ISOTOPIC UNKNOTTING IN $F \times I$

BY

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ABSTRACT. This paper is essentially a generalization of Unknotting in $M^3 \times I$, by E. M. Brown. The major results in this paper concern the existence of ambient isotopies of unknotted arcs (families of arcs) properly embedded in $F \times I$.

I. Introduction. This paper is essentially a generalization of [1]. Loosely speaking when Brown shows the existence of a homeomorphism having certain properties, we show the existence of an isotopy having many of the same properties. Our major results concern the existence of an ambient isotopy of unknotted arcs in $F \times I$. The results in this paper come from the author’s doctoral thesis at Dartmouth College; however the proofs have been simplified by following some of the techniques used by Waldhausen in [6]. We also adopt much of the terminology used in [6]. The author would like to thank E. M. Brown of Dartmouth College for directing his dissertation and suggesting these problems. The author would also like to thank the referee for a number of helpful suggestions and for pointing out the necessity of an added hypothesis in the statement of Lemma 3.4.

II. Notation. In this paper all spaces are simplicial complexes and all maps are piecewise linear. We shall denote the boundary, closure, and interior of a subspace $X$ of a space $Y$ by $\text{bd}(X, Y)$, $\text{cl}(X, Y)$, and $\text{int}(X, Y)$ respectively. Since no confusion can result, we will be able to abbreviate these by $\text{bd}(X)$, $\text{cl}(X)$, and $\text{int}(X)$. The symbol $F$ will denote a compact, connected surface and $I$ the unit interval. A surface $F$ is properly embedded in a 3-manifold $M$ if $F \cap \text{bd}(M) = \text{bd}(F)$. A surface $F$ properly embedded in a 3-manifold $M$ is incompressible in $M$ if for every disk $D$ embedded in $M$ such that $D \cap F = \text{bd}(D)$, $\text{bd}(D)$ is nullhomotopic in $F$.

Let $F$ be a two-sided surface properly embedded in the 3-manifold $M$. Then the manifold $M'$, obtained by splitting $M$ along $F$, has by definition the properties:

1. $\text{bd}(M')$ contains surfaces $F_1$ and $F_2$ which are copies of $F$.
2. Identification of $F_1$ and $F_2$ gives a natural projection $(M', F_1 \cup F_2) \to (M, F)$.

We may also speak of splitting $M$ along a one-sided surface properly embedded in $M$.

An isotopy of a homeomorphism $h : X \to Y$ is a map $H : X \times I \to Y$ such that for $h_t = H|X \times \{t\}$, we have $h_0 = h$ and $h_t$ is a homeomorphism onto $Y$. An
isotopy of subspaces $Z_1$ and $Z_2$ in $X$ is an isotopy of the identity map on $X$ such that $h_1(Z_1) = Z_2$.

Let $M = F \times I$ and $p : M \to F$ be the natural projection map onto the factor $F$. A subspace $X$ of $M$ is called vertical if $X = p^{-1}(p(X))$. Let $\alpha$ be an arc properly embedded in $M$ such that $p(\text{bd}(\alpha))$ is a point in $\text{int}(F)$. We shall say that $\alpha$ is homeomorphically unknotted (or $h$-unknotted) if there is a homeomorphism $h : M \to M$ such that $h|_{\text{bd}(F) \times I \cup F \times \{0\}}$ is the identity and $h(\alpha)$ is vertical. We shall say that $\alpha$ is isotopically unknotted (or $i$-unknotted) if $\alpha$ is isotopic to a vertical subspace of $M$ leaving $\text{bd}(M)$ fixed. Similarly we can consider collections of arcs.

The major results in this paper concern necessary and sufficient conditions for an arc (collection of arcs) which are $h$-unknotted to be $i$-unknotted.

We shall try to follow Waldhausen’s principal of “induction on niceness” (see [6, p. 58]). That is, after we convince ourselves that there is no obstruction to achieving some niceness, we take up our problem again assuming that niceness.

Let $G_1$ and $G_2$ be incompressible surfaces properly embedded in a 3-manifold $M$. We suppose that $M$ is irreducible (every embedded 2-sphere bounds a 3-ball).

We suppose that there is a simple loop $\lambda \subset G_1 \cap G_2$ such that $\lambda \neq \text{bd}(G_1)$. If $\lambda$ is nullhomotopic in $G_1$ (and thus in $G_2$), $\lambda$ bounds a disk both in $G_1$ and in $G_2$. We may assume that $\lambda$ bounds an “innermost” disk on $G_1$ and thus that the union of the disks bounded by $\lambda$ on $G_1$ and $G_2$ is a 2-sphere $S^2$ embedded in $M$. Now it is not hard to show that there is an isotopy leaving $\text{bd}(M)$ fixed so that $h_1(G_1) \cap G_2$ is “less complicated” than $G_1 \cap G_2$. The isotopy above is found by pushing $G_1$ across the ball bounded by $S^2$. We will use this fact repeatedly in the proofs to follow.

III. Useful results. In this section we prove a number of results which will be useful in the proofs of the theorems on isotopic unknotting of arcs.

A restriction of the following lemma has also been proved by W. Heil in his thesis at Rice University.

**Lemma 3.1.** Let $F$ be a surface other than the 2-sphere and $M = F \times I$. In $M$, let $G$ be a system of properly embedded surfaces, such that each component of $G$ is either a disk which intersects $\text{bd}(F) \times I$ in two vertical arcs or an incompressible annulus which has one boundary curve in $\text{int}(F \times \{0\})$ and the other in $\text{int}(F \times \{1\})$. Then there is an isotopy constant on $F \times \{0\} \cup \text{bd}(F) \times I$ which makes $G$ vertical.

**Proof.** In case $F$ is not the projective plane the proof of this lemma is the same as that of Lemma 3.4 in [6] except that we use Theorem 2.1 or Theorem 3.3 in [4] to do the work of Baer's theorem because Baer's theorem applies only to orientable manifolds.

We now assume that $F$ is the projective plane. Let $G_1$ be a component of $G$. Since $F$ is without boundary, $G_1$ is an annulus. Let $\lambda_1 \times \{0\} = G_1 \cap F \times \{0\}$ and $\lambda_1 \times \{1\} = G_1 \cap F \times \{1\}$. We observe that neither $\lambda_1$ nor $\lambda$ separates a regular neighborhood of itself in $F$. Thus $\lambda_1 \cap \lambda \neq \emptyset$. It is easily seen that after a
deformation constant on $F \times \{0\}$, we may assume that $\lambda_1 \cap \lambda$ is a single point. After a small deformation constant on $\text{bd}(M)$, we may assume that $\lambda \times I \cap G_1$ consists of $\lambda \times \{0\}$, a number of simple closed loops in the interior of $\lambda \times I$, and a number of simple arcs with their endpoints lying in $\lambda \times \{0, 1\}$. Since one of the arcs in $\lambda \times I \cap G_1$ runs from $\lambda \times \{0\}$ to $\lambda \times \{1\}$, every simple loop in $\lambda \times I \cap G_1 \cap \text{int}(M)$ is nullhomotopic in $\lambda \times I$. Since $F \times I$ is irreducible, after a deformation constant on $\text{bd}(M)$ we may assume that $\lambda \times I \cap G_1$ consists of $\lambda \times \{0\}$ together with a number of arcs with their endpoints in $\lambda \times \{0, 1\}$. If such an arc has both its endpoints on $\lambda \times \{0\}$, it together with an arc in $\lambda \times \{0\}$ bounds a disk in $\lambda \times I$. Thus, after a deformation leaving $\text{bd}(M)$ fixed, we may assume that $\lambda \times I \cap G_1$ consists of $\lambda \times \{0\}$ and a single arc running from $\lambda \times \{0\}$ to $\lambda \times \{1\}$.

One observes that the arc in $\lambda \times I \cap G_1$ is isotopic in $\lambda \times I$ to a vertical arc mod $\lambda \times \{0\}$ and that one can cover each stage of the isotopy with a 3-ball meeting $\lambda \times I$ in a disk. After another deformation leaving $F \times \{0\}$ fixed, we may assume that the arc running from $\lambda \times \{0\}$ to $\lambda \times \{1\}$ is vertical. Finally after a deformation leaving $F \times \{0\}$ fixed, we may assume that $G_1 = \lambda \times I$ since one obtains a 3-ball by splitting $M$ along $\lambda \times I$.

Since $M - \lambda \times I$ is simply connected, it can be seen that $G_1$ was the only component of $G$ and the proof of Lemma 3.1 is complete.

**Lemma 3.2.** Let $F$ be a surface other than the 2-sphere. Let $M = F \times I$ and $h : M \to M$ a homeomorphism which is the identity on $F \times \{0\} \cup \text{bd}(F) \times I$. Then there is an isotopy constant on $F \times \{0\} \cup \text{bd}(F) \times I$ of $h$ to the identity.

**Proof.** The proof of Lemma 3.2 is essentially the same as that given for Lemma 3.5 in [6] except that we refer to Lemma 3.1 above.

**Lemma 3.3.** Let $F$ be a surface other than the 2-sphere. Let $M = F \times I$. Let $D_1$ and $D_2$ be disks properly embedded in $M$ such that $\text{bd}(D_1) = \text{bd}(D_2)$. Then there is an isotopy of the subspaces $D_1$ and $D_2$ constant on $\text{bd}(M)$.

**Proof.** Since every 2-sphere embedded in $M$ bounds a 3-cell in $M$, this is an easy application of the techniques used to prove Lemmas 3.1 and 3.2 above.

Let $\lambda$ be a simple loop properly embedded in a surface $F$. Then $\lambda$ is nonorientable if it fails to separate a regular neighborhood of itself in $F$.

**Lemma 3.4.** We suppose that our surface $F$ is not the 2-sphere or projective plane. Let $\lambda$ be a simple non-nullhomotopic loop properly embedded in $F$ such that $F - \lambda$ is not the interior of an annulus. Let $A$ be an annulus properly embedded in $M = F \times I$ such that $\text{bd}(A) = \lambda \times \{0, 1\}$. Then the subspaces $\lambda \times I$ and $A$ are isotopic under a deformation constant on $\text{bd}(M)$.

**Proof.** Case 1. We suppose that $\lambda$ is orientable. After a small deformation leaving $\text{bd}(M)$ fixed, we may assume that $A \cap \lambda \times I$ consists of the common boundary of $A$ and $\lambda \times I$ together with a number of arcs with their endpoints in $\lambda \times \{0, 1\}$ and a number of simple closed loops properly embedded in $\lambda \times I$. After
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a deformation constant on \( \text{bd}(M) \), we may assume that \( A \cap \lambda \times I \) does not contain any simple loops which are nullhomotopic in \( A \). Thus there are no arcs in \( A \cap \lambda \times I \) with both endpoints in \( \lambda \times \{j\} \) where \( j = 1 \) or \( 2 \). Similarly there are not two disjoint arcs running from \( \lambda \times \{0\} \) to \( \lambda \times \{1\} \). Since \( \lambda \) separates a regular neighborhood of itself, we may assume that \( \lambda \times I \cap A \) is a collection of disjoint, simple, non-nullhomotopic loops. We split \( M \) along \( \lambda \times I \) to obtain \( M_t = F_1 \times I \). Then there is a subannulus \( A_1 \) of \( A \) with \( \text{bd}(A_1) \subset \text{bd}(F_1) \times I \). Since \( F - \lambda \) is not the interior of an annulus, \( \text{bd}(A_1) \) lies on a single component of \( \text{bd}(F_1) \times I \). Now it is a consequence of the lemma on p. 91 in [6] that there is an annulus \( A_2 \) in \( \text{bd}(F_1 \times I) \) such that \( A_1 \cup A_2 \) bounds a solid torus \( T \) and that we can push \( A_1 \) across the projection of \( T \) with a deformation constant on \( \text{bd}(M) \). Thus we can assume that there are exactly two simple loops in \( A \cap \lambda \times I \). Now it is a consequence of the lemma on p. 91 in [6] that there is an annulus \( A_2 \) in \( \text{bd}(F_1 \times I) \) such that \( A_1 \cup A_2 \) bounds a solid torus \( T \) and that we can push \( A_1 \) across the projection of \( T \) with a deformation constant on \( \text{bd}(M) \). Thus we can assume that there are exactly two simple loops in \( A \cap \lambda \times I \). Now we can assume that \( F \) is a solid torus.

Case 2. \( \lambda \) is nonorientable.

Let \( A_1 \) be a simple loop properly embedded in \( F \) such that
1. \( \lambda_1 \) bounds a moebius band \( B_1 \) in \( F \).
2. \( \lambda_1 \subset \text{int}(B_1) \).

After a deformation constant on \( \text{bd}(M) \) we may assume that \( A_1 \cap \lambda \times I \) is a collection of simple loops. Since \( \lambda \) and \( \lambda_1 \) are not freely homotopic in \( F \), every loop in \( A_1 \cap \lambda \times I \) is nullhomotopic both in \( A \) and in \( \lambda \times I \). Thus, after a deformation leaving \( \text{bd}(M) \) fixed, we may assume that \( A_1 \cap \lambda \times I \) is empty. Now we can assume that \( F \) is a moebius band.

Let \( \beta \) be an arc properly embedded in \( F \) which fails to separate \( F \) and which meets \( \lambda \) in a single point. After a deformation constant on \( \text{bd}(M) \), we may assume that \( A_1 \cap \beta \times I \) is a vertical arc. Next we cut \( M \) along \( \beta \times I \). This allows us to apply Lemma 3.3 to complete the proof of Case 2.

Thus the proof of Lemma 3.4 is complete.

Lemma 3.5. Let \( F \) be a closed surface other than the klein bottle or torus. Let \( \lambda \) be a simple non-nullhomotopic loop embedded in \( F \). We suppose that \( \lambda \) separates \( F \). Let \( h : F \times I \rightarrow F \times I \) be a homeomorphism such that
1. \( h \mid \text{bd}(F \times I) = \text{id} \).
2. \( h(\lambda \times I) = \lambda \times I \).

Let \( x \) be a point in \( \lambda \). Then the loop \( \phi_\mu((x) \times I) \) is homotopic to a point in \( \lambda \).

Proof. Let \( F_1 \) and \( F_2 \) be the closures of the components of \( F - \lambda \). Since \( F \) is not the klein bottle either \( \pi_1(F_1) \) or \( \pi_1(F_2) \) is not abelian. Thus we may assume that \( \pi_1(F_1) \) is free on two or more generators and that \([\lambda] = a_1a_2a_1^{-1}a_2^{-1} \cdots a_{2n-1}a_{2n}a_{2n-1}^{-1}a_{2n}^{-1} \) where \( a_1 \cdots a_{2n} \) are generators of \( \pi_1(F_1 \times \{0\}, (x, 0)) \) (or \([\lambda] = a_1^2 \cdots a_2^2 \) if \( F_1 \) is not orientable). Let \( \mu \) be a loop in \( F_1 \).

Observe that \( h \mid \mu \times \{0, 1\} = \text{id} \). Consider the loop \( \nu \) defined by
Then \( v \) is homotopic to a point in \( F_1 \times I \). Now \( h : F_1 \times I \to F_1 \times I \) induces an automorphism of \( \pi_1(F_1 \times I,(x,0)) \). We see

\[
p_M v = [p_M v] = [\mu][p_M((x) \times I)][\mu]^{-1}[p_M((x) \times I)]^{-1} = 1.
\]

Thus \([\mu]\) and \([p_M((x) \times I)]\) commute for all \( \mu \subset F_1 \). Thus the loop \( p_M((x) \times I) \) is homotopic to a point and Lemma 3.5 follows.

**Theorem 3.6.** Suppose that the surface \( F \) is not the 2-sphere, projective plane, klein bottle, or torus and that \( M = F \times I \). Let \( h : M \to M \) be a homeomorphism such that \( h \mid \text{bd}(M) = \text{id} \). Then there is an isotopy of \( h \) to the identity which is constant on \( \text{bd}(M) \).

**Proof.** We suppose that \( \text{bd}(F) \) is not empty. Let \( \beta_i, i = 1, \ldots, n \), be disjoint simple arcs properly embedded in \( F \) such that we obtain a disk by splitting \( F \) along \( \bigcup_{i=1}^n \beta_i \). It is a consequence of Lemma 3.3 that we may assume that \( h(\bigcup_{i=1}^n \beta_i \times I) = \bigcup_{i=1}^n \beta_i \times I \). After a deformation leaving \( \text{bd}(M) \) fixed, we may assume that \( h \mid \beta_i \times I = \text{id} \) for \( i = 1, \ldots, n \). The theorem is now an immediate consequence of Alexander's theorem.

We now assume that \( F \) is without boundary. Let \( \lambda \) be a simple non-nullhomotopic loop properly embedded in \( F \). We suppose that \( \lambda \) separates \( F \). It is a consequence of Lemma 3.4 that we may assume \( h(\lambda \times I) = \lambda \times I \). Let \( x \) be a point in \( \lambda \). Since by Lemma 3.5 the loop \( ph((x) \times I) \) is nullhomotopic in \( \lambda \), after a deformation leaving \( \text{bd}(M) \) fixed, we may assume that \( h \mid (x) \times I = \text{id} \) and \( h(\lambda \times I) = \lambda \times I \). After another deformation constant on \( \text{bd}(M) \), we may assume that \( h \mid \lambda \times I = \text{id} \). We may now split \( M \) along \( \lambda \times I \) to reduce the case when \( F \) is without boundary to the bounded case. The theorem follows.

**IV. Unknotting of arcs.** In this section we investigate the relationship between homeomorphically unknotted arcs and isotopically unknotted arcs. The following result is Theorem 6.1 in [1].

**Theorem 4.1.** Let \( F \) be a surface and \( M = F \times I \). Let \( \alpha_i, i = 1, \ldots, n \), be a collection of disjoint simple arcs properly embedded in \( M \) such that \( p \text{bd}(\alpha_i) \) is a single point in \( \text{int}(F) \) for \( i = 1, \ldots, n \). Then there is homeomorphism \( h : M \to M \) such that

1. \( h \mid \text{bd}(F) \times I \cup F \times \{0\} = \text{id} \).
2. \( h(\bigcup_{i=1}^n \alpha_i) \) is vertical.
if and only if the natural homomorphism induced by inclusion

\[ \pi_1 \left( F \times \{0\} - \bigcup_{i=1}^{n} \alpha_i \right) \to \pi_1 \left( M - \bigcup_{i=1}^{n} \alpha_i \right) \]

is an epimorphism.

The above theorem gives us a relationship between the property "homeomorphically unknotted" and an algebraic map.

Let \( \alpha \) be an arc properly embedded in \( F \times I \) and \( p \text{ bd}(\alpha) \) be a single point in \( \text{int}(F) \). Then \( \alpha \) is embedded monotonically in \( M \) if \( \alpha \) meets \( F \times \{t\} \) in a single point for \( t \) in \( I \). A level preserving isotopy \( H : F \times I \times I \to F \times I \) is an isotopy such that \( h_s(F \times \{s\}) \subset F \times \{s\} \) for \( s \) in \( I \).

**Theorem 4.2.** We assume that \( F \) is not the 2-sphere. Let \( \alpha \) be an arc properly embedded in \( M = F \times I \) such that \( p(\text{bd}(\alpha)) \) is a single point in \( \text{int}(F) \). Then there is an isotopy constant on \( \text{bd}(M) \) carrying \( \alpha \) to a monotone arc if and only if \( \alpha \) is homeomorphically unknotted.

**Proof.** Suppose \( \alpha \) is homeomorphically unknotted. Let \( \{x\} = p \text{ bd}(\alpha) \). After a small deformation constant on \( \text{bd}(M) \), we may assume that \( \alpha_2 = \alpha \cap F \times [t_0, 1] = \{x\} \times [t_0, 1] \) where \( t_0 < 1 \) is sufficiently near 1. Let \( \alpha_1 = \text{cl}(\alpha - \alpha_2) \). Consider the diagram of groups promised by van Kampen's theorem and shown in Figure 1.

\[ \begin{array}{ccc}
\pi_1(F \times \{t_0\} - \alpha) & \xrightarrow{f_1} & \pi_1(M - \alpha) \\
\downarrow & & \downarrow \\
\pi_1(F \times [0, t_0] - \alpha_1) & \xrightarrow{f_3} & \pi_1(F \times [t_0, 1] - \alpha_2) \\
\downarrow & & \downarrow \\
\pi_1(F \times \{0\} - \alpha_1) & \xrightarrow{f_4} & \pi_1(M - \alpha) \\
\end{array} \]

Figure 1

Now \( f_1 \) and \( f_2 \) are monomorphisms and \( f_4 \) is onto; thus \( f_1 \) is onto by 4.2 in [3]. It follows from 4.1 above that there is a homeomorphism \( k : F \times [0, t_0] \to F \times [0, t_0] \) such that
It is an immediate consequence of Lemma 3.2 that there is an isotopy, constant on \( \text{bd}(F) \times [0, t_0] \cup F \times \{0\} \) of \( k \) to the identity. It is not hard to use this isotopy together with a level preserving isotopy constant on \( \text{bd}(F) \times [t_0, 1] \cup F \times \{1\} \) which “pays for the motion on \( F \times \{t_0\} \)” to construct the isotopy required in the theorem. We may omit the details of this construction.

Suppose \( \alpha \) is monotonically embedded in \( M \). Let \( \alpha(t) = \alpha \cap F \times \{t\} \). If \( F \) is a disk, it is clear that there exists a deformation retraction of \( F \times I - \alpha \) onto \( F \times \{0\} - \alpha(0) \) which slides the boundary vertically downward and lets the missing point follow the track of \( \alpha \). It follows that the natural map from \( \pi_1(F \times \{0\} - \alpha(0)) \) into \( \pi_1(M - \alpha) \) induced by inclusion is an epimorphism and \( \alpha \) is \( h \)-unknotted.

In case \( F \) is not a disk, we can find a finite sequence of disks \( D_1, \ldots, D_n \) covering the projection of \( \alpha \) to \( F \). We may assume that \( \alpha(t) = \alpha \cap F \times \{t\} \) for \( t \) in \( [(i - 1)/n, i/n] \), \( i = 1, \ldots, n \), by using the Lebesgue number. We define a deformation retraction of \( F \times [0, i/n] - \alpha \) to \( F \times [0, (i - 1)/n] - \alpha \) on \( D_i \times [(i - 1)/n, i/n] - \alpha \) as the one above and sliding \( (F - D_i) \times [(i - 1)/n, i/n] \) vertically downward.

We compose finitely many of these deformation retractions to deformation retract \( F \times I - \alpha \) to \( F \times \{0\} - \alpha(0) \). It is now clear that the natural map \( \pi_1(F \times \{0\} - \alpha(0)) \to \pi_1(F \times I - \alpha) \) is onto and \( \alpha \) is \( h \)-unknotted. This completes the proof of Theorem 4.2.

**Theorem 4.3.** Let \( F \) be a surface not the 2-sphere. Let \( \alpha \) be an arc properly embedded in \( M = F \times I \) such that \( p \text{ bd}(\alpha) \) contains a single point in \( \text{int}(F) \). Then \( \alpha \) is isotopically unknotted if and only if \( \alpha \) is homeomorphically unknotted and the loop \( p(\alpha) \) is homotopic to a point.

**Proof.** Clearly we need only prove that \( \alpha \) is isotopically unknotted in case \( \alpha \) is \( h \)-unknotted and the loop \( p(\alpha) \) is nullhomotopic. We will divide the proof of Theorem 4.3 into six lemmas.

**Lemma 4.4.** Let \( F \) be a disk. Let \( \alpha \) be embedded in \( F \times I \) as above. Let \( \alpha \) be homeomorphically unknotted in \( F \times I \). Then \( \alpha \) is isotopically unknotted.

**Proof.** Let \( h : F \times I \to F \times I \) be a homeomorphism such that \( h(\alpha) \) is vertical and \( h | \text{bd}(F) \times I \cup F \times \{0\} = \text{id} \). Then \( k(m) = ph(m, 1) \) defines a homeomorphism from \( F \) onto \( F \). By Theorem 5.2 in [4], there is an isotopy of \( k \) to the identity constant on \( \text{bd}(F) \cup p \text{ bd}(\alpha) \). Thus after a deformation, we may assume that \( h | \text{bd}(F \times I) = \text{id} \). The desired result is now a consequence of Alexander’s theorem or Theorem 3.6 above.

**Lemma 4.5.** Let \( F \) be an annulus or a möbius band. Let \( \alpha \) be embedded in \( F \times I \) as in Theorem 4.3. Let the loop \( p(\alpha) \) be nullhomotopic in \( F \). Let \( \alpha \) be \( h \)-unknotted in \( F \times I \). Then \( \alpha \) is isotopically unknotted in \( F \times I \).
Proof. It is a consequence of Theorem 4.2 that we may assume that \( a \) is embedded monotonically in \( F \times I \). Let \( \beta_1 \) and \( \beta_2 \) be disjoint simple arcs properly embedded in \( F \) such that \( F - \beta_j \) is connected for \( j = 1, 2 \); and \( p \ \text{bd}(a) \) is not contained in \( \beta_1 \cup \beta_2 \). After a deformation constant on \( \text{bd}(F \times I) \), we may assume that \( J = a \cap (\beta_1 \cup \beta_2) \times I \) is a finite collection of points and that \( a \) crosses \( (\beta_1 \cup \beta_2) \times I \) at each point in \( J \).

It can be checked that the monotonic arc constructed in 4.2 projects to a trivial loop if \( a \) does. We assume that \( J \) is not empty. Since the loop \( p(a) \) is nullhomotopic, there must be an arc \( \alpha_1 \subset a \) with both its endpoints on one component of \( (\beta_1 \cup \beta_2) \times I \) such that \( \alpha_1 \cap (\beta_1 \cup \beta_2) \times I = \text{bd}(\alpha_1) \). Since \( a \) is monotonic in \( F \times I \), it is possible to find a level preserving deformation constant on \( \text{bd}(F \times I) \) carrying \( a \) to \( a' \) so that \( a' \cap (\beta_1 \cup \beta_2) \times I = J - \text{bd}(\alpha_1) \). It follows that after a deformation constant on \( \text{bd}(F \times I) \), we may assume that \( J \) is empty. Thus we may split \( F \times I \) along \( \beta_i \times I \). Lemma 4.5 is now a consequence of Lemma 4.4.

Lemma 4.6. Let \( F \) be a torus. Let \( a \) be embedded in \( F \times I \) as in Theorem 4.3. Let \( a \) be \( h \)-unknotted in \( F \times I \). Let the loop \( p(a) \) be nullhomotopic. Then \( a \) is isotopically unknotted in \( F \times I \).

Proof. Let \( \lambda_1 \) and \( \lambda_2 \) be disjoint, non-nullhomotopic simple loops properly embedded in \( F \) such that \( p \ \text{bd}(a) \) is not contained in \( \lambda_1 \cup \lambda_2 \). As in the proof of Lemma 4.5 above we can arrange \( a \) so that it is embedded monotonically in \( F \times I \) and so that it meets \( (\lambda_1 \cup \lambda_2) \times I \) in a finite collection of points. We can find an arc \( \alpha_1 \subset a \) such that \( \alpha_1 \cap (\lambda_1 \cup \lambda_2) \times I = \text{bd}(\alpha_1) \) and \( \text{bd}(\alpha_1) \) is contained in a single component of \( (\lambda_1 \cup \lambda_2) \times I \). It is now easy to find a level preserving deformation constant on \( \text{bd}(F \times I) \) which carries \( a \) to a monotonic arc \( a' \) such that \( a' \cap (\lambda_1 \cup \lambda_2) \times I = \alpha \cap (\lambda_1 \cup \lambda_2) \times I - \text{bd}(\alpha_1) \). Thus after a deformation constant on \( \text{bd}(F \times I) \), we can assume that \( \alpha \cap (\lambda_1 \cup \lambda_2) \times I \) is empty. By cutting \( F \times I \) along \( \lambda_1 \times I \), we reduce the proof of Lemma 4.6 to an application of Lemma 4.5.

Lemma 4.7. Let \( F \) be the projective plane. Let \( a \) be embedded in \( F \times I \) as in Theorem 4.3. Suppose the loop \( p(a) \) is nullhomotopic and \( a \) is \( h \)-unknotted in \( F \times I \). Then \( a \) is isotopically unknotted in \( F \times I \).

Proof. The proof of Lemma 4.7 is similar to that of Lemmas 4.4, 4.5, and 4.6. One may again assume that \( a \) is embedded monotonically in \( F \times I \). Let \( \lambda \) be a simple non-nullhomotopic loop properly embedded in \( F \) which does not contain \( p \ \text{bd}(a) \). Then we can find a level preserving isotopy constant on \( \text{bd}(F \times I) \) carrying \( a \) to \( a' \) such that \( a' \cap \lambda \times I \) is a finite collection of points and \( a' \) crosses \( \lambda \times I \) at each point in \( a' \cap \lambda \times I \). After another level preserving isotopy constant on \( \text{bd}(F \times I) \), we may assume that \( p(\alpha \cap \lambda \times I) \) is a single point. We will denote this point by \( x_0 \). We cut \( F \times I \) along \( \lambda \times I \) to obtain \( D \times I \) where \( D \) is a disk. Let
$P : D \times I \to F \times I$ be the natural projection map. Since the loop $p(\alpha)$ is nullhomotopic, we can find an arc $\alpha_1 \subset \alpha$ such that $\alpha_1 \cap \lambda \times I = \text{bd}(\alpha_1)$ and so that both endpoints of the arc in $p^{-1}(\alpha_1)$ lie in the same component of $P^{-1}((x_0) \times I)$. It is not hard to find a level preserving deformation of $F \times I$ constant on $\text{bd}(F \times I)$ carrying $\alpha$ to $\alpha'$ such that $\alpha' \cap \lambda \times I = \alpha \cap \lambda \times I - \text{bd}(\alpha_1)$. Thus we may assume that $\alpha \cap \lambda \times I$ is empty. Lemma 4.7 now follows from an application of Lemma 4.4 to the space obtained by cutting $F \times I$ along $\lambda \times I$.

**Lemma 4.8.** Let $F$ be a surface other than the 2-sphere. Let $F = F_1 \cup F_2$ and $F_1 \cap F_2 = c$ be a component of $\text{bd}(F_1)$. We assume that $F_1$ and $F_2$ are closed in $F$ and that $F_2$ is not a disk. Let $\alpha$ be an arc properly embedded in $\text{int}(F_1) \times I$ such that $p(\alpha)$ contains a single point. Then $\alpha$ is $h$-unknotted in $F_1 \times I$ if and only if $\alpha$ is $h$-unknotted in $F \times I$.

**Proof.** We need only show that if $\alpha$ is $h$-unknotted in $F \times I$ it is also $h$-unknotted in $F_1 \times I$. Since $\alpha$ is $h$-unknotted in $F \times I$, the natural map

$$\phi : \pi_1(F \times \{0\} - \alpha) \to \pi_1(F \times I - \alpha)$$

induced by inclusion is onto by 4.1. Consider the diagram of groups promised by van Kampen’s theorem and shown in Figure 2 where all maps are induced by inclusion. Observe that

$$h_1$$

$\text{and } h_2$ are monomorphisms and $f_0$ is an epimorphism. Since the natural map

$$\phi : \pi_1(F_1 \times \{0\} - \alpha) \to \pi_1(F \times I - \alpha)$$

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is an epimorphism, it follows from Proposition 2.4 in [2] that \( f_1 \) is an epimorphism. The lemma follows immediately from Theorem 4.1.

**Lemma 4.9.** Let \( F \) be a surface other than the 2-sphere, projective plane, disk, or torus. Let \( \alpha \) be embedded in \( F \times I \) as in Theorem 4.3. Let \( \alpha \) be \( h \)-unknotted and the loop \( p(\alpha) \) be homotopic to a point. Then \( \alpha \) is isotopically unknotted.

**Proof.** By assumption there exists a homeomorphism \( k : F \times I \rightarrow F \times I \) such that \( k \mid \partial(F) \times I \cup F \times \{0\} = \text{id} \) and \( k(\alpha) \) is vertical. Define \( \hat{k} : F \rightarrow F \) by \( \hat{k}(m) = pk^{-1}(m,1) \). Let \( x \) be the point in \( p(\partial(\alpha)) \). Now \( \hat{k} \) is homotopic to the identity homeomorphism via the homotopy \( H_t(m) = pk^{-1}(m,t) \). We claim that \( \hat{k} \) is homotopic to the identity under a homotopy constant on \( \{x\} \cup \partial(F) \). Let \( K : I \times I \rightarrow F \) be an extension of the map defined on \( \partial(I \times I) \) by \( K(t,0) = pa(t), K(t,s) = x \) for \( s \neq 0 \) where \( a(t) = k^{-1}(x,t) \). Such an extension exists since the loop \( p(\alpha) \) is homotopic to a point. Consider the space \( F \times I \times I \). Define a map \( \tilde{H} \) on the subspace

\[
F \times I \times \{0\} \cup \{x\} \times I \times I \cup F \times \{0,1\} \times I \cup \partial(F) \times I \times I
\]
on to \( F \) by

1. \( \tilde{H}(m,t,0) = H_t(m) \) for \( m \) in \( F \) and \( t \) in \( I \),
2. \( \tilde{H}(x,t,s) = K(t,s) \) for \( t \) in \( I \) and \( s \) in \( I \),
3. \( \tilde{H}(m,0,s) = m \) for \( m \) in \( F \) and \( s \) in \( I \),
4. \( \tilde{H}(m,1,s) = \hat{k}(m) \) for \( m \) in \( F \) and \( s \) in \( I \),
5. \( \tilde{H}(m,s,t) = m \) for \( m \) in \( \partial(F) \), \( t \) in \( I \) and \( s \) in \( I \).

If we let \( A = \partial(F) \times I \cup F \times \{0,1\} \cup \{x\} \times I \) and \( X = F \times I \), we see that \( \tilde{H} \) is defined on \( (X \times \{0\} \cup A \times I) \subset X \times I \). It follows from the homotopy extension theorem that \( \tilde{H} \) can be extended to \( X \times I \). We assume that \( \tilde{H} \) is defined on \( X \times I \). Define \( L_t : F \rightarrow F \) by \( L_t(m) = \tilde{H}(m,t,1) \). Then \( L_t \) is a homotopy of \( \hat{k} \) to the identity which is constant on \( \partial(F) \cup \{x\} \) and our claim is established.

Let \( \lambda \) be a simple non-nullhomotopic loop properly embedded in \( F \). Assume that \( x \) is in \( \lambda \) and that \( \lambda \) does not bound a möbius band in \( F \). Now \( L_t : \lambda \rightarrow F \) is a homotopy of \( \lambda \) to \( \hat{k}(\lambda) \) leaving \( x \) fixed. Then by 4.1 in [4] there is an isotopy of \( F \) carrying \( \lambda \) to \( \hat{k}(\lambda) \) which is constant on \( \{x\} \cup \partial(F) \). Thus we may assume that \( k(\lambda) = \lambda \) and \( k(\alpha) = \{x\} \times I \). Note that \( k^{-1}(\lambda \times I) \) is an annulus \( A \) properly embedded in \( F \times I \) such that \( p(\partial(A)) \) is a simple non-nullhomotopic loop properly embedded in \( F \) and that \( \alpha \subset A \). By Lemma 3.4 we may assume that \( A = \lambda \times I \). Let \( R \subset F \) be a möbius band or annulus embedded in \( \text{int}(F) \) which contains \( \lambda \) in its interior. We claim that (1) \( \alpha \) is \( h \)-unknotted in \( R \times I \) and (2) the loop \( p(\alpha) \subset R \) is nullhomotopic in \( R \). It is easy to see that (1) is a consequence of Lemma 4.8. Since the loop \( p(\alpha) \) is nullhomotopic in \( F \) and \( \partial(R) \) is not nullhomotopic in \( F \), the proof of (2) is a triviality. Thus the proof of Lemma 4.9 can be completed by applying Lemma 4.5. This completes the proof of Theorem 4.3.
Theorem 4.10. Let \( \lambda_1 \) and \( \lambda_2 \) be simple loops properly embedded in the surface \( F \). Let \( \lambda_i \) be non-nullhomotopic and freely homotopic to \( \lambda_2 \). Then \( \lambda_1 \times \{0\} \cup \lambda_2 \times \{1\} \) bounds an annulus properly embedded in \( F \times I \).

Proof. Theorem 2.1 or Theorem 3.3 in [4] guarantees that there exists an isotopy \( H : F \times I \to F \) such that

1. \( H_0 = \text{id} \).
2. \( H_1(\lambda_1) = \lambda_2 \).
3. \( H_t | \text{bd}(F) = \text{id} \).

The desired annulus is the image of \( \lambda_1 \times I \) under the map \( f \) defined by

\[
f(m, t) = (H(m, t), t) \quad \text{for } m \in F \text{ and } t \in I.
\]

We will denote the Euler characteristic of a manifold \( F \) by \( \chi(F) \).

Theorem 4.11. Let \( F \) be a surface such that \( \chi(F) < 0 \). Let \( \alpha_i \) for \( i = 1, \ldots, n \) be a nonempty collection of disjoint simple arcs properly embedded in \( F \times I \) such that \( \partial \alpha_i \) contains a single point for \( i = 1, \ldots, n \). We assume that the collection \( \alpha_i \), \( i = 1, \ldots, n \), is \( h \)-unknotted. We assume that for each \( \lambda \subset F - \bigcup_{i=1}^n \partial \alpha_i \), \( \lambda \times \{0\} \) is freely homotopic to \( \lambda \times \{1\} \) in \( F \times I - \bigcup_{i=1}^n \alpha_i \). Then there is an isotopy constant on \( \partial(F \times I) \) moving \( \bigcup_{i=1}^n \alpha_i \) to a vertical set, i.e. the collection \( \alpha_i \), \( i = 1, \ldots, n \), is \( i \)-unknotted.

Proof. Let \( h : F \times I \to F \times I \) be a homeomorphism such that \( h(\alpha_i) \) is vertical for \( i = 1, \ldots, n \). Now \( h(\lambda \times \{0\}) \) is freely homotopic to \( h(\lambda \times \{1\}) \). It is a consequence of Lemma 4.10 that \( h(\lambda \times \{0,1\}) \) bounds an annulus properly embedded in \( F \times I - \bigcup_{i=1}^n h(\alpha_i) \) and thus \( \lambda \times \{0,1\} \) bounds an annulus properly embedded in \( F \times I - \bigcup_{i=1}^n \alpha_i \).

It is easily shown using standard arguments that if \( \lambda_1 \) and \( \lambda_2 \) are disjoint simple loops properly embedded in \( F - \bigcup_{i=1}^n \partial \alpha_i \), \( \lambda_1 \times \{0\} \) and \( \lambda_2 \times \{0\} \) bound disjoint annuli properly embedded in \( F \times I - \bigcup_{i=1}^n \alpha_i \).

We claim that the loop \( p(\alpha_i) \) is nullhomotopic for \( i = 1, \ldots, n \). Let \( \{x_i\} = p(\partial \alpha_i) \) for \( i = 1, \ldots, n \). Let \( \lambda_1 \) and \( \lambda_2 \) be simple non-nullhomotopic loops such that \( \lambda_1 \cap \lambda_2 = \{x_1\} \) and if \( [\lambda_1]^m = [\lambda_2]^n \) in \( \pi_1(F, x_1) \), \( m = n = 0 \). Let \( R_j \) be a möbius band or annulus such that \( \lambda_j \subset R_j \subset F \) and \( R_j \cap \{x_i, i = 1, \ldots, n\} = \{x_i\} \) for \( j = 1, 2 \). We assume that \( R_1 \) is an annulus. Then by the argument above \( \partial(R_i) \times \{0,1\} \) bounds a pair of disjoint annuli \( A_1, A_2 \) properly embedded in \( F \times I \). It is a consequence of 3.4 that we may assume that \( A_1 \) and \( A_2 \) are vertical sets. Thus the loop \( p(\alpha_i) \) is a representative for \( [\lambda_j]^m \) in \( \pi_1(F, x_1) \) for some \( m \). It is easily seen that we would have reached the same conclusion if \( R_i \) had been a möbius band.

Similarly the loop \( p(\alpha_1) \) is a representative of \( [\lambda_2]^n \) in \( \pi_1(F, x_1) \) for some \( n \). It follows that the loop \( p(\alpha_i) \) is nullhomotopic in \( F \). Repetition of this argument establishes our claim.

We now pick disjoint möbius bands or annuli \( R_1, \ldots, R_n \) such that \( \{x_i\} \subset R_i \) and \( \partial(R_i) \) is not nullhomotopic for \( i = 1, \ldots, n \). Then after an argument similar
to the one given above, we see that we may assume that \( \alpha_i \subset \text{int}(R_i) \times I \) for \( i = 1, \ldots, n \). We may apply Lemma 4.8 to show that \( \alpha_i \) is \( h \)-unknotted in \( R_i \times I \) for \( i = 1, \ldots, n \). We see that the loop \( \rho(\alpha_i) \) is nullhomotopic in \( R_i \) since \( \rho(\alpha_i) \) is nullhomotopic in \( F \) and \( \text{bd}(R_i) \) is not nullhomotopic in \( F \). Theorem 4.11 now follows from Theorem 4.3.

**Theorem 4.12.** Let \( F \) be a surface such that \( \chi(F) = 0 \). Let \( \alpha_i, i = 1, \ldots, n, \) be a collection of disjoint arcs properly embedded in \( F \times I \) such that \( p \text{bd}(\alpha_i) \subset \text{int}(F) \) contains a single point. Suppose that the collection \( \alpha_i, i = 1, \ldots, n, \) is \( h \)-unknotted. Suppose also for every loop \( \lambda \subset F \times \bigcup_{i=1}^{n} p \text{bd}(\alpha_i), \lambda \times \{0\} \) is freely homotopic to \( \lambda \times \{1\} \) in \( F \times I \times \bigcup_{i=1}^{n} \alpha_i \). Then if the loop \( p(\alpha_i) \) is nullhomotopic, the collection \( \alpha_i, i = 1, \ldots, n, \) is \( i \)-unknotted.

**Proof.** It is a consequence of Theorem 4.3 that \( \alpha_1 \) can be assumed to be a vertical arc in \( F \times I \). Thus Theorem 4.12 becomes a consequence of 4.11.

There are a number of theorems which the reader may formulate which we could state in case \( F \) is a disk relating \( h \)-unknottedness and \( i \)-unknottedness. One simply finds hypotheses of the sort used in 4.11 and 4.12 which are sufficient to confine the unknotted arcs to product cells in \( F \times I \).

**References**


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