SOME STABLE RESULTS ON THE COHOMOLOGY OF THE CLASSICAL INFINITE-DIMENSIONAL LIE ALGEBRAS

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ABSTRACT. In this paper we compute the cohomology of various classical infinite-dimensional Lie algebras generalizing results of Gel'fand-Fuks for the Lie algebra of all formal power series vector fields.

1. Let \( k \) be a field of characteristic zero and \( x_1, x_2, \ldots, y_1, y_2, \ldots, \) etc. indeterminants. For us the classical infinite-dimensional Lie algebras will mean simply the Lie algebras occurring on the following list.

(I)_n Formal power series vector fields. This is just all expressions of the form \( \sum_{i=1}^{n} f_i(\partial/\partial x_i) \), where \( f_i \in k[[x_1, \ldots, x_n]] \). The bracket is the usual Lie bracket.

(II)_n Divergence zero formal power series vector fields. The subalgebra of (I)_n consisting of all \( \vec{f} = \sum f_i(\partial/\partial x_i) \) for which \( \text{div} \vec{f} = \sum (\partial f_i/\partial x_i) = 0 \).

(III)_n Divergence constant formal power series vector fields. The subalgebra of (I)_n consisting of all \( \vec{f} \) for which \( \text{div} \vec{f} \in k \).

(IV)_{2n} The Poisson algebra. As a vector space this is \( k[[x_1, \ldots, x_n, y_1, \ldots, y_n]] \), and the bracket operation is the Poisson bracket:

\[
\{f, g\} = \sum \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial y_j}.
\]

(V)_{2n} The Hamiltonian algebra. The Poisson algebra over its one-dimensional center.

(VI)_{2n} The derivation algebra of the Hamiltonian algebra. Since there is just one outer derivation, this contains the Hamiltonian algebra as a codimension one ideal.

(VII)_{2n+1} The contact algebra. As a vector space this is \( k[[x_1, \ldots, x_n, y_1, \ldots, y_n, z]] \). The bracket operation is the Lagrange bracket:

\[
[f, g] = (f, g)_{x,y} + \frac{\partial f}{\partial z} \left( g - \sum y_i \frac{\partial g}{\partial y_i} \right) - \frac{\partial g}{\partial z} \left( f - \sum y_i \frac{\partial f}{\partial y_i} \right).
\]

In this paper we will compute the cohomology of each of these algebras (with coefficients in \( k \)) for the range \( 0 \leq i \leq n \). It turns out that in this range the answer in each case is independent of \( n \) and rather simple. In certain cases our
results can be considerably improved. For example Gel'fand and Fuks computed the entire cohomology ring of $(I)_n$ in [2]; and, with slight modifications, their computation also works for $(III)_n$. (See [7].) A complete picture of the range $0 \leq i \leq 2n$ has been obtained by the second author for all the above examples except $(II)_n$ and $(VII)_{2n+1}$. (See [7] and the remarks at the end of §4.) However, these improved results seem to require much more delicate arguments than ours. (2)

2. Let $L$ be a Lie algebra over $k$, $(3)$ $A$ an abelian subalgebra, and $S(A)$ the symmetric algebra over $A$. We note that $L$ and $L^*$ are both modules for $S(A)$. In fact $A$ acts on both $L$ and $L^*$ by the adjoint representation; therefore, so does the universal enveloping algebra of $A$, which happens to be $S(A)$ since $A$ is abelian.

We begin with the basic theorem:

**Theorem 1.** Let $L$ be a Lie algebra over $k$, $A$ a finite-dimensional abelian subalgebra, and $\alpha$ an element of $L$. Assume that $L^*$ is free as a module over $S(A)$. Assume also that $\alpha$ acts semisimply on $L$ and that $A$ is contained in a nonzero eigenspace. Then $H^i(L, k) = 0$ for $0 < i \leq \dim A$.

**Proof.** Let $m = \dim A$, and consider the Koszul complex:

$$
0 \to S(A) \to S(A) \otimes A^* \to \ldots \to S(A) \otimes \Lambda^m(A^*) \otimes \Lambda^m(A^*) \to 0
$$

(2.1)

where $S^+(A)$ is the maximal ideal of $S(A)$ of elements of degree $> 0$. Taking the tensor product over $S(A)$ with $A^*$, which is free over $S(A)$, we get an exact sequence:

$$
0 \to N(L^*) \otimes A^* \xrightarrow{d} \ldots
$$

(2.2i)

$$
\xrightarrow{d} N(L^*) \otimes \Lambda^\alpha A^* \to N(L^*)/S^+(N(L^*)) \otimes \Lambda^\alpha A^* \to 0.
$$

Let $\delta$ be the Hochschild coboundary operator for $\Lambda L^*$. We can amalgamate the complexes (2.2i) into a double complex, whose $j$th column is

$$
0 \to \Lambda^j L^* \otimes \Lambda^\alpha A^* \xrightarrow{\delta} \ldots \to \Lambda^j L^* \otimes \Lambda^\alpha A^* \to 0.
$$

(2.2ii)

(2) In [b] Rozenfel'd has announced complete results for $(III)_n$ and $(VI)_{2n}$. However it is not clear as he claims that the methods of proof in [2] extend in $(VI)_{2n}$. In [a] Gel'fand et al. show that $(V)_{2n}$ has nontrivial cohomology in dimensions 7 and 10.

(3) If dim $L = \infty$ we assume $L$ is topologized and $L^{**} = L$. See, for example, [3].
have just described breaks up into a countable number of subcomplexes. The \( a \)th such complex is described in Figure 1.

\[
0 \to (N^{a+1}L^*)_a \to (N^{a+1}L^*)_{a+1} \otimes A^* \to \ldots \to \left( \frac{N^{a+1}L^*}{S^+ N^{a+1}L^*} \right)_{a+m} \otimes \Lambda^a A^* \to 0
\]

\[
0 \to (NL^*)_a \to (NL^*)_{a+1} \otimes A^* \to \ldots \to \left( \frac{NL^*}{S^+ NL^*} \right)_{a+m} \otimes \Lambda^a A^* \to 0
\]

\[
0 \to (N^{-1}L^*)_a \to (N^{-1}L^*)_{a+1} \otimes A^* \to \ldots \to \left( \frac{N^{-1}L^*}{S^+ N^{-1}L^*} \right)_{a+m} \otimes \Lambda^a A^* \to 0
\]

**Figure 1**

As we have already observed, the rows of Figure 1 are exact. We claim that the columns are exact except for the extreme right-hand column and the column indexed by \( a + j = 0 \). In fact, the standard identity: \((ad\alpha)\omega = \delta(\alpha \omega) + \alpha \delta \omega\) implies that the only eigenspace of \( AL^* \) with nontrivial cohomology is the zero eigenspace.

Let us consider Figure 1 with \( a = 0 \). All the rows are exact, and all the columns are exact except for the extreme right-hand column. Therefore, we get an isomorphism between the cohomology of the first column in position \( i \) (which is just \( H^i(L,k) \)) and the cohomology of the last column in position \( i - m \). If \( i < m \) this cohomology is zero. (It is zero in dimension zero since the complex in question is a relative complex.) Thus \( H^i(L,k) = 0 \) for \( i \leq m \), proving our theorem.

As corollaries we get:

**Corollary 1.** The cohomology of the algebra \((I)_a\) is zero in dimensions \( 0 < i \leq n \).

**Proof.** Just take \( A \) to be the subalgebra consisting of the constant vector fields, \( \sum c_i(\partial/\partial x_i) \), \( c_i \in k \), and take \( a = \sum_{i=1}^n x_i(\partial/\partial x_i) \).

**Corollary 2.** The cohomology of the algebra \((IV)_{2n}\) is zero in dimensions \( 0 < i \leq n \).
Proof. Take $A$ to be the subalgebra consisting of all linear forms $\sum_{i=1}^{n} c_i x_i$, $c_i \in k$, and take $\alpha = \sum x_i y_i$.

**Corollary 3.** The cohomology of the algebra $(\mathcal{VH})_{2n+1}$ is zero in dimensions $0 < i \leq n$.

**Proof.** Take $A$ and $\alpha$ to be the same as in the preceding example.

3. To apply Theorem 1 to example $(\mathcal{V})_n$ we will show that the dual algebra is free over $S(A)$ where $A$ is the subalgebra spanned by $\partial/\partial x_i$, $i = 1, \ldots, n - 1$. It is not hard to prove this directly, but we prefer to deduce it from a result which is applicable to other examples besides those discussed here.

We begin with a standard theorem in commutative algebra.

**Theorem.** Let $M$ be a graded module over the polynomial ring $k[x_1, \ldots, x_n]$. Then the following are equivalent.

(a) The Koszul cohomology of $M$ with respect to $x_1, \ldots, x_n$ is zero except in dimension zero.

(b) $M$ is free as a module over $k[x_1, \ldots, x_n]$.

(c) For each $j$, $0 \leq j \leq n$, $x_j$ is a nonzero divisor of $M/(x_1, \ldots, x_{j-1})M$.

See [1, Chapter VIII, Theorem 6.1] and [6, Chapter IV, Proposition 3].

Using the equivalence of (a) and (b) one can prove some general results relating the vanishing of the cohomology of a Lie algebra, $L$, to the vanishing of its Koszul cohomology with respect to an abelian subalgebra $A$. The Koszul cohomology is in turn closely related to the Spencer cohomology of $L$. (See, for example, [4].)

To show that the cohomology of example $(\mathcal{V})_n$ vanishes in dimensions $0 < i < n$ we will use the equivalence of (b) and (c).

Condition (c) dualizes to the following condition:

For each $j$, let $L_j = \{ f, \text{div } f = \partial f / \partial x_1 = \cdots = \partial f / \partial x_{j-1} = 0 \}$; then the map $L_j \to L_j, f \to \partial f / \partial x_j$, is surjective. To check this condition we note first of all that $L_j$ is just all divergence free vector fields in the variables $x_j, \ldots, x_n$; so it is enough to check the condition for $j = 1$. Given $\hat{f} \in L_1$, we can find a $\overrightarrow{g}$ such that $(\partial/\partial x_1)\overrightarrow{g} = \hat{f}$. Since $\text{div } \hat{f} = 0$, $(\partial/\partial x_1)\text{div } \overrightarrow{g} = 0$; so $\text{div } \overrightarrow{g}$ is a power series in $x_2, \ldots, x_n$. Let $\overrightarrow{g_1}$ be a power series vector field in $x_2, \ldots, x_n$ such that $\text{div } \overrightarrow{g_1} = \text{div } \overrightarrow{g}$, then $(\partial/\partial x_1)(\overrightarrow{g} - \overrightarrow{g_1}) = \hat{f}$ and $\text{div}(\overrightarrow{g} - \overrightarrow{g_1}) = 0$. We have, therefore, proved

**Proposition 1.** If $L$ is the Lie algebra $(\mathcal{V})_n$ then $L^\ast$ is free over $S(A)$ where $A$ is the subalgebra spanned by $\partial/\partial x_i$, $i = 1, \ldots, n - 1$.

If $\alpha = (\sum_{i=1}^{n-1} x_i (\partial/\partial x_i)) - (n - 1)x_n (\partial/\partial x_n)$, the hypotheses of Theorem 1 are satisfied so we have:

**Corollary.** The cohomology of the algebra $(\mathcal{V})_n$ is zero in dimensions $0 < i < n$. 

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Remark. One can show that the $n$th cohomology group of $(II)_n$ is one-dimensional with the volume form, $dx_1 \wedge \ldots \wedge dx_n$, as its generator. See [7].

4. We will finally compute the cohomology of the algebras $(III)_n$, $(V)_{2n}$ and $(VI)_{2n}$. We begin by describing a spectral sequence due to Hochschild and Serre: Let $L$ be a Lie algebra over $k$ and $M$ an ideal in $L$. The Hochschild-Serre spectral sequence has as its $E_2$ term $H(L, M)$ and has $E_2^{ij} = H^j(L/M, H^i(M))$. (See for example [1, Chapter XVI, §6]).

If $L$ is the Poisson algebra and $M$ is its one-dimensional center, then $H^i(M) = k$ when $i = 0, 1$ and zero otherwise; so $E_2^{ij} = E_2^{i+1} = H^j(L/M, k)$, and the other terms are zero. Since $H^j(L, k) = 0$ in dimensions $0 < j < n$ the only way these $E_2$ terms can cancel out is for $d_2 : E_2^{i+1} \rightarrow E_2^{i+2}$ to be bijective for $i + 2 < n$. So we have proved

**Proposition 2.** The cohomology of the Hamiltonian algebra $(V)_{2n}$ is equal to $k$ in all even dimensions in the range $0 \leq i \leq n$ and equal to zero in all odd dimensions in this range.

Remark. The Poisson algebra is a nontrivial central extension of the Hamiltonian algebra, so it defines an element in $H^2$ of the Hamiltonian algebra. (See, for example, [5].) It is not hard to see that the $i$th power of this 2-dimensional element is a basis for the cohomology in dimension $2i$ when $2i < n$.

Next we apply the Hochschild-Serre spectral sequence to the Hamiltonian algebra and to its derivation algebra. Let $M$ be the Hamiltonian algebra and $L$ its derivation algebra. It is easy to see that the two-dimensional generator of $H(M)$ is not invariant with respect to $L/M$, so $H^j(L/M, H^i(M))$ is zero except when $j = 0$; and $H^j(L/M, H^0)$ is equal to $k$ for $i = 0, 1$ and zero otherwise, since $L/M$ is one-dimensional. We conclude with

**Proposition 3.** The cohomology of the algebra $(VI)_{2n}$ is equal to $k$ in dimensions zero and one and is zero in the range $1 < i < n$.

A simpler computation of the same kind shows

**Proposition 4.** The cohomology of the algebra $(III)_n$ is equal to $k$ in dimensions zero and one, and is zero in the range $1 < i < n$.

Remark. The generator of the first cohomology group of $(III)_n$ is the one cocycle $f \rightarrow \text{div } f$. The restriction of this is the generator of the first cohomology group of $(VI)_{2n}$.

To conclude, we note that Gel'fand and Fuks have proved the cohomology of $(I)_n$ vanishes in the range $0 < i < 2n$. See [2]. The second author, in his thesis, has proved that the cohomology of $(IV)_{2n}$ also vanishes in this range. (The proofs require rather complicated techniques from classical invariant theory.)
Bibliography


Added in proof.


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