SURJECTIVE STABILITY IN DIMENSION 0
FOR $K_2$ AND RELATED FUNCTORS

BY

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ABSTRACT. This paper continues the investigation of generators and relations for Chevalley groups over commutative rings initiated in [14]. The main result is that if $A$ is a semilocal ring generated by its units, the groups $L(\Phi, A)$ of [14] are generated by the values of certain cocycles on $A^* \times A^*$. From this follows a surjective stability theorem for the groups $L(\Phi, A)$, as well as the result that $L(\Phi, A)$ is the Schur multiplier of the elementary subgroup of the points in $A$ of the universal Chevalley-Demazure group scheme with root system $\Phi$, if $\Phi$ has large enough rank. These results are proved via a Bruhat-type decomposition for a suitably defined relative group associated to a radical ideal. These theorems generalize to semilocal rings results of Steinberg for Chevalley groups over fields, and they give an effective tool for computing Milnor's groups $K_2(A)$ when $A$ is semilocal.

Let $\Phi_l$ be a reduced irreducible root system of rank $l$ and $A$ a commutative ring with 1. There is an exact sequence

$$1 \rightarrow L(\Phi_l, A) \rightarrow \text{St}(\Phi_l, A) \rightarrow E(\Phi_l, A) \rightarrow 1$$

where $\text{St}(\Phi_l, A)$ is the Steinberg group [14, (3.7)] and $E(\Phi_l, A)$ is the elementary subgroup of the points in $A$ of the universal Chevalley-Demazure group scheme with root system $\Phi_l$ [14, (3.3)]. If $\Phi_m$ is a second such root system, containing $\Phi_l$ as a subsystem generated by a connected subgraph of the Dynkin diagram of $\Phi_m$, there are induced homomorphisms $\theta(l, m): L(\Phi_l, A) \rightarrow L(\Phi_m, A)$, and Steinberg [17] has shown these are surjective for all $m \geq l \geq 1$ when $A$ is a field. In this paper I will prove that this is true for any semilocal ring $A$ with at most one residue field isomorphic to $\mathbb{F}_2$. I will also show, in this case, that the groups $L(\Phi, A)$ are generated by the values of certain cocycles on $A^* \times A^* \times A^*$ and that (1) is a central extension (and not just stably central; cf. [14, (5.1)]), theorems again due to Steinberg [17] when $A$ is a field. These results were announced in [13].

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In general one conjectures that \( \theta(l, m) \) is surjective for all \( m \geq l \geq d \), where \( d \) is a fixed positive integer related to the dimension of the maximal ideal space of \( A \); the theorem proved here may thus be thought of as the dimension 0 case of a surjective stability theorem for \( L(\Phi_l, \cdot) \). If \( \Phi_l \) belongs to one of the infinite families \( A_n, B_n, C_n, D_n \), one deduces, under the same hypotheses, the surjectivity of

\[
\theta(l, \infty) : L(\Phi_l, A) \to L(\Phi_\infty, A) = \lim_{l \to \infty} L(\Phi_l, A).
\]

This reveals one motivation of the present research, since \( L(A_\infty, \cdot) \) is Milnor's algebraic \( K_2 \) functor [9].

The paper proceeds as follows. Let \( \mathfrak{q} \subset A \) be an ideal, and write \( (1 + \mathfrak{q})^* \) for the units congruent to 1 modulo \( \mathfrak{q} \). In §1 I define pairings (''relative Steinberg symbols'')

\[
\{, \} : A^* \times (1 + \mathfrak{q})^* \to L(\Phi_l, \mathfrak{q})
\]

and recall some of their properties. In §2 I prove, when \( \mathfrak{q} \subset \text{rad } A \), a normal form for the relative group \( \text{St}(\Phi, \mathfrak{q}) \) analogous to the Bruhat decomposition of the Chevalley groups over fields [17, 7.6]. This implies that the groups \( L(\Phi_l, \mathfrak{q}) \) are generated by the relative symbols of §1, and, therefore, that \( L(\Phi_l, \mathfrak{q}) \to L(\Phi_m, \mathfrak{q}) \) is surjective for all \( m \geq l \geq 1 \). Combining this with Steinberg's theorem for fields yields the above-mentioned results for semilocal rings. In addition the theorems of this section allow one to deduce a presentation for \( E(\Phi, A) \) of such a semilocal ring.

In §3 I compute \( L(\Phi_l, A) \) for various local rings, using the results of §§1 and 2. In §4 I apply these results to the problem of surjective stability for the maps

\[
H_2(\text{SL}_2(A), \mathbb{Z}) \to H_2(E(\Phi_l, A), \mathbb{Z}).
\]

The reader primarily interested in \( K_2 \) should note the following. Milnor's groups \( E_{n+1}(A), \text{St}_{n+1}(A) \) are the groups \( E(A_n, A), \text{St}(A_n, A) \) of this paper \((n \geq 2)\), and \( K_2(A) = L(A_\infty, A) \). The symbols \( \{, \} \) are always bilinear in this case. A positive root \( \alpha \in A_n \) is to be identified with a pair \((ij), 1 \leq i < j \leq n + 1; -\alpha \) then corresponds to \((ji)\).

Milnor's \( K_2 \) theory exists for noncommutative rings as well, and most of the results of §2 remain true in this case, provided certain elements in \( A^* \) lie in \( [A^*, A^*] \). I have omitted a discussion of these points since the surjective stability theorem for \( K_2 \) of noncommutative semilocal rings has recently been obtained by Dennis [3], based on work of Silvester [12].

When \( A = K \) is a field, Matsumoto [8] has shown that the maps \( \theta(l, m) \) are injective as well. This injective stability theorem remains true for radical ideals in the semilocal rings considered here, and will be the subject of a subsequent paper [15].
I would like to thank Professor Hyman Bass, who directed the Columbia University doctoral thesis which contained a preliminary version of these results, for his advice and encouragement. I would also like to thank M. Léon Motchane and the Institut des Hautes Études Scientifiques for their hospitality during the first stages of this research.

Notation and terminology. The definitions, notations and terminology regarding root systems, Chevalley groups, Steinberg groups and their subgroups and relations are to be found in [14, §3]. However in this paper we always assume that the Chevalley-Demazure group schemes in question are universal [14, (3.3)]. If $\Phi_l \subset \Phi_m$ are reduced irreducible root systems, we say they are of the same type if they satisfy

(a) $\Phi_l$ is generated by a connected subgraph of the Dynkin diagram of $\Phi_m$.
(b) If $\Phi_m$ is symplectic, then $\Phi_l$ is also symplectic and at least one long root of $\Phi_m$ occurs in $\Phi_l$.

The inclusions $D_l \subset B_l$ violate (a) and the inclusions $A_{l-1} \subset C_l$, $l > 2$, violate (b).

The reader is reminded that the relative groups used in this paper differ from those of [9] and [16] (cf. the warnings following [14, (3.13)]). However the results of this paper do apply to the relative groups of [16], as follows from [16, (1.1), (2.5) and (2.6)].

All rings are commutative with 1; all homomorphisms preserve 1. If $A$ is a ring, rad $A$ is its Jacobson radical and $A^*$ is its multiplicative group of units. A pair $(A, q)$ consists of a ring $A$ together with an ideal $q \subset A$; if $q \subset \text{rad} A$ we say $(A, q)$ is a radical pair. We write $(1 + q)^* = (1 + q) \cap A^*$. If $T$ is a subset of $A$, the subring of $A$ generated by $T$ is denoted $\mathbb{Z}[T]$.

Let $G$ be a group. For $\sigma, \tau \in G$ we write $\tau \sigma = \tau \sigma^{-1}$, $[\tau, \sigma] = \tau \sigma \cdot \sigma^{-1} = \tau \sigma \tau^{-1} = \sigma^{-1}$.

If $H, K$ are subgroups of $G$, $[H, K]$ is the subgroup generated by $[h, k], h \in H, k \in K$; in particular the commutator subgroup of $G$ is $[G, G]$. We write $G^{ab} = G/[G, G]$. If $G$ is finite, $|G|$ is its order.

Finally, $\mathbb{Z}$ denotes the rational integers and $\mathbb{F}_q$ a finite field with $q$ elements.

1. Relative Steinberg symbols and the subgroup $L(\Phi, A) \cap \hat{K}(\Phi, q)$. Recall [14, (3.12)] that $\hat{H}(\Phi, q)$ is the smallest normal subgroup of $\hat{K}(\Phi, q)$ containing all $\hat{h}_a(v), a \in \Phi, v \in (1 + q)^*$. $\hat{H}(\Phi, q)$ is a subgroup of $St(\Phi, q)$ (cf. (2.7)(a)).

Definition. Let $\alpha \in \Phi, u, v \in A^*$, and set

\[ |u, v|_{\alpha} = \hat{h}_a(u) \hat{h}_a(v)^{-1} \hat{h}_a(v)^{-1}. \]

The subgroup of $\hat{H}(\Phi, A)$ generated by all $|u, v|_{\alpha}, |u, u|_{\alpha}$, where $u \in A^*, w \in (1 + q)^*$ and $\alpha$ ranges over $\Phi$ is denoted $D(\Phi, q)$. $D(\Phi, q)$ is a subgroup of $St(\Phi, q)$ (cf. (2.7)(a)).
It follows from relation (R8) that for all $\alpha, \beta \in \Phi$,
\begin{equation}
\{u^{(\beta, \alpha)}, v\}_\beta = [\hat{\delta}_\alpha(u), \hat{\delta}_\beta(v)].
\end{equation}
Thus if there is an $\alpha \in \Phi$ with $\langle \beta, \alpha \rangle = 1$, we have $\{u, v\}_\beta \in [\hat{\mathbb{H}}(\Phi, A), \hat{\mathbb{H}}(\Phi, q)]$ $\subset \hat{\mathbb{H}}(\Phi, q)$. This will be the case except when $\Phi$ is symplectic and $\beta$ is long.

The following proposition summarizes various well-known identities satisfied by $\mathbb{H}_\alpha$. Proofs may be found in [8, 5.5–5.7], [10, 3.2, 3.9, Appendix] and [18, Lemma 39 and Theorem 12].

(1.1) Proposition. Let $\alpha \in \Phi$, $u, v, w \in A^*$. Then $\{u, v\}_\alpha^{-1} = \{v, u\}_\alpha$. Writing $\mathbb{H}, \mathbb{I} = \mathbb{I}, \mathbb{W}$, the following identities hold in $\mathcal{D}(\Phi, A)$:

(51) $\{u, 1\}_\alpha = \{1, u\}_\alpha = 1$.

(52) $\{u, v\}_\alpha \{uv, w\}_\alpha = \{u, v\}_\alpha \{u, w\}_\alpha \{v, w\}_\alpha$.

(53) $\{u, v\}_\alpha = \{u^{-1}, v^{-1}\}$.

(54) $\{u, v\}_\alpha = \{u, u^{-1}\} \{v, u\}_\alpha$.

(55) $\{u, v\}_\alpha = \{u, u^{-1}\} \{v, u\}_\alpha$ if $1 - u \notin A^*$.

(56) $\{u, v\}_\alpha = \{u, u^{-1}\} \{v, u\}_\alpha$ if $1 - u \in A^*$.

(57) If $\mathbb{R}, \mathbb{I}$ generate a cyclic subgroup of $A^*$, then $\{u, v\}_\alpha = \{u, v\}_\alpha$.

(58) If $\mathbb{R}, \mathbb{I}$, then $\{u, v^2\} = 1$.

Moreover, if $\Phi$ is nonsymplectic or if $\alpha$ is short,

(S02) $\{u, v\}_\alpha = \{u, v\}_\alpha$.

(S03) $\{u, v\}_\alpha = \{v, u\}_\alpha$.

Remarks. 1. The above identities are not independent. For example, (S1)–(S4) imply (S6)–(S8), and if $\Phi$ is nonsymplectic or if $\alpha$ is short, (S1) (S5) (S02) (S03) imply the others. (Cf. [10, Appendix].)

2. Identity (S5), which is of great importance for computations when $A$ is a field, is valueless when $u \in (1 + q)^*$ (since in that case $1 - u \notin A^*$ if $q \notin A$). A new identity which can sometimes be used to replace (S5) in such computations when $q \subset \text{rad} A$ will be proved in (2.8).

(1.2) Definition. A relative Steinberg symbol on the pair $(A, q)$ with values in an abelian group $C$ is a mapping

$\{, \}_\alpha : A^* \times (1 + q)^* \to C$

satisfying (S1)–(S5) of (1.1) and (2.8). When $q = A$, we call $\{, \}$ a Steinberg symbol. If (S02) holds, we call $\{, \}$ a (relative) bilinear Steinberg symbol. We sometimes abbreviate "Steinberg symbol" to "symbol."

In this paper the word symbol will always refer to one of the symbols $\{, \}$ with values in $\mathcal{D}(\Phi, q)$ constructed above.

Let $\hat{\mathbb{H}}(\Phi, q)$ be the subgroup of $\mathcal{S}_r(\Phi, q)$ generated by $\mathcal{D}(\Phi, q)$ and all $\hat{\delta}_\alpha(v)$, $\alpha \in \Phi$, $v \in (1 + q)^*$. 

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(1.3) Proposition. (a) $D(\Phi, q)$ is a central subgroup of $St(\Phi, A)$.

(b) $\hat{H}(\Phi, q) \subset \hat{K}(\Phi, q)$, and

$$[\hat{H}(\Phi, A), \hat{H}(\Phi, q)] \subset L(\Phi, A) \cap \hat{H}(\Phi, q) \subset L(\Phi, A) \cap \hat{K}(\Phi, q) \subset D(\Phi, q),$$

with equality if $\Phi$ is nonsymplectic or if every element of $(1 + q)^*$ is a square.

(c) $D(\Phi, q)$ is generated by all $\{u, v\}, u \in A^*, v \in (1 + q)^*$ for any fixed long root $\alpha$. Hence if $\Phi_i \subset \Phi_m$ are reduced irreducible root systems of the same type, the homomorphism $D(\Phi_1, q) \to D(\Phi_m, q)$ is surjective for all $m \geq l \geq 1$, including $m = \infty$ if $\Phi$ is classical.

Since $H(A)$ is an abelian subgroup of $E(\Phi, A)$ [18, Lemma 28(b)], $D(\Phi, q)$ is a subgroup of $\hat{H}(\Phi, A) \cap L(\Phi, A)$, and the latter group is central in $St(\Phi, A)$ [18, p. 39, Corollary 1]. This also proves $[\hat{H}(\Phi, A), \hat{H}(\Phi, q)] \subset L(\Phi, A) \cap \hat{H}(\Phi, q)$, since $\hat{H}(\Phi, q)$ is normal in $\hat{H}(\Phi, A)$.

If $u \in A^*, v \in (1 + q)^*$, then

$$\hat{h}_\alpha(u)\hat{h}_\beta(v)\hat{h}_\alpha(u)^{-1} = \hat{h}_\beta(u^{(\beta, \alpha)})\hat{h}_\beta(u^{(\beta, \alpha)})^{-1} = \{u^{(\beta, \alpha)}, v\} \hat{h}_\beta(v) \in \hat{K}(\Phi, q).$$

Since $D(\Phi, q)$ is central in $St(\Phi, q)$ by (a), this shows that $\hat{K}(\Phi, q)$ is a normal subgroup of $\hat{H}(\Phi, q)$, hence $\hat{H}(\Phi, q) \subset \hat{K}(\Phi, q)$. Thus $L(\Phi, A) \cap \hat{H}(\Phi, q) \subset L(\Phi, A) \cap \hat{K}(\Phi, q)$.

Given $\hat{h} \in \hat{K}(\Phi, q)$, it follows from [17, 7.7] that we may write $\hat{h} = d\hat{b}_1(u_1) \cdots \hat{b}_l(u_l)$ where $d \in D(q), \hat{b}_i(u_i) = \hat{h}_\alpha(u_i), \alpha_i \in A, \text{ and } u_i \in (1 + q)^*$. Then if

$$1 = \pi(\hat{h}) = b_1(u_1) \cdots b_l(u_l)$$

we must have $u_i = 1$ for all $i$, since $E(\Phi, A)$ is a subgroup of a universal Chevalley group [18, Corollary to Lemma 28]. Hence $\hat{b}_i(u_i) = 1$ for all $i$; that is, $\hat{h} = d \in D(q)$ proving the last inclusion of (b).

Now if $\Phi$ is nonsymplectic, it follows from (2) that $D(\Phi, q) \subset [\hat{H}(\Phi, A), \hat{H}(\Phi, q)]$, and the inclusions in (b) are equalities. If $\Phi$ is symplectic, we may assume $\langle \beta, \alpha \rangle = 2$ and (2) becomes

$$\{u^2, v\} = [\hat{h}_\alpha(u), \hat{h}_\beta(v)].$$

By (1.1), $\{u^2, v\} = \{u, v^2\}$; thus it follows from (3) that if every $v \in (1 + q)^*$ is a square, again

$$D(\Phi, q) \subset [\hat{H}(\Phi, A), \hat{H}(\Phi, q)]$$

which completes the proof of (b).

For fixed $\beta$, let $D_\beta$ be the subgroup of $D(\Phi, q)$ generated by all $\{u, v\}$, $u \in A^*, v \in (1 + q)^*$. Let $\sigma = \sigma_\alpha$ be an element of the Weyl group of $\Phi$. Then relation (R5) and (a) imply
for some $\eta = \pm 1$. This proves $D_\beta \subset D_{\sigma\beta}$ and, by symmetry, $D_\beta = D_{\sigma\beta}$. Since the Weyl group acts transitively on roots of the same length, we have shown that if $\alpha$ and $\beta$ have the same length, $D_\alpha = D_\beta$.

Suppose then that $\beta$ is short and choose a long root $\alpha$ such that $(\beta, \alpha) = 1$. Then by (2)

\[(u, v)_\beta = [\hat{h}_\alpha(u), \hat{h}_\beta(v)] = [\hat{h}_\beta(v), \hat{h}_\alpha(u)]^{-1} = \{v, u\alpha, u\}^{-1}
\]

which proves $D_\beta \subset D_\alpha$. Since by (1.1)(S6) $\{v, u\alpha = \{u\alpha, v\}$ we have shown $D_\alpha = D(\Phi, q)$, proving the first part of (c); the rest of (c) is now an easy corollary.

**Remark.** In view of (1.3) we will usually write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_\alpha$; in that case it is to be understood that the symbol in question is taken with respect to a fixed long root $\alpha$.

2. The relative Bruhat decomposition for a radical ideal.

(2.1) **Lemma.** Let $\alpha \in \Delta$.

(a) $\hat{U}(\Phi, q) = \hat{U}(\Phi_+ - \{\alpha\}, q) \cdot \hat{U}(\alpha, q)$.

(b) $\hat{U}(\Phi_+ - \{\alpha\}, q)$ is normalized by $St_\alpha(A)$.

The set of roots $\Phi_+ - \{\alpha\}$ (resp. $\Phi_- - \{\alpha\}$) is an ideal in the closed sets of roots $\Phi_+$ and $(\Phi_+ - \{\alpha\}) \cup \{\alpha\}$ (resp. $\Phi_-$ and $(\Phi_- - \{\alpha\}) \cup \{\alpha\}$). The lemma thus follows from [18, Lemmas 16, 17, 18, 36].

**Definition.** Set $\hat{M}(\Phi, q) = \hat{U}(\Phi, q)\hat{K}(\Phi, q)\hat{U}(\Phi, q)$, a subset of $St(\Phi, q)$ (cf. (2.7)). Recall from (1.3) that if $\Phi$ is nonsymplectic or if $(1 + q)* = (1 + q)^*$, then $\hat{K}(\Phi, q) = \hat{H}(\Phi, q)$, and that in any case, $\hat{K}(\Phi, q)$ is the product of the central subgroup $D(\Phi, q)$ with the group generated by all $\hat{h}_\alpha(v), v \in (1 + q)^*$. Thus $\pi(\hat{K}(\Phi, q)) = \hat{H}(\Phi, q)$.

(2.2) **Lemma.** $\hat{U}(\Phi, q)\hat{K}(\Phi, q)\hat{M}(\Phi, q) = \hat{M}(\Phi, q) = \hat{M}(\Phi, q)\hat{K}(\Phi, q)\hat{U}(\Phi, q)$.

This follows from relation (R6) which shows that $\hat{H}(\Phi, q)$, and therefore also $\hat{K}(\Phi, q)$, normalizes $\hat{U}(\Phi, q)$ and $\hat{U}(\Phi, q)$.
(2.3) Theorem. (a) The product map
\[ \hat{U}^{-}(\Phi; q) \times \hat{K}(\Phi; q) \times \hat{U}(\Phi; q) \rightarrow \text{St}(\Phi; q) \]
is injective.
(b) \( L(\Phi; A) \cap \hat{M}(\Phi; q) \subseteq \hat{K}(\Phi; q) \).
(c) \( \hat{M}(\Phi; q) = \text{St}(\Phi; q) \) implies \( q \subseteq \text{rad } A \).

Suppose \( \hat{u}, \hat{u}' \in \hat{U}(q), \hat{v}, \hat{v}' \in \hat{U}^{-}(q) \) and \( \hat{k}, \hat{k}' \in \hat{K}(q) \). Then if \( \hat{v} \hat{k} \hat{u} = \hat{v}' \hat{k}' \hat{u}' \), we have
\[ \pi(\hat{v}' \hat{u}'^{-1}) = \pi(\hat{k}' \hat{u}'^{-1} \hat{u}^{-1} \hat{k}^{-1}) \in U^{-}(A) \cap U(A)H(A) = \{1\} \]
by [18, Lemma 21]. Hence \( \hat{v} = \hat{v}' \), since \( \pi[U^{-}(A)] \) is an isomorphism [18, Lemma 36]. Similarly \( \hat{u} = \hat{u}' \), and therefore \( \hat{k} = \hat{k}' \), proving (a).

Now suppose \( \pi(\hat{v} \hat{k} \hat{u}) = 1 \). Then \( \pi(\hat{v}) = \pi(\hat{u}^{-1} \hat{k}^{-1}) \in U^{-}(A) \cap U(A)H(A) = \{1\} \)
implies \( \hat{v} = 1 \); hence \( \hat{u} = 1 \) also, proving (b).

Finally, it is easily checked in \( SL(2; A) \) that \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in U^{-1}HU \) implies \( a \in A^* \). Moreover, \( \phi^{-1}_a(U^{-1}HU) \subseteq U^*HU \), where the decomposition on the right is in \( SL(2; A) \) and \( \phi_a: SL(2; A) \rightarrow E_a(A) \) is the homomorphism of [14, (3.6)].

Applying these remarks to
\[ \left( \begin{array}{cc} 1 + q & -q \\ q & 1 - q \end{array} \right) \in \phi^{-1}_a(\pi(x_\alpha(1)x_{-\alpha}(q)x_\alpha(-1))) \]
for any \( q \in q \), we see that \( \hat{M}(q) = \text{St}(q) \) implies \( (1 + q) \subseteq A^* \) and therefore, \( q \subseteq \text{rad } A \). This proves (c).

The key result of this section is the following partial converse to (2.3)(c):

(2.4) Theorem. Let \((A, q)\) be a radical pair and assume \( A = \mathbb{Z}[A^*] \). Then \( \text{St}(\Phi; q) = \hat{M}(\Phi, q) \).

(2.5) Theorem. Let \((A, q)\) be a radical pair with \( A = \mathbb{Z}[A^*] \), and suppose \( \Phi_l \subseteq \Phi_m \) are reduced irreducible root systems of the same type. Then \( L(\Phi_m; q) \)
is generated by all \( \{u, v\} \in A^*, v \in (1 + q)^* \) for any fixed long root \( \alpha \), and the homomorphisms \( L(\Phi_l; q) \rightarrow L(\Phi_m; q) \) are surjective for all \( m > l \geq 1 \), including \( m = \infty \) if \( \Phi_m \) is classical.

If, in addition, \( \Phi_m \) and \( A \) satisfy one of the hypotheses of [14, Theorem 5.3], \( \text{St}(\Phi_m; (0, q)) \) is the universal \( E(\Phi_m, A) \)-covering [14, §2] of \( E(\Phi_m; q) \).

This theorem is a corollary of (2.3)(b), (2.4) and (1.3).

Note. The hypothesis \( A = \mathbb{Z}[A^*] \) is innocent. It is fulfilled, for example, by semilocal rings having at most one residue field with 2 elements [14, (4.2)] (in particular, by local rings) and by group rings.

The proof of (2.4) will be based on a series of lemmas.
Lemma. Let \( \alpha \in \Delta, t \in A \). Then \( x_\alpha(t) \) normalizes \( M(q) \) if and only if \( x_\alpha(t) \hat{U}(-\alpha, q)x_\alpha(-t) \subset \hat{M}(q) \).

The "only if" is clear. For the converse, we assume \( \alpha \in \Delta \) (the case \( \alpha \in -\Delta \) is similar). By (2.1)(a\(-\)), we have
\[
\hat{M}(q) = \hat{U}(\Phi_- - \{\alpha\}, q) \cdot \hat{U}(-\alpha, q) \cdot \hat{K}(q) \cdot \hat{U}(q).
\]
Since \( x_\alpha(t) \) normalizes \( \hat{U}(\Phi_- - \{\alpha\}, q) \) by (2.1)(b\(\)), and also normalizes \( \hat{U}(q) \), it suffices to prove
\[
x_\alpha(t) \cdot \hat{U}(-\alpha, q) \hat{K}(q) \cdot x_\alpha(-t) \subset \hat{M}(q)
\]
and, in view of the hypothesis and (2.2), that would follow from
\[
x_\alpha(t) \cdot \hat{K}(q) \cdot x_\alpha(-t) \subset \hat{K}(q)\hat{U}(q)
\]
which is true since \( \hat{K}(q) \subset \hat{H}(A) \) and \( \hat{H}(A) \) normalizes \( \hat{U}(q) \) by relation (R6).

Proposition. Let \( u, v \in A^* \), \( \alpha \in \Phi \). The following identities hold in \( St(\Phi, A) \):

\[
\begin{align*}
\{u, v\} & \overset{\alpha}{\hat{\sigma}} \hat{\sigma}(v) \\
(a) & = \hat{x}_\alpha(u^{-1}(1 - v^{-1})) \cdot x_\alpha(-u^{-1}) \cdot x_\alpha(u(v - 1)) \cdot x_\alpha(u(v^{-1} - 1)), \\
& = x_\alpha(-u^{-1}) \cdot x_\alpha(u(v - 1)) \\
(b) & = x_\alpha(u^{-1}(v^{-1} - 1)) \{u, v\} \hat{\sigma}(v) x_\alpha(u(1 - v^{-1})), \\
& = x_\alpha(-u) \cdot x_\alpha(u^{-1}(1 - v)) \\
(c) & = x_\alpha(u^{-1}(v^{-1} - 1)) \{u, v\} \hat{\sigma}(v) x_\alpha(u(1 - v^{-1})).
\end{align*}
\]

Proof. (a)
\[
\begin{align*}
\{u, v\} & \overset{\alpha}{\hat{\sigma}} \hat{\sigma}(v) = \hat{x}_\alpha(u v) \hat{x}_\alpha(u^{-1}) = \hat{w}_\alpha(u v) \hat{w}_\alpha(-u) \\
& = \hat{w}_\alpha(-u^{-1} - 1) \hat{w}_\alpha(-u) \\
& = x_\alpha(-u^{-1} - 1) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
& = x_\alpha(-u^{-1} - 1) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
& = x_\alpha(-u^{-1} - 1) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
& = x_\alpha(u^{-1}(v^{-1} - 1)) x_\alpha(-u^{-1}) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
& = x_\alpha(u^{-1}(v^{-1} - 1)) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
& = x_\alpha(u^{-1}(v^{-1} - 1)) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
& = x_\alpha(u^{-1}(v^{-1} - 1)) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
& = x_\alpha(u^{-1}(v^{-1} - 1)) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
& = x_\alpha(u^{-1}(v^{-1} - 1)) \cdot x_\alpha(u v) \hat{w}_\alpha(-u) \\
(b) & \text{ follows immediately from (a)}.
\]

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(c) In (b) exchange $\alpha$ with $-\alpha$ and $u$ with $u^{-1}$; then take the inverse of each side. The identities $\tilde{b}_{-\alpha}(v)^{-1} = \tilde{b}_{\alpha}(v)$ and $|u^{-1}, v_{-\alpha}|_{\alpha} = |v, u^{-1}l_{\alpha}| = |u, v_{\alpha}|$ complete the proof.

(2.8) Corollary. Let $\alpha \in \Phi$, $q \in \text{rad } A$. For all $u, v, u', v' \in A^*$ such that $u + v = u' + v'$, the symbol $\lambda$, $|\alpha|$ satisfies the identity

\[
|u, (1 + qv)/(1 + qv')|_{\alpha}^1 |v, 1 + qv|_{\alpha}^1 |1 + qv', - (1 + qz)|_{\alpha}^{-1}
\]

(Sq)

\[
= |u', (1 + qz)/(1 + qz')|_{\alpha}^1 |v', 1 + qz'|_{\alpha}^1 |1 + qz', - (1 + qz)|_{\alpha}^{-1}
\]

where $z = u + v = u' + v'$. Moreover if $z \in A^*$, both sides of (S9) equal $|z, 1 + qz|_{\alpha}$.

Since $u + v = u' + v'$, we must have

(1) \[\lambda_{\alpha}(-v)\lambda_{\alpha}(q) = \lambda_{\alpha}(-z)\lambda_{\alpha}(q) = \lambda_{\alpha}(-u')\lambda_{\alpha}(-v')\lambda_{\alpha}(q).\]

We will use (2.7) to put (1) into $\hat{M}(q)$; (S9) will then follow by comparing the terms in $\hat{K}(q)$ which are uniquely determined according to (2.3)(a).

Write $w = 1 - qv \in A^*$. Then $q = v^{-1}(1 - w)$ and $w^{-1} - 1 = qw^{-1}$; applying (2.7)(c) with $u = v, v = w$ yields

(2) \[\lambda_{\alpha}(-v)\lambda_{\alpha}(-q) = \lambda_{\alpha}(qw^{-1})|v, w|_{\alpha}^1 \lambda_{\alpha}(qw^{-1})(-qv^2 - w^{-1}).\]

Similarly write $x = 1 - quw^{-1} = w^{-1}(1 - qz) \in A^*$; then $x^{-1} - 1 = qw(1 - qz)^{-1}$ and we have

(3) \[\lambda_{\alpha}(u)\lambda_{\alpha}(qw^{-1}) = \lambda_{\alpha}(q(1 - qz)^{-1})|u, x|_{\alpha}^1 \lambda_{\alpha}(x)|\lambda_{\alpha}(-qz^2(1 - qz)^{-1}).\]

Combining (2) and (3), and simplifying using relation (R6) and the definition of $\lambda_{\alpha}$, we get the identity

(4) \[\lambda_{\alpha}(-u)\lambda_{\alpha}(-v)\lambda_{\alpha}(q) = \lambda_{\alpha}(q(1 - qz)^{-1})|u, x|_{\alpha}^1 \lambda_{\alpha}(v, w)|_{\alpha}^1 \lambda_{\alpha}(1 - qz)|\lambda_{\alpha}(-qz^2(1 - qz)^{-1}).\]

(It should be noted that in deriving (4) we need only the weaker hypotheses $u, v, 1 - qv, 1 - qu, 1 - qz \in A^*$; this will be important in (2.9) below.) We perform a similar calculation for $\lambda_{\alpha}(-u')\lambda_{\alpha}(-v')\lambda_{\alpha}(q)$; the identity follows by comparing the terms in $\hat{K}(q)$ (noting that $\tilde{b}_{-\alpha}(1 - qz)$ depends only on $z$) and replacing $q$ by $-q$.

Finally if $z \in A^*$, we may use (2.7)(c) to compute $\lambda_{\alpha}(-u)\lambda_{\alpha}(-q)$ directly; comparing $\hat{K}(q)$ terms, we see that $|z, 1 + qz|_{\alpha}$ must equal both sides of (S9).

(2.9) Corollary. Let $u, v \in A^*$, $\alpha \in \Phi$ and write $p = u - 1, q = v - 1$. Then if $pq = 0, |1 + q, 1 + pl_{\alpha}^1| = |\lambda_{\alpha}(q), \lambda_{\alpha}(p)|$.

We will compute the right-hand side using (4) above. Make the substitutions $-u = u, -v = -1, q = -q$ in (4); then $z = -p, 1 = qz = 1 - qp = 1, x^{-1} = w = 1 + q$, and
\[ x^{-a}(p) = x^{-a}(u) x^{-a}(-1) \]

Therefore

\[ [x^{-a}(q), x^{-a}(p)] = \{ -u, x_{-a} x^{-1}, x_{-a}^{-1} \}. \]

But (1.1) implies

\[ \{ -u, x_{-a} x^{-1}, x_{-a}^{-1} \}. \]

and therefore

\[ [x^{-a}(q), x^{-a}(p)] = \{ -1, x_{-a} x^{-1}, x_{-a}^{-1} \} = \{ 1 + q, 1 + p \}. \]

which yields the desired result by interchanging \( a \) and \( -a \).

(2.10) Proposition. Let \((A, q)\) be a radical pair. Then \(\hat{M}(q)\) is a normal subgroup of \(\text{St}(\Phi, Z[A^*])\).

Let us first show that (2.10) completes the proof of (2.4). The hypotheses of (2.4) imply that \(\text{St}(\Phi, A) = \text{St}(\Phi, Z[A^*])\); thus by (2.10), \(\hat{M}(q)\) is a normal subgroup of \(\text{St}(\Phi, A)\) containing all \(\hat{U}(\alpha, q)\). Therefore \(\text{St}(\Phi, q) \subseteq \hat{M}(q)\). But \(\hat{M}(q) \subseteq \text{St}(\Phi, q)\), whence (2.4).

Now let us prove (2.10). \(\text{St}(\Phi, Z[A^*])\) is generated by all \(x_{\alpha}(t), \alpha \in \Delta, t \in A^*\). By (2.6), the set \(\hat{M}(q)\) is normalized by \(\text{St}(\Phi, Z[A^*])\) if and only if \(x_{\alpha}(t) x^{-a}(q) \in \hat{M}(q)\) for all \(\alpha \in \Delta, t \in A^*, q \in q\). Since \(q \subseteq Z[A^*]\), this follows from (2.7)(b) and (c).

Now since \(\hat{U}^{-1}(q) \subseteq \text{St}(\Phi, Z[A^*])\), we have

\[ \hat{M}(q) \hat{M}(q) = \hat{M}(q) \hat{U}^{-1}(q) \hat{K}(q) \hat{U}(q) = \hat{U}^{-1}(q) \hat{M}(q) \hat{K}(q) \hat{U}(q) = \hat{M}(q) \]

by (2.2). Therefore \(\hat{M}(q)\), being the monoid generated by 3 groups, is a group.

Remark. In showing \(\hat{M}(q) = \text{St}(\Phi, q)\) for a radical pair \((A, q)\), the restriction \(A = Z[A^*]\) was needed only in verifying (2.6). In \(\text{SL}(2, A)\), however, it is easy to show that

\[ e_{\alpha}(t) \text{U}(-\alpha, q) e_{\alpha}(-t) \subseteq U^{-1}(q) H(q) U(q); \]

this is simply the matrix equation

\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
q & 1
\end{pmatrix}
\begin{pmatrix}
1 & -t \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
u & 0 \\
qu^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
1 & -t^2 qu^{-1} \\
0 & u^{-1}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}
\]

where \(u = 1 + tq \in A^*\), since \(q \subseteq \text{rad} A\). We conclude

(2.11) Corollary. Let \((A, q)\) be a radical pair. Then

\[ E(\Phi, q) = U^{-1}(q) H(q) U(q). \]
(2.12) Lemma. If \( \text{rk } \Phi \geq 2 \), \( \text{St}(\Phi, \cdot) \) preserves finite products. If \( \text{rk } \Phi = 1 \), \( \text{St}(\Phi, A) \times \text{St}(\Phi, B) \cong \text{St}(\Phi, A \times B)/C \), where \( C \) is the normal subgroup generated by all \( \{x_\alpha((a, 0)), x_{-\alpha}(0, b)\} \).

There is always a surjective homomorphism \( p: \text{St}(\Phi, A \times B) \rightarrow \text{St}(\Phi, A) \times \text{St}(\Phi, B) \) induced by the projections of \( A \times B \) onto its factors. Now \( \text{St}(\Phi, A) \times \text{St}(\Phi, B) \) is generated by all \( (x_\alpha(a), 1), (1, x_\alpha(b)) \), and we may define a map \( s \) backwards by

\[
(x_\alpha(a), 1) \mapsto x_\alpha((a, 0)), \quad (1, x_\alpha(b)) \mapsto x_\alpha((0, b)).
\]

To show this defines an inverse isomorphism to \( p \), we must check that the defining relations of \( \text{St}(\Phi, A) \times \text{St}(\Phi, B) \) are preserved by \( s \). These relations are

(i) the defining relations of \( \text{St}(\Phi, A) \) applied to the generators \( (x_\alpha(a), 1) \),
(ii) the defining relations of \( \text{St}(\Phi, B) \) applied to the generators \( (1, x_\alpha(b)) \),
(iii) \( [x_\alpha((a, 0)), 1, x_\alpha((0, b))] = 1 \) for all \( \alpha, \beta \in \Phi, a \in A, b \in B \).

It is clear that \( s \) preserves (i) and (ii). Moreover relation (R2) in \( \text{St}(\Phi, A \times B) \) shows that \( s \) preserves (iii) whenever \( \beta \neq -\alpha \). Hence the induced map \( \tilde{s}: \text{St}(A) \times \text{St}(B) \rightarrow \text{St}(A \times B)/C \) is an isomorphism, since \( p(C) = 1 \). This completes the proof when \( \text{rk } \Phi = 1 \).

If \( \text{rk } \Phi \geq 2 \), there exist \( \beta, \gamma \in \Phi, \beta, \gamma \neq -\alpha, \) such that

\[
x_{-\alpha}((0, b)) = [x_\beta((0, 1)), x_\gamma((0, b))]y
\]

where \( y \in \hat{U}(S, (0, B)) \), for some \( S \subseteq \Phi \) with \(-\alpha \notin S\). Hence

\[
[x_\alpha((a, 0)), x_{-\alpha}((0, b))] = [x_\alpha((a, 0)), x_\beta((0, 1)), x_\gamma((0, b)))]y = 1
\]

which proves \( C = 1 \) and the lemma.

(2.13) Theorem. Let \( A \) be a semilocal ring with at most one residue field isomorphic to \( \mathbb{F}_2 \), and suppose \( \Phi_1 \subseteq \Phi_m \) are reduced irreducible root systems of the same type. Then the homomorphisms \( \theta(l, m): L(\Phi_l, A) \rightarrow L(\Phi_m, A) \) are surjective for all \( m \geq l \geq 1 \), including \( m = \infty \) if \( \Phi_m \) is classical.

If \( l \geq 2 \), \( L(\Phi_l, A) \) is the central subgroup generated by all \( u, v\alpha, u, v \in A^\times \), for any fixed long root \( \alpha \). This is also true when \( l = 1 \), provided either that \( A \) has no residue field isomorphic to \( \mathbb{F}_2 \) or that \( A \) is a local ring.

If, in addition, \( \Phi_1 \) and \( A \) satisfy one of the hypotheses of [14, Theorem 5.3], \( \text{St}(\Phi_1, A) \) is the universal covering of \( E(\Phi_1, A) \) and \( L(\Phi_1, A) \cong H_2(E(\Phi_1, A), \mathbb{Z}) \).

Write \( \overline{A} = A/\text{rad } A \), a finite product of fields. Steinberg [17] has shown that \( L(\Phi, k) = D(\Phi, k) \) when \( k \) is a field. Since \( E(\Phi, \cdot) \) preserves finite products, it follows from (2.12) that \( L(\Phi, \overline{A}) = D(\Phi, \overline{A}) \) if \( \text{rk } \Phi \geq 2 \), and that \( L(\Phi, \overline{A}) \) is generated by \( D(\Phi, \overline{A}) \) and \( C \) when \( \text{rk } \Phi = 1 \), where \( C \) is the normal subgroup generated by all
\[ [x_a((0, \cdots, k_j, \cdots, 0)), x_{-a}((0, \cdots, k_j, \cdots, 0))] \]

(the appropriate generalization of the subgroup \( C \) of (2.12) when \( \bar{A} \) is a product of more than 2 factors).

Now suppose \( rk \Phi = 1 \). Then if \( A \) is local, \( L(\Phi, \bar{A}) = D(\Phi, \bar{A}) \) by Steinberg [17]. If \( A \) is semilocal but has no residue field isomorphic to \( F_r \), we want to show \( C \subset D(\Phi, \bar{A}) \), and it clearly suffices to consider the case \( \bar{A} = k \times k' \), a product of two fields. Then by (2.9),

\[ [x_a((a, 0)), x_{-a}((0, b))] = \{(l + a, 1), (1, l + b)\}_{a \in D(\Phi, \bar{A})} \]

provided neither \( a \) nor \( b \) equals \(-1\). But even if \( a = -1 \),

\[ [x_a((-1, 0)), x_{-a}((0, b))] = [x_{-a}((0, b)), x_a((1, 0))] \]

\[ = \{(1, 1 + b), (2, 1)\}_{a \in D(\Phi, \bar{A})} \]

and a similar argument applies if \( b = -1 \). Hence if \( -1 \not= 1 \), \( C \subset D(\Phi, \bar{A}) \).

Thus our hypotheses imply \( L(\Phi, q) = D(\Phi, q) \); since \( A^* \to \bar{A}^* \) is surjective, so is \( D(\Phi, q) \to L(\Phi, q) \). But our hypotheses also imply (2.5) for \( q = \text{rad } A \); therefore \( L(\Phi, q) = D(\Phi, q) \) and the second part of the theorem follows from the exact sequence

\[ 1 \to L(\Phi, q) \to L(\Phi, A) \to L(\Phi, \bar{A}) \to 1 \]

together with (1.3).

The first part of the theorem is a consequence of the second and (1.3), and the last part follows from [14, (5.3)].

(2.14) Corollary. Let \( A \) be a semilocal ring with at most one residue field isomorphic to \( F_r \). If \( rk \Phi = 1 \), assume further that either \( A \) is local, or that \( A \) has no residue field isomorphic to \( F_r \). Then \( E(\Phi, A) \) has a presentation by generators \( e_\alpha(t), \alpha \in \Phi, t \in A, \) and relations (R1), (R2) (resp. (R3)) if \( rk \Phi = 1 \) and

\[ (C) b_\alpha(u)b_\alpha(v) = b_\alpha(uv), \quad u, v \in A^*, \alpha \in \Phi. \]

The proof is the same as [18, Theorem 8(b)] in view of (2.13).

Note. Theorems related to (2.14) have been proved by Silvester [11], [12], and Wardlaw [19].

(2.15) Proposition. Let \( \wp, q \) be ideals of \( A \).

(a) If \( rk \Phi = 1 \), assume \( L(\Phi, q) \) is central in \( St(\Phi, A) \). Then if \( St(\Phi, q) \) is generated by \( M(q) \),

\[ [St(\Phi, A), [St(\Phi, q), St(\Phi, \wp)] \subset St(\Phi, \wp q)]. \]
(b) Suppose \( r_k > 1 \) and that \( 2 \in \mathbb{A}^* \) if \( \Phi = C_2 \). If either \( \text{St}(\Phi, q) \) is generated by \( \bar{M}(q) \) or \( \text{St}(\Phi, \beta^2) \) is generated by \( \bar{M}(\beta^2) \), then

\[
[\text{St}(\Phi, q), \text{St}(\Phi, \beta^2)] \subseteq \text{St}(\Phi, \beta q).
\]

Suppose \( M, N \) are normal subgroups of a group \( G \), and define

\[
(M : N) = \{ g \in G : [g, N] \subseteq M \}.
\]

It follows from the commutator formulas of [14, (2.1)] that \((M : N)\) is a normal subgroup of \( G \). The conclusions of the proposition are thus equivalent to

\[
(a') \text{St}(\beta \Phi, \beta q) \subseteq ((\text{St}(\beta q) : \text{St}(\beta q)) : \text{St}(\beta q)),
\]

\[
(b') \text{St}(\beta^2) \subseteq (\text{St}(\beta q) : \text{St}(\beta q)).
\]

The groups on the right in \((a')\) and \((b')\) are normal in \( \text{St}(\Phi, \beta) \); therefore by [14, (2.1)] it suffices to prove

\[
(a'') \widehat{\text{U}}(\alpha, p) \subseteq ((\text{St}(\beta q) : \text{St}(\beta q)) : \text{St}(\beta q)),
\]

\[
(b'') \widehat{\text{U}}(\alpha, p^2) \subseteq (\text{St}(\beta q) : \text{St}(\beta q))
\]

for one root \( \alpha \) of each length.

If \( \beta \neq -\alpha \), (R2) implies that

\[
[\widehat{\text{U}}(\alpha, p), \widehat{\text{U}}(\beta, q)] \subseteq \text{St}(\beta q).
\]

Suppose \( r_k \Phi > 1 \) and that \( 2 \in \mathbb{A}^* \) if \( \Phi = C_2 \). Then (R2) implies the existence of \( \beta, \gamma \in \Phi \) such that

\[
\widehat{\text{U}}(\alpha, p^2) \subseteq [\widehat{\text{U}}(\beta, q), \widehat{\text{U}}(\gamma, q)] \cdot \widehat{\text{U}}(S, p^2)
\]

where \( S \subseteq \Phi \) and \( \alpha \notin S \). Therefore

\[
[\widehat{\text{U}}(\alpha, p^2), \widehat{\text{U}}(-\alpha, q)] \subseteq [\widehat{\text{U}}(\beta, q), \widehat{\text{U}}(\gamma, q)] \cdot \widehat{\text{U}}(S, p^2), \widehat{\text{U}}(-\alpha, q)] \subseteq \text{St}(\beta q).
\]

(The last inclusion follows from [14, (2.1)] and (5).)

Finally, \( \widehat{K}(\Phi, q) \) is generated by elements of the form \( \{u, v|_{\beta} \widehat{h}_{\beta}(v) \}, u \in \mathbb{A}^*, v \in (1 + q)^* \). Therefore since \( \{u, v|_{\beta} \) is central, relation (R6) implies

\[
[x_{\alpha}(p), \{u, v|_{\beta} \widehat{h}_{\beta}(v) \}] = [x_{\alpha}(p), \widehat{h}_{\beta}(v)] = x_{\alpha}(p' q')
\]

for some \( p' \in \beta, q' \in q \), which implies that

\[
[\widehat{\text{U}}(\alpha, p), \widehat{K}(q)] \subseteq \text{St}(\beta q).
\]

Clearly \((b'')\) is a consequence of \((5), (6), (7)\); this is true under either hypothesis of \((b)\) since \((b')\) is equivalent to

\[
\text{St}(q) \subseteq (\text{St}(\beta q) : \text{St}(\beta^2))
\].
From (5) and (7) we also conclude that
\[ \left[ \hat{U}(\alpha, \mathfrak{p}), \text{St}(q) \right] = \text{St}(pq) \cdot \left[ \hat{U}(\alpha, \mathfrak{p}), \hat{U}(-\alpha, q) \right]. \]

It is easily checked, moreover, that in \( SL(2, A) \)
\[ \left[ U(\alpha, \mathfrak{p}), U(-\alpha, q) \right] \subset E(pq) \]
and therefore
\[ \left[ \hat{U}(\alpha, \mathfrak{p}), \text{St}(q) \right] \subset \text{St}(pq) \cdot (L(\Phi, q) \cap \text{St}_A(A)). \]

Since \( L(\Phi, A) \cap \text{St}_A(A) \) is central in \( \text{St}(\Phi, A) \) (by \[14, (5.1)\] if \( \text{rk } \Phi > 1 \) and by hypothesis if \( \text{rk } \Phi = 1 \)), (a) is proved.

(2.16) Corollary. Let \((A, q)\) be a radical pair and assume \( A = Z[A^*] \). If \( \mathfrak{p} \subset A \) is an ideal such that \( pq = 0 \), then \( \left[ \text{St}(\Phi, \mathfrak{p}), \text{St}(\Phi, q) \right] \) is central in \( \text{St}(\Phi, A) \).
Moreover if \( \text{rk } \Phi > 1 \) and \( 2 \in A^* \) if \( \Phi = C_2 \), then for all \( i \geq 2 \),
\[ \left[ \text{St}(\Phi, \mathfrak{p}^i), \text{St}(\Phi, q) \right] = \left[ \text{St}(\Phi, \mathfrak{p}^i), \text{St}(\Phi, q^i) \right] = 1. \]

(2.17) Corollary. Let \((A, q)\) be as in (2.15) and suppose further that \( q^{n+1} = 0 \). Then \( \Gamma = \left[ \text{St}(\Phi, q^i), \text{St}(\Phi, q^j) \right] \) is central in \( \text{St}(\Phi, A) \) if \( i + j \geq n + 1 \).
If \( \text{rk } \Phi > 1 \) and if \( 2 \in A^* \) if \( \Phi = C_2 \), \( \Gamma \) is trivial when \( i + j \geq n + 2 \).

3. Some computations for local rings.

(3.1) Proposition. For any pair \((A, q)\), the sequence
\[ 1 \to L(\Phi, q) \to L(\Phi, A) \to L(\Phi, A/q) \]
is exact.

Except for the "1" on the left, this is just [16, (3.2)]. Exactness at the left holds because the group \( L(\Phi, q) \) used here is the image under the natural homomorphism of the group \( L(\Phi, q) \) of [16], and is therefore a subgroup of \( L(\Phi, A) \).

(3.2) Proposition [17, 3.3]. If \( k \) is an algebraic extension of a finite field, \( L(\Phi, k) = 1 \).

(3.3) Proposition. (a) For every positive integer \( m \) not divisible by 4, \( L(\Phi, Z/mZ) = 1 \), provided \( \text{rk } \Phi \geq 2 \).
(b) For every integer \( n \geq 2 \), the groups \( L(\Phi, Z/2^{n+1}Z) \) and \( L(\Phi, Z/2^nZ) \) are isomorphic and are generated by the symbol \( | - 1, - 1 \), which has order at most 2 if \( \Phi \) is nonsymplectic.

Proof. (a) Since \( L(\Phi, ) \) commutes with finite products, the Chinese remainder theorem implies we may assume \( m = p^n \), \( p \) a prime; we may further assume \( n > 1 \) and \( p \neq 2 \) by (3.2). Since \( Z/p^nZ \) satisfies the hypotheses of (2.13), it follows from (3.2) and from (3.1) with \( q = \text{rad } (Z/p^nZ) = pZ/p^nZ \) that \( L(\Phi, Z/p^nZ) \) is isomorphic
to $L(\Phi, p\mathbb{Z}/p^n\mathbb{Z})$ which, according to (2.5), is generated by all $\{u, v\}, u \in (\mathbb{Z}/p^n\mathbb{Z})^*$, $v \in (1 + p\mathbb{Z}/p^n\mathbb{Z})$.

Now $(\mathbb{Z}/p^n\mathbb{Z})^*$ is a cyclic group of order $(p - 1)p^{n-1}$, isomorphic to the direct product $(\mathbb{Z}/p\mathbb{Z})^* \times (1 + p\mathbb{Z}/p^n\mathbb{Z})$. Hence (1.1)(S7), (S8) imply $\{u, v^2\} = 1$ ($u, v$ as above). Since $p$ is odd, every element of $1 + p\mathbb{Z}/p^n\mathbb{Z}$ is a square, which proves (a).

(b) Again the hypotheses of (2.13) are satisfied. It follows from (1.1)(S1) that $\{-1, -1\}$ is the only possibly nontrivial symbol in $L(\Phi, \mathbb{Z}/4\mathbb{Z})$, and if $\Phi$ is nonsymplectic, (1.1)(S9) implies that the order of this symbol is at most 2. Since $(\mathbb{Z}/2^n\mathbb{Z})^* \to (\mathbb{Z}/2^n\mathbb{Z})^*$ is surjective, we have, by (2.13) and (3.1), an exact sequence

$$1 \to L(\Phi, 2^n\mathbb{Z}/2^{n+1}\mathbb{Z}) \to L(\Phi, \mathbb{Z}/2^{n+1}\mathbb{Z}) \to L(\Phi, \mathbb{Z}/2^n\mathbb{Z}) \to 1$$

for all $n \geq 1$ and all $\Phi$. Thus to complete the proof of (b) it suffices to show

$$L(\Phi, 2^n\mathbb{Z}/2^{n+1}\mathbb{Z}) = 1 \quad \text{for} \quad n \geq 2.$$

Let $n \geq 2$. According to (2.5), $L(\Phi, 2^n\mathbb{Z}/2^{n+1}\mathbb{Z})$ is generated by the symbols $\{1 + 2^n, u\}, u \in (\mathbb{Z}/2^{n+1}\mathbb{Z})^*$, $\{1 + 2^n, v\}, v \in (\mathbb{Z}/2^{n+1}\mathbb{Z})^*$. Now $(\mathbb{Z}/2^{n+1}\mathbb{Z})^*$ is the direct product of the group $(1 + 2^n)$ with the cyclic group of order $2^{n-1}$ generated by the residue class of 5 modulo $2^n$. Moreover, an easy induction argument shows that for all $n \geq 2$,

$$(1) \quad 1 + 2^n = 5^s \mod 2^{n+1}, \quad s = 2^{n-2}.$$

Now assume $n \geq 3$. Then $1 + 2^n$ is a square and (1.1)(S6) implies that $L(\Phi, 2^n\mathbb{Z}/2^{n+1}\mathbb{Z})$ is generated by the two symbols $\{1 + 2^n, -1\}, \{1 + 2^n, 5\}$; since $\{1 + 2^n, -1\} = \{1 + 2^n, 1 + 2^n\} = \{1 + 2^n, 5\}$ by (1), this group is generated by the single symbol $\{1 + 2^n, 5\}$. Again applying (1) and computing in $L(\Phi, \mathbb{Z}/2^{n+1}\mathbb{Z})$, we have $\{1 + 2^n, 5\} = \{5^5, 5\} = 1$ by (1.1)(S8).

Now suppose $n = 2$. Then it follows from (2.5) and (1.1)(S1) and (S4) that $L(\Phi, 4\mathbb{Z}/8\mathbb{Z})$ is also generated by $\{5, -1\}$. Take $q = 2$, $u = u' = -1$, $u' = v = 5$ in (2.8) to conclude that, in $L(\Phi, \mathbb{Z}/8\mathbb{Z})$, $1 = \{5, -1\}$.

Note. For the functor $K_2 = \lim_{\longleftarrow} L(A)$, this proposition was proved by Milnor [9] using his computation of $K_2(\mathbb{Z})$ (cf. [11], [19]) and results of Mennicke, Bass, Lazard and Serre [1] on the congruence subgroup problem.

(3.4) Proposition. Let $A$ be an artinian ring such that $A^*$ is cyclic, and suppose $\text{rk} \Phi \geq 2$. Then $L(\Phi, A) = 1$, except possibly when $A$ has a direct factor isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

Eldridge and Fischer [4] have shown that if $A$ is artinian and $A^*$ is finitely generated, then $A$ is finite. Moreover, a finite ring is a finite product of primary rings $A_1, \ldots, A_n$ (rings with a unique prime ideal); if $A^*$ is cyclic, $A_i^*$ must also be cyclic for $i = 1, \ldots, n$ with $|A_i^*|$ and $|A_j^*|$ relatively prime for $i \neq j$. Gilmer
[5] has determined all primary rings with cyclic groups of units; they are
(a) $F_q$, $q$ a prime power,
(b) $\mathbb{Z}/pm\mathbb{Z}$, $p$ an odd prime, $m > 1$,
(c) $\mathbb{Z}/4\mathbb{Z}$,
(d) $F_p[X]/(X^2)$, $p$ prime,
(e) $F_2[X]/(X^3)$,
(f) $\mathbb{Z}[X]/(4, 2X, X^2 - 2)$.

Since $L(\Phi, )$ commutes with finite products, it suffices to compute $L(\Phi, A)$
when $A$ is one of the rings in (a)–(f) and we may apply (2.13). Propositions 3.2
and 3.3 above settle cases (a)–(c). In (d), (e), (f) we let $x$ denote the residue
class of $X$ in $A$.

In (d) we use (3.1), with $q = \text{rad } A = 1 + Ax$, and (3.2) to conclude that
$L(\Phi, A) \cong L(\Phi, 1 + Ax)$. If $\zeta$ is a generator of $F_p^*$, $A^*$ is the product of the cy-
clic group $\langle \zeta \rangle$ of order $p - 1$ with the cyclic group $\langle 1 + x \rangle = 1 + Ax$ of order $p$.
If $p$ is odd, $1 + x$ is a square, and $L(\Phi, 1 + Ax)$ is generated by $\{\zeta, 1 + x\}$ and
$\{1 + x, 1 + x\}$ according to (2.5) and (1.1)(S6). That these symbols are trivial fol-
lows from (1.1)(S6), (S8).

If $p = 2$ in (d), $\zeta = 1$ and $L(\Phi, 1 + Ax)$ is generated by
\[\{1 + x, 1 + x\} = \{1 + x, - (1 + x)\} = 1\]
by (S4) of (1.1).

In (e) and (f), $A^*$ is cyclic of order 4, generated by $1 + x$, and $L(\Phi, A)$ is
generated by $\{1 + x, 1 + x\}$. In (e) we have
\[\{1 + x, 1 + x\} = \{1 + x, - (1 + x)\} = 1,\]
and in (f)
\[\{1 + x, 1 + x\} = \{1 + x, (1 + x)^{-1}\} = \{1 + x, - (1 + x)\} = 1,\]
which completes the proof of (3.4).

Our next objective is to generalize Proposition 3.3. Throughout the rest of
this section we will assume $A$ is a local ring whose maximal ideal $\mathfrak{p}$ is principal
and generated by $\mu$. We further assume that $A/\mathfrak{p}$ is a finite field containing $q = p^s$
elements.

For $n \geq 0$, the group of units $(A/\mathfrak{p}^{n+1})^*$ is the direct product $\langle \zeta \rangle \times (1 +$ $\mathfrak{p}/\mathfrak{p}^{n+1})$, where $\zeta \in (A/\mathfrak{p}^{n+1})^*$ is of order $q - 1$ and maps to a generator of $(A/\mathfrak{p})^*$
\[\cong (F_q)^*\]. Since $A$ and $A/\mathfrak{p}^{n+1}$ are local, they are generated by their units.

(3.5) Lemma. For all $n \geq 0$ and $1 \leq i \leq n + 1$, the additive group $\mathfrak{p}^i/\mathfrak{p}^{n+1}$
and the multiplicative group $1 + \mathfrak{p}^i/\mathfrak{p}^{n+1}$ have exponent $p^{n-i+1}$. Hence if $p$ is
odd, every element of $1 + \mathfrak{p}^i/\mathfrak{p}^{n+1}$ is a square.

The map $a \mapsto \overline{a} \mu^n$ induces, for all $n \geq 0$, an isomorphism of additive groups
\[A/\mathfrak{p} \cong \mathfrak{p}^n/\mathfrak{p}^{n+1}\]
where we write $\overline{a}$ for the residue class of $a \in A$ modulo $\mathbb{F}_p$. Since $(\mathbb{F}_p/\mathbb{F}_{p^1})^2 = 0$, $1 + \mathbb{F}_p/\mathbb{F}_{p^1} \simeq \mathbb{F}_p/\mathbb{F}_{p^1}$ and both, therefore, have exponent $p$. The lemma follows by descending induction on $i$ and the exact sequences

$$
0 \to \mathbb{F}_{i+1}/\mathbb{F}_{p^1} \to \mathbb{F}_i/\mathbb{F}_{p^1} \to \mathbb{F}_{i+1}/\mathbb{F}_{p^1} \to 0,
$$

$$
1 \to (1 + \mathbb{F}_{i+1}/\mathbb{F}_{p^1}) \to (1 + \mathbb{F}_i/\mathbb{F}_{p^1}) \to (1 + \mathbb{F}_{i+1}/\mathbb{F}_{p^1}) \to 1.
$$

(3.6) Lemma. Let $k$ be a finite field. Every element of $k$ is a sum of squares. Every element of $k$ is a sum of fourth powers if and only if $k \neq \mathbb{F}_2$.

Let $k = \mathbb{F}_q$, $q = p^n$, and let $d$ be a positive nonzero integer. The subset $S$ of $k$ consisting of sums of $d$th powers is closed under addition, multiplication and subtraction, since $-1 = p - 1 = 1^d + \cdots + 1^d$. Hence $S$, being a subdomain of a finite field, is a subfield of $k$, and $S = \mathbb{F}_r$, $r = p^m$ for some $m$ dividing $n$. In particular, $p^m - 1$ divides $p^n - 1$ with quotient $c$.

Choose an $x \in k^*$ of order $p^n - 1$. Then $x^d \in S$ and thus $x^d(p^m - 1) = 1$, which implies $p^n - 1 | d(p^m - 1)$. Hence $c(p^m - 1) | d(p^m - 1)$ and $c | d$. If $d = 2$, then $c = 1$ or 2. If $c = 2$, then

$$
2p^m - 2 = p^n - 1, \quad p^m(2 - p^{n-m}) = 1, \quad p = 1.
$$

Thus $c = 1$ and $n = m$.

If $d = 4$ we must have $c = 1, 2$ or 4, and we have seen above that $c = 2$ leads to a contradiction. If $c = 4$, then

$$
p^m(4 - p^{n-m}) = 3, \quad p = 3, m = 1, n = 2,
$$

and it is easily checked that $(\mathbb{F}_3)^4 = \mathbb{F}_3$.

Note. I would like to thank Armand Brumer who supplied the neat proof of this lemma.

(3.7) Corollary. The symbols \{1 + s, 1 + t\}, $s \in \mathbb{F}_p/\mathbb{F}_{p^1}$, $t \in \mathbb{F}_p/\mathbb{F}_{p^1}$ generate $D(\Phi, \mathbb{F}_p/\mathbb{F}_{p^1})$.

Recall from (1.3) that $D(\Phi, \mathbb{F}_p/\mathbb{F}_{p^1})$ is the subgroup of $L(\Phi, \mathbb{F}_p/\mathbb{F}_{p^1})$ generated by all \{1, 1 + t\}, $u \in (A/\mathbb{F}_{p^1})^*$, $t \in \mathbb{F}_p/\mathbb{F}_{p^1}$. Write $u = \zeta^i(1 + s)$, $s \in \mathbb{F}_p$, where $\zeta$ is of order $q - 1$. Then if $p$ is odd, $1 + s$ is a square by (3.5), and if $p = 2$, $\zeta^i$ is a square. In either case (1.1)(S6) implies

$$
\{u, 1 + t\} = \{\zeta^i(1 + t), 1 + t\}\{1 + s, 1 + t\}
$$

and we must show $\{\zeta^i, 1 + t\} = 1$. Suppose $1 + t$ is a square and let $v \in 1 + \mathbb{F}_p/\mathbb{F}_{p^1}$, $v^2 = 1 + t$. Then $v$ has exponent $p$ by (3.5) and $\zeta^i$ has order prime to $p$. Hence $\zeta^i$ and $v$ generate a cyclic subgroup of $(A/\mathbb{F}_{p^1})^*$ and $\{\zeta^i, 1 + t\} = 1$ by (1.1)(S7) and (S8). If $1 + t$ is not a square, we must have $p = 2$ and $\zeta^i$ is a square; a similar argument applied to $(\zeta^i)^{1/2}$ and $1 + t$ again yields $\{\zeta^i, 1 + t\} = 1$.  

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Lemma. If \( \text{rk } \Phi = 1 \), assume \( A/\mathfrak{p} \not\cong \mathbb{F}_q \). Then \( L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1}) \) is generated by all

\[
[1 + u \bar{\mu}_i, 1 + u \bar{\mu}_n], \quad 1 \leq i \leq n,
\]

where \( u \) is a power of \( \zeta \) and \( \bar{\mu} \) denotes the image of \( \mu \) in \( A/\mathfrak{p}^{n+1} \).

Moreover if \( \Phi \not\equiv A_1, C_2 \), or if \( \Phi = C_2 \) and \( p \) is odd, then these symbols are trivial except possibly when \( i = 1 \).

We begin by proving that the additive group \( \mathfrak{p}^m/\mathfrak{p}^{n+1} \) is generated by all \( \xi \mathfrak{p}^k, m \leq k \leq n \), where \( \xi \) is an even power of \( \zeta \) (resp. \( \xi \) is a fourth power of \( \zeta \) if \( A/\mathfrak{p} \not\cong \mathbb{F}_q \)). By (3.6) this is true if \( m = n \), for \( \mathfrak{p}^n/\mathfrak{p}^{n+1} \) is isomorphic to \( A/\mathfrak{p} \). By definition of \( \zeta \), \( \mathfrak{p}^{m-1}/\mathfrak{p}^{n+1} \) is generated by all \( v \bar{\mu}_k, m - 1 \leq k \leq n \), where \( v \) is a power of \( \zeta \). According to (3.6), \( v \equiv a_1 + \cdots + a_r \) modulo \( \mathfrak{p}/\mathfrak{p}^{n+1} \) where the \( a_i \) are even (resp. fourth) powers of \( \zeta \). Therefore \( v \bar{\mu}_k = a_1 \bar{\mu}_k + \cdots + a_r \bar{\mu}_k + b \) for some \( b \in \mathfrak{p}^m/\mathfrak{p}^{n+1} \); by descending induction on \( m \), \( b \) is of the desired form.

Our hypothesis on \( p \) assures us, by (2.5), that \( L(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1}) = D(\Phi, \mathfrak{p}^n/\mathfrak{p}^{n+1}) \) and is generated, according to (3.7), by all

\[
[1 + s, 1 + \xi \mathfrak{p}^{-n}]_a, \quad s \in \mathfrak{p}/\mathfrak{p}^{n+1},
\]

where \( \xi = b_1 + \cdots + b_r \) is a sum of even (resp. fourth) powers of \( \zeta \), and \( a \) is any fixed long root. (The "resp." statements hold under the hypothesis \( A/\mathfrak{p} \not\cong \mathbb{F}_q \).)

Now if \( \Phi \) is nonsymplectic, there is a \( \beta \in \Phi \) with \( (\alpha, \beta) = 1 \), where \( \alpha \) is the root occurring in (2). We now show that the same is true if \( \Phi = C_l, l \geq 2 \), and \( p \) is odd. In that case \( 1 + s = (1 + s')^2 \) for some \( s' \in \mathfrak{p}/\mathfrak{p}^{n+1} \) by (3.5), and we have, by (4) of §1 and (1.1)(S°3),

\[
[1 + s, 1 + t]_a = [1 + s', 1 + t]_a,
\]

where \( \gamma \in \Phi \) is a short root such that \( (\alpha, \gamma) = 2 \), \( (\gamma, \alpha) = 1 \). Replacing \( \alpha \) by \( \gamma \) in (2), we are done.

Because \( (\mathfrak{p}/\mathfrak{p}^{n+1})(\mathfrak{p}^n/\mathfrak{p}^{n+1}) = 0 \), we may apply (2.9), (2.17), and the commutator identities of [14, (2.1)] to conclude

\[
[1 + s, 1 + \xi \mathfrak{p}^{-n}]_a = [x_{-a}(s), x_{a}(\xi \mathfrak{p}^{-n})]
\]

\[
= [x_{-a}(s), x_{a}(b_1 \bar{\mu}^n) \cdot \cdots \cdot x_{a}(b_r \bar{\mu}^n)]
\]

\[
= [x_{-a}(s), x_{a}(b_1 \bar{\mu}^n)] \cdot \cdots \cdot [x_{-a}(s), x_{a}(b_r \bar{\mu}^n)]
\]

\[
= [1 + s, 1 + b_1 \bar{\mu}^n]_a \cdot \cdots \cdot [1 + s, 1 + b_r \bar{\mu}^n]_a
\]

which shows we may assume in (2) that \( \xi \) itself is an even (resp. fourth) power of \( \zeta \) (and not just a sum of such powers).
Conjugating
\[ l_1 + s, 1 + \xi \bar{\mu}^n \] yields
\[ l_1 + s, 1 + \xi \bar{\mu}^n \] by \( \hat{h}_\alpha(\xi^{1/2}) \). And
\[ l_1 + s, 1 + \xi \bar{\mu}^n \] \[ = [x_{-\alpha}(\xi^{1/2} s), x_{\alpha}(\bar{\mu}^n)] \] \[ = \{l_1 + s, 1 + \bar{\mu}^n\}_\alpha, \]
and \( L(\Phi, \|\mu\|^{n+1}) \) is thus generated by all
\[ \{l_1 + s, 1 + \bar{\mu}^n\}_\alpha, \ s \in \|\mu\|^{n+1}. \]

Now we may write \( s = a_1 \bar{\mu} + \cdots + a_n \bar{\mu}^n \), where each \( a_i \) is a sum of even (resp. fourth) powers of \( \xi \). Arguing as for (3) above, we have
\[ [x_{-\alpha}(a_1 \bar{\mu}), x_{\alpha}(\bar{\mu}^n)] \] \[ = \{l_1 + a_1 \bar{\mu}, 1 + \bar{\mu}^n\}_\alpha, \] \[ = \{l_1 + a_1 \bar{\mu}, 1 + \bar{\mu}^n\}_\alpha, \]
and a further argument of this type shows we may assume each \( a_i \) is itself an even (resp. fourth) power of \( \xi \). We conclude, therefore, from (4) and (5) that
\[ L(\Phi, \|\mu\|^{n+1}) \] is generated by the symbols
\[ \{l_1 + a_i \bar{\mu}, 1 + \bar{\mu}^n\}_\alpha, \ \ 1 \leq i \leq n, \]
where \( a \) is an even (resp. fourth) power of \( \xi \).

Now if \( \Phi \) is nonsymplectic, or if \( p \) is odd and \( \Phi = C_p, \ l \geq 2 \), take \( \beta \) so that \( (\alpha, \beta) = 1 \) and let \( \nu \) be a power of \( \xi \) such that \( \nu^2 = a \). If \( \Phi = A_1 \), or if \( p = 2 \) and \( \Phi = C_2, \ l \geq 2 \), take \( \beta = \alpha \) and let \( \nu \) be a power of \( \xi \) such that \( \nu^4 = a \) (these choices are possible by our hypotheses and the previous discussion). Conjugating (6) by \( \hat{h}_\beta(\nu) \) yields
\[ [x_{-\alpha}(a \bar{\mu}^i), x_{\alpha}(\bar{\mu}^n)] \]
\[ = \{l_1 + a \bar{\mu}^i, 1 + \bar{\mu}^n\}_\alpha, \]
where \( u = \nu^{(\alpha, \beta)} \) is a power of \( \xi \) as desired.

Finally if \( \Phi \notin A_1, C_2 \), or if \( \Phi = C_2 \) and \( p \) is odd, it follows from (2.9) and (2.17) that for \( i > 1 \),
\[ [l_1 + u \bar{\mu}^i, 1 + u \bar{\mu}^n] = [x_{-\alpha}(u \bar{\mu}^i), x_{\alpha}(u \bar{\mu}^n)] = 1. \]

(3.9) Lemma. For every \( u \in A^* \) and all \( n \geq 1 \),
\( (1 + u \mu^k)\mu^{n-k} \equiv 1 + u \mu^{n-k} \mod \|\mu\|^{n+1}, \quad 2 \leq k \leq n. \)
If \( p \neq 2 \), this congruence holds for \( k = 1 \) as well.
Our induction hypothesis implies

\[(1 + u\mu)^{p^{n-k-1}} \equiv 1 + u\rho^{n-k-1}\mu^k \mod \rho^n\]

and, therefore, for some \(s \in \rho^n/\rho^{n+1}\),

\[(1 + u\mu)^{p^{n-k-1}} \equiv 1 + u\rho^{n-k-1}\mu^k + s
\equiv (1 + u\rho^{n-k-1}\mu^k)(1 + s) \mod \rho^{n+1}\]

since \(s\mu^k = 0\).

Thus modulo \(\rho^{n+1}\) we have

\[(1 + u\mu)^{p^{n-k}} = ((1 + u\mu)^{p^{n-k-1}})^p
\equiv (1 + u\rho^{n-k-1}\mu^k)^p(1 + s)^p
\equiv (1 + u\rho^{n-k-1}\mu^k)^p
\equiv 1 + u\rho^{n-k}\mu^k + \sum_{i=2}^{p} \binom{p}{i} (u\rho^{n-k-1}\mu^k)^i\]

since \(1 + \rho^n/\rho^{n+1}\) has exponent \(p\) by (3.5), and it suffices to show

\[\binom{p}{i} p^{n-k-i} - i\mu^k \equiv 0 \mod \rho^{n+1}\]

for \(2 \leq i \leq p\).

According to (3.5), \(\rho^{ki}/\rho^{n+1}\) has additive exponent \(p^{n-ki+1}\). Since \(\binom{p}{i}\) is divisible by \(p\) if \(2 \leq i \leq p - 1\), we must have

\[ni - ki - i + 1 \geq n - ki + 1, \quad 2 \leq i \leq p - 1,\]
\[np - kp - i \geq n - kp + 1.\]

That is, we must have

\[i \geq n/(n-1), \quad 2 \leq i \leq p - 1,\]
\[p \geq (n + 1)/(n - 1).\]

These identities are satisfied except when \(n = 1\) (in which case the lemma is trivial) and when \(p = 2, n = 2\).

This completes the proof when \(p\) is odd. If \(p = 2\), the lemma holds for \(n = 2, k = 2\) and hence by induction for all \((n, k)\) with \(n \geq 2, k \geq 2\). The cases \((n, 1)\), \(n \geq 1\) are true exceptions.

(3.10) Theorem. Let \(A\) be a local ring whose residue field is a finite field with \(q = p^2\) elements and whose maximal ideal \(\rho\) is principal, generated by \(\rho = \mu\), the image of \(p\) in \(A\). If \(\text{rk} \Phi = 1\), assume that \(A/\rho \neq \mathbb{F}_q\). Then for all \(n \geq 0\) and all odd primes \(p\), \(L(\Phi, A/\rho^{n+1}) = 1\). Moreover, if \(p = 2\), the groups \(L(\Phi, A/\rho^{n+1})\) and \(L(\Phi, A/\rho^n)\) are isomorphic for all \(n \geq 2\) and are generated by the \(2^s - 1\) symbols \(\{1 + \zeta^i \rho, 1 + \zeta^j \mu\}, 1 \leq i \leq 2^s - 1, \zeta \in (A/\rho^{n+1})^*\) has

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order $2^s - 1$ and maps to a generator of $A/\hat{p}$. Each of these symbols has order at most 2.

Since $\bar{p} = \bar{\mu}$ generates $\hat{p}/\hat{p}^{n+1}$ (we identify $\bar{p} \in A$ with its image in $A/\hat{p}^{n+1}$), (3.9) implies, for $p$ odd, that

$$1 + u\bar{\mu}^n = 1 + u\bar{p}^{-n-i}\bar{\mu}^i = (1 + u\bar{\mu}^i)\rho^{n-i}$$

and it follows from (3.8) that $L(\Phi, \hat{p}/\hat{p}^{n+1})$ is generated by all

$$\{1 + u\bar{\mu}^i, (1 + u\bar{\mu}^i)\rho^{n-i} \ | \ 1 \leq i \leq n,$$

where $u$ is a power of $\zeta$. Since $p$ is odd, (3.5) implies that $1 + u\bar{\mu}^i$ is a square, and

$$\{1 + u\bar{\mu}^i, (1 + u\bar{\mu}^i)\rho^{n-i} \ | \ 1 \leq i \leq n,$$

by (1.1)(S6), (S7) and (S8). The first part of the theorem now follows by induction on $n$ from (3.2) and the exact sequence

$$(8) \quad 1 \to L(\Phi, \hat{p}/\hat{p}^{n+1}) \to L(A/\hat{p}^{n+1}) \to L(A/\hat{p}^{n}) \to 1.$$

Suppose, then, that $p = 2$. The above argument still applies if $2 \leq i \leq n$, and we conclude that

$$\{1 + u\bar{\mu}^i, 1 + u\bar{\mu}^n \} = \{1 + u\bar{\mu}^i, (1 + u\bar{\mu}^i)^2\rho^{n-i} \} = 1$$

so long as $2 \leq i \leq n$ and $(1 + u\bar{\mu}^i)^2\rho^{n-i}$ is a square; that is, when $n - i \geq 1$. Thus these symbols are trivial whenever $n \geq i + 1 \geq 3$ and $i \geq 2$.

If $i = 1$, it follows from the argument of (3.8) that we may assume $u = \zeta^2$ is an even power of $\zeta$. Then, since we may take $\bar{\mu} = 2$, we have

$$\{1 + \zeta^{2k}\bar{\mu}, 1 + \zeta^{2k}\bar{\mu}^n \} = [x_{-\alpha}(\zeta^{2k}\bar{\mu}), x_{\alpha}(\zeta^{2k}\bar{\mu}^n)]$$

$$= x_{-\alpha}(-\bar{\mu})^{\alpha}(\xi) [x_{-\alpha}(\zeta^{2k}\bar{\mu}), x_{\alpha}(\zeta^{2k}\bar{\mu}^n)]$$

$$= x_{-\alpha}(\xi) [x_{-\alpha}(\bar{\mu}), x_{\alpha}(\zeta^{4k}\bar{\mu}^n)] = [x_{\alpha}(\zeta^{4k}\bar{\mu}^n), x_{-\alpha}(\bar{\mu})]$$

$$= \{1 + \zeta^{4k}\bar{\mu}^n, 1 - \bar{\mu}\rho^{-1} \} = \{(1 + \zeta^{4k}\bar{\mu}^2)^{2n-2}, -1\rho^{-1} \} = 1$$

if $n - 2 \geq 1$; that is if $n \geq 3$. Thus we have shown that $L(\Phi, \hat{p}/\hat{p}^{n+1}) = 1$ for all $n \geq 3$.

Finally suppose $n = 2$, and continue to take $\bar{\mu} = 2$. Then the characteristic of $A/\hat{p}^3$ is 8, and for any $u \in A^*$,

$$\{1 + 4u, 1 + 4u \} = [x_{-\alpha}(4u), x_{\alpha}(4u)]$$

$$= \hat{w}_{\alpha}^1 [x_{-\alpha}(4u), x_{\alpha}(4u)] = [x_{\alpha}(4u), x_{-\alpha}(4u)] = \{1 + 4u, 1 + 4u\rho^{-1} \}.$$
Thus \(|1 + 4u, 1 + 4u| = 1\) for any \(u \in A^*\). Now \(L(\Phi, p^2/p^3)\) is generated by the symbols \(|1 + 4u, 1 + 4u|, |1 + 2u, 1 + 4u|\). But

\[
|1 + 4u, 1 + 4u| = |x_{-\alpha}(4u), x_{\alpha}(4u)| = |x_{-\alpha}(2u), x_{\alpha}(4u)| = 1 + 2u, 1 + 4u^2
\]

and we may take the symbols \(|1 + 2u, 1 + 4u|, u = \zeta^2k\), as generators. But (9), (10), (11) then imply

\[
|1 + 2\zeta^2k, 1 + 4\zeta^2k| = |1 + 4\zeta^{4k}, -1|^{-1}
\]

\[
= |1 + 4\zeta^{2k}, 1 + 4\zeta^{4k}|^{-1} = |1 + 4\zeta^{4k}, 1 + 4\zeta^{4k}|
\]

\[
= |x_{-\alpha}(4\zeta^{4k}), x_{\alpha}(4\zeta^{4k})| = |x_{-\alpha}(2\zeta^{4k}), x_{\alpha}(4\zeta^{4k})|^2
\]

\[
= |1 + 2\zeta^{4k}, 1 + 4\zeta^{4k}|^2 = |1 + 4\zeta^{8k}, -1|^{-2} = 1.
\]

(Note that the last 3 lines of this computation follow from (9) by substituting \(2k\) for \(k\).)

Thus by (8), \(L(\Phi, A/p^n + 1) \approx L(\Phi, A/p^n)\) for all \(n \geq 2\) as stated. If \(n = 1\), then (8) and (3.2) imply \(L(\Phi, A/p^2) \approx L(\Phi, p/p^2)\) is generated by the symbols \(|1 + w, 1 + w|\) where \(w = \zeta^i, 1 \leq i \leq 2^s - 1\). Since the characteristic of \(A/p\) is 4, an argument similar to (10) shows that each of these symbols has order at most 2.

**3.11 Corollary.** Under the hypothesis of (3.10) assume further that \(p\) is nilpotent. Then if \(p\) is odd, \(L(\Phi, A) = 1\), and if \(p = 2\), \(L(\Phi, A)\) is generated by the \(2^s - 1\) symbols \(|1 + \zeta^i, 1 + \zeta^i|, 1 \leq i \leq 2^s - 1\), which have order at most 2.

The corollary follows from the theorem, since if \(p^{n+1} = 0, A/p^{n+1} = A\).

**3.12 Corollary.** Let \(\mathcal{O}\) be the ring of integers in an algebraic number field and let \(0 \neq \mathfrak{p} \subset \mathcal{O}\) be a prime ideal which is unramified over \(p\mathcal{O} = \mathfrak{p} \cap \mathbb{Z}\). If \(rk \Phi = 1\), assume that \(\mathcal{O}/\mathfrak{p} \neq F_q\). Then if \(p\) is odd, \(L(\Phi, \mathcal{O}/p^n + 1) = 1\) for all \(n \geq 0\). Moreover, if \(p = 2\), the groups \(L(\Phi, \mathcal{O}/p^n + 1)\) are isomorphic for all \(n \geq 1\) and are generated by the \(2^s - 1\) symbols \(|1 + 2\zeta^i, 1 + 2\zeta^i|, 1 \leq i \leq 2^s - 1\), where \(|\mathcal{O}/\mathfrak{p}| = 2^s\) and \(\zeta \in (\mathcal{O}/p^n + 1)^*\) has order \(2^s - 1\) and maps to a generator of \((\mathcal{O}/p)^*\). These symbols have order at most 2.

This follows from (3.11) with \(A = \mathcal{O}/p^{n+1}\).

**Note.** For the groups of type \(A_p, l \geq 2\), this corollary is due to Christofides [2].

4. Stability for \(H_2(E(\Phi, A), \mathbb{Z})\). Throughout this section, \(A\) denotes a local ring with maximal ideal \(\mathfrak{p}\). We set \(k = A/\mathfrak{p}\), but do not assume that \(k\) is finite or that \(\mathfrak{p}\) is principal, as in §3.

We fix an \(l > 1\) (depending on \(\Phi\) and \(A\)) such that \(L(\Phi, A) \approx H_2(E(\Phi, A), \mathbb{Z})\)
and write $\Phi = \Phi_f$. It follows from [14, Theorem 5.3] that for a given $A$ and $\Phi$ there is an $l_0 \geq 1$ such that every $l \geq l_0$ satisfies this condition, and it is clear that $l_0$ depends only on $\Phi$ and $A/\text{rad } A = k$.

We abbreviate the functors $St(A, \cdot)$ and $L(A, \cdot)$ by $St(A)$ and $L(A)$ and we write $H_i(G)$ for the homology groups $H_i(G, \mathbb{Z})$ of the group $G$, $i = 1, 2$. Recall that the functor $E(A, \cdot)$ is $\text{SL}_2(A)$.

(4.1) Theorem. $H_2(\text{SL}_2(A)) \to H_2(E(\Phi, A))$ is surjective whenever $|k| \geq 4$.

Apply the homology spectral sequence [6] to the diagram of group extensions

\[
1 \to L_1(A) \to St_1(A) \to \text{SL}_2(A) \to 1
\]

\[
1 \to L(\Phi, A) \to St(\Phi, A) \to E(\Phi, A) \to 1
\]

to obtain the following commutative diagram with exact rows:

\[
\begin{array}{c}
H_2(\text{SL}_2(A)) \phi \to L_1(A) \to St_1(A)_{ab} \to \text{SL}_2(A)_{ab} \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
H_2(E(\Phi, A)) \xrightarrow{\cong} L(\Phi, A) \\
\downarrow \\
1
\end{array}
\]

(1)

The surjectivity of $L_1(A) \to L(\Phi, A)$ is a consequence of (2.13). If $|k| \geq 4$, there exists $u \in A^*$ with $u^2 - 1 \in A^*$ and by [14, (4.4)], $St_1(A)_{ab} = 0$. Thus the theorem follows from (1).

We shall require the following unpublished result of Bass.

(4.2) Lemma. Let $q \subset A$ be the ideal generated by all $u^2 - 1$, $u \in A^*$. If $k = F_2$, assume that $p$ is principal, generated by $\mu$. Then $St_1(A)_{ab} \cong St_1(A/q)_{ab}$ and both groups are quotients of $A/q$. Moreover, $q = A$ except in the following cases:

- $k = F_3$, $q = p$, $A/q = F_3$,
- $k = F_2$, $2 \in pA$, $q = 2A$, $A/q = \mathbb{Z}/2^n\mathbb{Z}$, $n = 1, 2$ or $3$,
- $k = F_2$, $\mu A = 2A$, $q = \mu^2A$, $A/q \cong F_2[\mathbb{X}]/(\mathbb{X}^2)$.

Denote the image in $St_1(A)_{ab}$ of $g \in St_1(A)$ by $[g]$, and set $(t) = [x_{\alpha}(t)]$ for $t \in A$. It follows from relation (R1) that $t \mapsto (t)$ is a homomorphism $A^+ \to St_1(A)_{ab}$. By relation (R3)

\[
\tilde{w}_\alpha(u)x_{-\alpha}(t)\tilde{w}_\alpha(-u) = x_{\alpha}(-u^2t), \quad u \in A^*,
\]

we have $[x_{-\alpha}(t)] = (-u^2t)$; hence $t \mapsto (t)$ is surjective. Moreover by (R6)

\[
[x_{\alpha}(t)] = x_{\alpha}((u^2 - 1)t)
\]
and therefore \( \langle t \rangle = 0 \) for \( t \in q \). This proves that \( \text{St}_1(A)^{ab} \) is a quotient of \( A/q \) and that \( \text{St}_1(q) \subset \text{St}_1(A), \text{St}_1(A) \). Hence there is a surjective homomorphism \( \text{St}_1(A/q) \rightarrow \text{St}_1(A)^{ab} \) which factors through \( \text{St}_1(A/q)^{ab} \); the projection \( \text{St}_1(A)^{ab} \rightarrow \text{St}_1(A/q)^{ab} \) is an inverse to this induced homomorphism.

Now let us determine the ideal \( q \). Since \( A \) is local, \( q = A \) if and only if \( |k| \geq 4 \). If \( k = F_3 \), we have \( A^* = \{1 + x, x - 1, x \in \mathfrak{p}\} \). Hence if \( u \in A^*, u^2 - 1 = x(2 + x) \) or \( x(x - 2) \) for some \( x \in \mathfrak{p} \); since \( 2 + x, 2 - x \in A^* \), this proves \( q = \mathfrak{p} \).

If \( k = F_2 \), write \( 2A = \mu^eA \) with \( e = \infty \) if \( 2A = 0 \). If \( e = 1 \) we may assume \( \mu = 2 \), and \( (1 + 2x)^2 - 1 = 4x + 4x^2 \equiv 0 \mod 8A \). Taking \( x = 1 \), we see that \( q = 8A \) and, therefore, that \( A/q \cong \mathbb{Z}/2^n\mathbb{Z}, n = 1, 2 \) or 3. If \( e > 1 \), write \( 2 = \mu^e v, v \in A^* \).

Then \( (1 + \mu)^2 - 1 = 2\mu + \mu^2 = \nu\mu^{e+1} + \mu^2 = \mu^2(1 + \nu\mu^{e-1}) \).
Since \( 1 + \nu\mu^{e-1} \in A^*, q = \mu^2 A \) and \( A/q \cong F_2[X]/(X^2) \) as desired.

(4.3) Theorem. The map
\[ H_2(SL_2(A)) \rightarrow H_2(E(\Phi, A)) \]
is surjective if \( k \approx F_3 \).

It suffices, by (1), to show that \( L_1(A) \rightarrow \text{St}_1(A)^{ab} \) is 0, and this map factors, by (4.2), as
\[ L_1(A) \twoheadrightarrow \text{St}_1(A)^{ab} \]
\[ 1 \rightarrow L_1(A/q)^{ab} \rightarrow \text{St}_1(A/q)^{ab} \]

But \( L_1(A/q) = L_1(F_3) = 1 \) by (3.2).

(4.4) Lemma. Let \( \{u, v\} \in L_1(A) \). Then \( \{u, v\} = (3(u - 1)(v - 1)) \) in \( \text{St}_1(A)^{ab} \). Moreover, \( \{u, v\} \) lies in the image of \( H_2(SL_2(A)) \) if and only if \( \{u, v\} = 1 \).

Since \( [x_{-a}(t)] = (-u^{-1}t) \) (cf. the proof of (4.2)), taking \( t = -u^{-1} \), we have \( [x_{-a}(-u^{-1})] = (u) \). Hence \( [\hat{\omega}(u)] = [x_a(u)x_{-a}(-u^{-1})x_a(u)] = (3u) \) and \( [\hat{b}_a(u)] = [\hat{\omega}_a(u)\hat{\omega}_a(-1)] = (3(u - 1)) \). Finally,\n\[ \{u, v\} = [\hat{b}_a(uv)\hat{b}_a(u)\hat{b}_a(v)^{-1}] \]
\[ = (3(uv - 1) - 3(u - 1) - 3(v - 1)) = (3(uv - 1 - u + 1 - v + 1)) = (3(u - 1)(v - 1)) \).

Now consider the commutative diagram
\[ 1 \rightarrow L_1(A) \twoheadrightarrow \text{St}_1(A) \rightarrow SL_2(A) \\
\]
\[ 1 \rightarrow L_1(A)/\phi(H_2(SL_2(A))) \rightarrow \text{St}_1(A)^{ab} \rightarrow SL_2(A)^{ab} \rightarrow 1 \]

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Its columns and top row are clearly exact. Since the bottom row is obtained by factoring out the image of $H_2(SL_2(A))$ from the top row of (1), it too is exact. The second part of the lemma follows easily from (2).

(4.5) Proposition. The map $H_2(SL_2(Z/2^nZ)) \to L_1(Z/2^nZ)$ is surjective for $n = 1, 2$ but not for $n \geq 3$. Therefore the map

$$H_2(SL_2(Z/4Z)) \to H_2(E(\Phi, Z/4Z))$$

is surjective.

It is clear from (1) that the second statement is implied by the first. For $n = 1$, the first assertion is trivial since $L_1(Z/2Z) = 1$ by (3.2). Now $L_1(Z/4Z)$ is generated by the symbol $\{-1, -1\}$ whose image in $St_1(Z/4Z)^{ab}$ is $(3(-1-1)(-1-1)) = 1$. This completes the proof for $n = 2$ by (4.4).

Now suppose $n \geq 3$. According to (4.2), $St_1(Z/2^nZ)^{ab} \cong St_1(Z/8Z)^{ab}$ for all $n \geq 3$; thus (1) implies that

$$\phi: H_2(SL_2(Z/2^nZ)) \to L_1(Z/2^nZ)$$

is surjective for $n = 3$ if and only if $\phi$ is surjective for all $n \geq 3$.

Suppose that this is the case. Then from (1) we have

$$St_1(Z/2^nZ)^{ab} \cong SL_2(Z/2^nZ)^{ab}$$

for all $n \geq 3$, and the same must be true for the 2-adic integers

$$St_1(\hat{Z}_2)^{ab} \cong SL_2(\hat{Z}_2)^{ab}.$$  

Hence $H_2(SL_2(\hat{Z}_2)) \to L_1(\hat{Z}_2) \to L_\infty(\hat{Z}_2) = K_2(\hat{Z}_2)$ is surjective by (1) and (2.13).

Dualizing, we have

$$\text{Hom}(H_2(SL_2(\hat{Z}_2)), Q/Z) \cong H^2(SL_2(\hat{Z}_2), Q/Z)$$

by the universal coefficient theorem [7, p. 77]. But $H^2(SL_2(\hat{Z}_2), Q/Z) = 0$ [1, Proposition 2]. Therefore if $\phi$ is surjective, we conclude that $K_2(\hat{Z}_2) = 0$; in particular $|-1, -1| = 0$ in $K_2(\hat{Q}_2)$. But it follows from results of Moore [10] and Matsumoto [8] that $|-1, -1| \neq 0$ in $K_2(\hat{Q}_2)$, whence the proposition.

(4.6) Corollary. The symbol $\{-1, -1\}$ is nontrivial in $L_1(Z/4Z)$.

Since $|-1, -1|$ generates $L_1(Z/4Z)$, if it is 1 we conclude from (3.1) that $L_1(Z/8Z) \approx L_1(4Z/8Z)$ is generated by the symbols $\{1 + 4a, 1 + 2b\}$, $a, b \in Z$. But in $St_1(Z/8Z)^{ab}$, $\{1 + 4a, 1 + 2b\} = (3(4a)(2b)) = 0$, which implies that $H_2(SL_2(Z/8Z)) \to L_1(Z/8Z)$ is surjective by (4.4). This contradicts (4.5).

Note. Despite (4.6), we cannot conclude that $|-1, -1| \neq 0$ in $K_2(Z/4Z)$ since $K_2(Z/4Z)$ is a quotient of $L_1(Z/4Z)$ by (2.13).
Added in proof. Much more extensive information on the functor $K_2 = \lim_{l \to \infty} L(A_l)$ has been obtained since this paper was written. Dennis ([20], [21]) has proved the conjecture of the Introduction, showing that when $\Phi$ is of type $A_l$, the maps $\theta(l, m)$ are surjective for all $m \geq l \geq d + 3$, where $d$ is the dimension of the maximal ideal space of $A$.

The results concerning $K_2$ of a semilocal ring (Theorem 2.13) have been completed by Stein and Dennis [24]. They have also proved ([22], [23]) that for nonsymplectic $\Phi$, the maps $\theta(l, m)$ are injective (and hence isomorphisms) when $A$ is a discrete valuation ring or a quotient thereof, and they have given a presentation of the $K_2$ of such a ring. These papers also compute $K_2$ of a ring of algebraic integers modulo any nonzero ideal, generalizing the results of §3. Among the consequences of this computation is the nontriviality of the symbol $[-1, -1] \in K_2(Z/4Z)$ (see the Note at the end of §4).

REFERENCES

3. R. K. Dennis, *Universal GE rings and the functor $K_2$* (unpublished manuscript; see [24]).


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