx\|^2 = \|T(x^*)\|^2 = \tau(Tx(x^*))
= \tau(T(x(Tx^*))) \leq \tau(T)\tau(x(Tx^*)) \leq \tau(T)\|x\|\|Tx\|
$$

by (4) above;

thus $\|Tx\| \leq \tau(T)\|x\|$ for each $x$ in $A$. Thus $\|T\| \leq \tau(T)$, so equality obtains.

Thus far the trace-class $\mathcal{A}$ and its norm $\tau$ have behaved much the same as in the $H^*$-algebra setting. Now however we notice some differences: $\tau$ is not an algebra norm on $\mathcal{A}$, and is incomplete. One may attribute these failings to the lack of the same properties of the norm $\|\|$ on $A$, as we see from the next result.

2.5 Theorem. The following statements are equivalent:

1. Multiplication in $\mathcal{A}$ is jointly $\tau$-continuous.
2. There is a constant $M > 0$ such that $\tau(ab) \leq M\tau(a)\tau(b)$ for every $a$ and $b$ in $\mathcal{A}$.
3. There is a constant $K > 0$ such that $\|a\| \leq K\tau(a)$ for each $a$ in $\mathcal{A}$.
4. $\tau$ is a complete norm on $\mathcal{A}$.
5. $\|\|$ is a complete norm on $A$ (so $A = H$).
6. Multiplication in $A$ is jointly continuous.
7. $A$ is trivially renormable to be an $H^*$-algebra.

Proof. (5), (6), and (7) are equivalent by Lemma 4.5 of [14]. The equivalence of (1) and (2) is a simple matter. If (2) is true, then for each $a$ in $\mathcal{A}$,

$$
\|a\|^2 = \tau(a^*a) \leq M\tau(a^*)\tau(a) = M\tau(a)^2
$$

so (3) holds. If (3) is true, then so is (6): for any $x$ and $y$ in $A$,

$$
\|xy\| \leq K\tau(xy) \leq K\|x\|\|y\| \quad \text{(using (4) of Theorem 2.4)}.
$$

Suppose now that $A$ is trivially renormable to be an $H^*$-algebra, and suppose the $H^*$-algebra norm on $A$ is $\|x\|_1 = c\|x\|$ (x in $A$). Then $(x, y)_1 = c^2(x, y)$ for all $x$ and $y$ in $A$, so $\tau_1(a) = c^2\tau(a)$ for all $a$ in $\mathcal{A}$ = $\tau_1(A)$. But $\tau_1(A) = \tau(A)$ is a Banach algebra in the norm $\tau_1$ (see [8] and [9]), hence $\tau$ itself is complete on the trace-class. Thus (7) $\Rightarrow$ (4). Finally, suppose (4) is true; then Lemma 2.3 implies that
for each \( b \) in \( \tau(A) \). An application of the uniform boundedness principle gives

\[
M = \sup \{ r(L_a) : a \in \tau(A), \tau(a) \leq 1 \} < +\infty.
\]

It follows that \( \tau(ab) = \tau(a/\tau(a)) \cdot \tau(a)b \leq Mr(\tau(a)b) = Mr(\tau(b)) \) for all \( a \neq 0 \) in \( \tau(A) \) and \( b \) in \( \tau(A) \); thus (2) is true. This completes the proof of the theorem.

3. The dual of the trace-class. What follows now is an attempt to extend two results of Schatten (see [10, pp. 46–48], as well as [9, Theorems 1 and 2] for the \( H^* \)-algebra case). One extends fully, the other only partially. We use the following notation: if \( \Psi \) is a linear functional on \( \tau(A) \) which is continuous with respect to the norm \( \tau \), we let \( \tau(\Psi) \) denote the sup norm of \( \Psi \):

\[
\tau(\Psi) = \sup \{ |\Psi(a)| : a \in \tau(A) \text{ and } \tau(a) \leq 1 \}.
\]

\( \tau(A)' \) shall denote the set of all \( \tau \)-continuous linear functionals on \( \tau(A) \). For example, \( \tau \in \tau(A)' \) and \( \tau(\tau) = 1 \).

3.1 Theorem. For \( T \) in \( \Lambda(A) \), define a functional \( \Psi_T \) on \( \tau(A) \) by \( \Psi_T(a) = \tau(Ta) \) (\( a \) in \( \tau(A) \)). The mapping \( T \mapsto \Psi_T \) is a linear isometry of \( \Lambda(A) \) onto \( \tau(A)' \):

\[
\tau(\Psi_T) = \|T\|.
\]

Proof. See [9, Theorem 2].

Let \( \mathcal{C}(A) \) denote the \( \mathcal{C}^* \)-algebra of \( A \), the operator norm closure in \( B(\mathcal{H}) \) (or in \( \Lambda(A) \)) of the space \{\( L_x : x \in A \)\}.

3.2 Theorem. For \( a \) in \( \tau(A) \), define a functional \( \phi_a \) on \( \mathcal{C}(A) \) by \( \phi_a(T) = \tau(Ta) \) (\( T \) in \( \mathcal{C}(A) \)). The mapping \( a \mapsto \phi_a \) is a linear isometry of \( \tau(A) \) into the space of continuous linear functionals on \( \mathcal{C}(A) \):

\[
\|\phi_a\| = \tau(a).
\]

Proof. By Lemma 2.3, \( \phi_a \) is a continuous linear functional on \( \mathcal{C}(A) \) and \( \|\phi_a\| \leq \tau(a) \). The mapping \( a \mapsto \phi_a \) is clearly linear, so it remains only to show that \( \tau(a) \leq \|\phi_a\| \) for each \( a \) in \( \tau(A) \). Use the Kaplansky density theorem (see [1, p. 46]) to obtain a sequence \( \{z_n\} \) in \( A \) with \( \|L_{z_n}\| \leq 1 \), for all \( n \), such that \( W \) is the limit in the strong operator topology of \( L_{z_n} \). By Theorem 1.3, \( [a] = \lim_{n \to \infty} z_n a \). Now let \( e_{\gamma} : \gamma \in \Gamma \) be any projection base for \( A \), and let \( F \) be any finite subset of \( \Gamma \). Put \( p = \Sigma_{\gamma \in F} e_{\gamma} \), a projection in \( A \). Since \( \|L_{p z_n}\| \leq \|L_p\| \|L_{z_n}\| \leq 1 \), we have

\[
\|\phi_a\| \geq \|\phi_a(L_{p z_n})\| = |\tau(pz_n, a)| = |\langle z_n a, p \rangle|
\]

for each \( n \). Letting \( n \to \infty \), we have \( \|\phi_a\| \geq \langle [a], p \rangle = \Sigma_{\gamma \in F} \langle [a], e_{\gamma} \rangle \). Since
$F$ is an arbitrary finite subset of $\Gamma$, this means that $\|\phi_a\| \geq \sum_{\gamma \in \Gamma} |(a, e_\gamma)| = r(a)$, thus proving the theorem.

The difference between Theorem 3.2 and Theorem 1 if $[9]$ is this: the image $\phi_A$ of the mapping $a \rightarrow \phi_a$ ($a \in r(A)$) need not be all of the dual of $C^*(A)$. Of course this cannot be so unless $r$ is a complete norm on the trace-class, which means that $A$ would have to be an $H^*$-algebra (after trivial renorming), by Theorem 2.5. However, $\phi_A$ need not even be dense in the dual of $C^*(A)$, as the following example shows:

3.3 Example. The notation for this example is that of $[3]$—see especially $S9, 10, 13, 19$ and $20$. Let $X$ denote an arbitrary nonvoid locally compact Hausdorff space and $(X, \mathcal{M}, \nu)$ a measure space of the kind discussed in $[3, S9, 10]$. The measure $\nu$ need not be $\sigma$-finite. For convenience, put $\mathcal{L}_p = \mathcal{L}_p(X, \mathcal{M}, \nu)$, for $1 \leq p \leq \infty$. All functions considered are $\mathcal{M}$-measurable. Then $\mathcal{L}_2 \cap \mathcal{L}_\infty$ is a commutative full Hilbert algebra under pointwise operations, the $\mathcal{L}_2$ inner product, and conjugation as involution. For any $f$ in $\mathcal{L}_2 \cap \mathcal{L}_\infty$, $g \in \mathcal{L}_2$, and $\|fg\|_2 \leq \|g\|_2 \|b\|_\infty$. Consequently, if $b$ in $\mathcal{L}_\infty$, we shall write $L_b(g) = bg$, $g \in \mathcal{L}_2$, noting that $L_b \in \mathcal{B}(\mathcal{L}_2)$ and in fact $\|L_b\| = \|b\|_\infty$. From this, one sees that the $C^*$-algebra of $\mathcal{L}_2 \cap \mathcal{L}_\infty$ is $L_\infty$. Using Theorem 19.30 of $[3]$ to construct a projection base for $\mathcal{L}_2 \cap \mathcal{L}_\infty$, one can show that $\text{tr}(f) = \int_X f \nu$, $f \in \mathcal{L}_2 \cap \mathcal{L}_\infty$; thus $\tau(\mathcal{L}_2 \cap \mathcal{L}_\infty) = \mathcal{L}_1 \cap \mathcal{L}_\infty$, and the trace-norm is the $\mathcal{L}_1$ norm.

If the mapping $a \rightarrow \phi_a$ of Theorem 3.2 sent $\mathcal{L}_1 \cap \mathcal{L}_\infty$ onto a dense subset of the dual of $C^*(\mathcal{L}_2 \cap \mathcal{L}_\infty)$, then it would extend to a linear isometry $f \rightarrow \tilde{\phi}_f$ of $\mathcal{L}_1$ onto $\mathcal{L}_\infty^*$ given by: $\tilde{\phi}_f(g) = \int_X fg \nu$, $f \in \mathcal{L}_1$, $g \in \mathcal{L}_\infty$. Using the special linear isometry of $\mathcal{L}_\infty$ onto $\mathcal{L}_1^*$ (see $[3, 19.31$ and $20.20]$), one shows easily that $\mathcal{L}_1$ would have to be reflexive. This is known to be false even for $X = [0, 1]$ and $l$ Lebesgue measure.

4. The trace-class and two norms. The trace-class of $A$ possesses two norms, $\|\|$ and $\tau$, neither of which is in general complete or an algebra norm. Multiplication in $r(A)$ is separately continuous with respect to each norm. There are two relationships between $\|\|$ and $\tau$: for any $x$ and $y$ in $A$,

$$\tau(xy) \leq \tau(x) \|y\|.$$

These two norms are not in general comparable—Theorem 2.5 shows that there is no constant $K$ such that $\|a\| \leq K\tau(a)$ for all $a$ in $\tau(A)$ unless $A$ is an $H^*$-algebra after trivial renorming, and the following result show that there is not generally any such reverse inequality.
4.1 Theorem. The following statements are equivalent:

1. There is a constant $K > 0$ such that, for every $a$ in $\tau(A)$, $\tau(a) \leq K\|a\|$.
2. There is a constant $K > 0$ such that, for every $x$ in $A$, $\|x\| \leq K\|L_x\|$.
3. There is a constant $M > 0$ such that, for every $a$ in $\tau(A)$, $\tau(a) \leq M\|a\|$.
4. $A$ is projection bounded from above.
5. $A$ has an identity.

Proof. If (1) is true then, for any $x$ in $A$,

$$\|x\|^2 = \tau(x^*x) \leq K\|x^*x\| \leq K\|L_x\|\|x\|$$

so $\|x\| \leq K\|L_x\|$. If (2) holds, then for any $a$ in $\tau(A)$,

$$\tau(a) = \|[a]\|^2 \leq K^2\|L_{[a]}\|^2 = K^2\|L_{[a]}\| = K^2\|L_{[a]}\|$$

so (3) is true, since $\tau(L_{[a]}) = \|L_{[a]}\|$ by (5) of Theorem 2.4. Suppose now (3) is true; if $e$ is any projection of $A$, then $e$ is in $\tau(A)$ and $\tau(e) = \|e\|^2$, $\tau(L_e) = \|L_e\| = 1$, therefore $\|e\| \leq M^{1/2}$. Thus $A$ is projection bounded from above. Suppose now there is a constant $c > 0$ such that $\|p\| \leq c$ for each projection $p$ of $A$. Let $\{e_\gamma : \gamma \in \Gamma\}$ be any projection base for $A$. If $F$ is any finite subset of $\Gamma$, then

$$\sum_{\gamma \in F} \|e_\gamma\|^2 = \sum_{\gamma \in F} \|e_\gamma\|^2 \leq c^2.$$ 

Thus $\Gamma$ must be countable, so let the projection base be denoted $\{e_n\}_{n=1}^{\infty}$. Then $\sum_{n=1}^{\infty} e_n^{1/n}$ is a Cauchy sequence, so it has a limit $e = \sum_{n=1}^{\infty} e_n$ in $H$. Note that $e^* = e$. Moreover $e$ is a bounded element of $H$: for any $y$ in $A$,

$$L_e(y) = L_y(e) = \lim_{m \to \infty} L_y\left(\sum_{n=1}^{m} e_n y\right) = \lim_{m \to \infty} \sum_{n=1}^{m} e_n y = y,$$

since $\{e_n\}_{n=1}^{\infty}$ is a projection base for $A$. Therefore $e \in A$, and clearly $ey = y = ye$ for all $y$ in $A$. Finally, if $A$ has an identity $1$, then for any $a$ in $\tau(A) = A$,

$$\tau(a) = \tau(a1) = \tau(a1) = (a, 1)$$

(note $1^* = 1$), so $\tau(a) = \|[a]\| \|1\| = \|1\|\|a\|$. Therefore (1) is true. This completes the proof of the theorem.

Yood [14, Theorem 4.2] gives other conditions on $A$ equivalent to those in the above theorem. For example, $A = C^+(A)$—that is, $\{L_x : x \in A\}$ is closed in the operator norm topology on $\mathcal{B}(H)$.

Having seen that these two norms on the trace-class need not be comparable, we attempt to discover what properties they have in common. To begin, we introduce some orthogonal complementation notation. If $S \subset \tau(A)$, put

$$S^+ = \{\xi \in H : (\xi, S) = 0\}$$

the orthogonal complement of $S$ in $H$,

$$S^\perp = \{x \in A : (x, S) = 0\} = S^+ \cap A$$

the orthogonal complement in $A$,

$$S^P = \{a \in \tau(A) : (a, S) = 0\} = S^\perp \cap \tau(A)$$

the orthogonal complement in $\tau(A)$. 

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It is a consequence of Proposition 2.7 of [7] that \( I^\perp \) is dense in \( I^+ \) if \( I \) is any one-sided ideal of \( A \).

Much of what now follows was inspired by similar results concerning \( H^* \)-algebras in [11]. Many of our statements are formulated for left ideals only, it being understood that the corresponding results for right ideals obtain. Regarding closure terminology, a subset of \( r(A) \) which is closed in the relative \( \| \| \) (resp. \( \tau \)) topology on \( r(A) \) shall be referred to as "\( \| \| \)-closed" (resp. "\( \tau \)-closed").

4.2 Lemma. If \( I \) is a left ideal of \( \mathfrak{A}(A) \), then
1. \( \overline{I}^A \) (the closure in \( A \) of \( I \)) is a closed left ideal of \( A \);
2. \( I^P \) is a \( \| \| \)-closed left ideal of \( r(A) \);
3. If \( I \) is \( \| \| \)-closed, then \( I \) is a left ideal of \( A \), and \( I^P \) is dense in \( I^+ \) in the norm \( \| \| \).

Proof. (1) and (2) are easily shown using separate continuity of multiplication. If \( I \) is \( \| \| \)-closed, then \( \overline{I}^A \cap r(A) \) is a left ideal of \( A \). Let \( x \in I^\perp \), and let \( \{e_\gamma : \gamma \in \Gamma\} \) be any projection base for \( A \). Then each \( e_\gamma x \in I^\perp \cap r(A) = I^P \), since \( (e_\gamma x, I) = (x, e_\gamma I) = \{0\} \). Therefore \( x = \sum_{\gamma \in \Gamma} e_\gamma x \in I^P \).

4.3 Lemma. If \( I \) is a \( \| \| \)-closed left ideal of \( \mathfrak{A}(A) \), then \( I = A \overline{I}^A = \{xy : x \in A \text{ and } y \in \overline{I}^A\} \).

Proof. Put \( M = \{xy : x \in A \text{ and } y \in \overline{I}^A\} \). Then \( M \subseteq A \overline{I}^A \subseteq \overline{I}^A \cap r(A) = I \). Now suppose \( a = xy \in I \) for some \( x \) and \( y \) in \( A \). Since \( A \) is orthocomplemented [14, Theorem 2.5] we can write \( y = y_1 + y_2 \) with \( y_1 \in \overline{I}^A \) and \( y_2 \in (\overline{I}^A)^\perp \). By Lemma 4.2, \( (\overline{I}^A)^\perp = I^\perp = I^P \); also \( \overline{I}^A \) and \( (\overline{I}^A)^\perp \) are left ideals of \( A \). Therefore \( xy_1 \in \overline{I}^A \) and \( xy_2 \in (\overline{I}^A)^\perp \). Hence \( a - xy_1 = xy_2 \in \overline{I}^A \cap (\overline{I}^A)^\perp = \{0\} \), so \( a = xy_1 \in M \).

4.4 Lemma. If \( \{e_\gamma : \gamma \in \Gamma\} \) is a projection base for \( A \), then for each \( a \) in \( \mathfrak{A}(A) \), \( a = \sum_{\gamma \in \Gamma} e_\gamma a = \sum_{\gamma \in \Gamma} e_\gamma a \) (convergence in the \( \tau \) norm).

Proof. Write \( a = xy \) for some \( x \) and \( y \) in \( A \). If \( F \) is any finite subset of \( \Gamma \), then

\[
\tau \left( a - \sum_{\gamma \in F} e_\gamma a \right) = \tau \left( xy - x \sum_{\gamma \in F} y e_\gamma \right) \leq \|x\| \left\| y - \sum_{\gamma \in F} y e_\gamma \right\|.
\]

This shows that \( \sum_{\gamma \in \Gamma} e_\gamma a \) is summable to \( a \) in the norm \( \tau \). The other equality is similarly shown.

4.5 Theorem. A left ideal \( I \) of \( r(A) \) is \( \tau \)-closed if and only if it is \( \| \| \)-closed.
Proof. Suppose $I$ is $\tau$-closed. We need to show that $I = \overline{I^A} \cap \tau(A)$. If $a \in \overline{I^A} \cap \tau(A)$, then for any $\epsilon > 0$ there is a projection $p$ of $A$ such that $\tau(a - pa) < \epsilon/2$, by Lemma 4.4. Also there is a $b \in I$ with $\|p\| \|a - b\| < \epsilon/2$. Then $pb \in I$ and

$$
\tau(a - pb) \leq \tau(a - pa) + \tau(pa - pb) \leq \epsilon/2 + \|p\| \|a - b\| < \epsilon.
$$

Therefore $a \in I$. Hence $I$ is $\| \|$-closed.

Conversely, suppose $I$ is $\| \|$-closed. $\tau(A)$ is a dual Hilbert algebra by Theorem 2.2 and Corollary 2.2 of [14], so $I = I^P$. But $I^P$ is a $\| \|$-closed right ideal of $\tau(A)$ whose left annihilator is $I^{P*P} = I^P = I$, which is consequently $\tau$-closed, since the separate $\tau$-continuity of multiplication in $\tau(A)$ forces any left or right annihilator in $\tau(A)$ to be $\tau$-closed.

4.6 Corollary. The $\| \|$-closure of any left ideal of $\tau(A)$ is equal to its $\tau$-closure.

For the definition of an orthocomplemented Hilbert algebra, see [14, Definition 2.3]. In the proof of Theorem 4.5 we used the fact that $\tau(A)$ is a dual Hilbert algebra; it is also orthocomplemented.

4.7 Theorem. If $I$ is any closed left ideal of $\tau(A)$, then $\tau(I) = I \oplus I^P$.

Proof. Let $J = \overline{I^A}$. Then $A = I \oplus J^\perp$ since $A$ is orthocomplemented [14, Theorem 2.5]. For any $a \in \tau(A)$, write $a = xy$ for some $x$ and $y$ in $A$. We can write $y = y_1 + y_2$ with $y_1 \in J, y_2 \in J^\perp$. Since $J^\perp = \overline{I^P}$ by Lemma 4.2, it follows from Lemma 4.3 that $xy_1 \in I$ and $xy_2 \in I^P$. Thus $a = xy_1 + xy_2 \in I \oplus I^P$.

Thus the trace-class of a full Hilbert algebra provides another example of an orthocomplemented Hilbert algebra which is not full (see [14, Example 2.6]).

A simple argument based on the orthocomplementation property in $A$ shows that the trace-class of any closed ideal $J$ of $A$ (which is also a full Hilbert algebra by Theorem 2.7 of [14]) is given by $\tau(J) = J \cap \tau(A)$. Using this we can obtain a characterization of the closed ideals of $\tau(A)$.

4.8 Theorem. If $J$ is a closed ideal of $A$, then $\tau(J)$ is a closed ideal of $\tau(A)$. Conversely, any closed ideal $I$ of $\tau(A)$ has the form $\tau(J)$ where $J = \overline{I^A}$.

Proof. If $J$ is a closed ideal of $A$, then $\tau(J) = J \cap \tau(A)$ is clearly an ideal of $\tau(A)$, and is closed in $\tau(A)$: the closure of $\tau(J)$ means the $\| \|$-closure by Corollary 4.6, so the closure of $\tau(J)$ is contained in $J \cap \tau(A) = \tau(J)$.

Suppose now that $I$ is a closed ideal of $\tau(A)$. Then $\overline{I^A}$ is a closed ideal of $A$, so $\tau(\overline{I^A}) = \overline{I^A} \cap \tau(A) = I$.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403

DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061