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THE RUDIN-KEISLER ORDERING OF P-POINTS

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ABSTRACT. The Stone-Čech compactification $\beta\omega$ of the discrete space ω of natural numbers is weakly ordered by the relation "D is the image of E under the canonical extension $\beta f : \beta\omega \to \beta\omega$ of some map $f : \omega \to \omega$." We shall investigate the structure, with respect to this ordering, of the set of P-points of $\beta\omega - \omega$.

1. **Introduction.** We shall be concerned with ultrafilters on ω. These can be considered from several different viewpoints. In the above abstract, we considered them topologically as points of $\beta\omega$. From the viewpoint of model theory, we may think of ultrafilters as "the things you use to construct ultraproducts." The abstract would then read as follows.

Let $\mathscr{N}$ be the model whose universe is ω and whose relations and functions are all the relations and functions on ω. The set of ultrafilters on ω is weakly ordered by the relation "D-prod $\mathscr{N}$ can be elementarily embedded into E-prod $\mathscr{N}$." We shall investigate the structure, with respect to this ordering, of the set of those ultrafilters D such that every nonstandard elementary submodel of D-prod $\mathscr{N}$ is cofinal with D-prod $\mathscr{N}$.

The two versions of the abstract describe the same ordering (called the Rudin-Keisler ordering) and the same set of ultrafilters (called the set of P-points).

For the most part, we shall consider ultrafilters directly as families of subsets of ω, but the model-theoretic interpretation will be used occasionally. Conversely, our theorems can be interpreted as asserting the existence of nonstandard models of arithmetic (i.e. elementary extensions of $\mathscr{N}$) with certain additional properties.

P-points were studied by W. Rudin [9], who proved, using the continuum hypothesis, that they exist and that any one of them can be mapped to any other by a homeomorphism of $\beta\omega - \omega$ onto itself. Further work on P-points was done by Choquet [4], [5] (who called them *ultrafiltres absolument* 1-*simples* in [4] and *ultrafiltres* δ-*stables* in [5]), by M. E. Rudin [8] and by Booth [3]. Booth considered two orderings of the set of ultrafilters, the Rudin-Keisler ordering and a stronger one called the Rudin-Frolik ordering. In the latter ordering, to which most of [3] is devoted, all P-points are minimal. Booth did not explicitly consider the Rudin-Keisler ordering of P-points, but, as we shall see, his Theorem 4.12 implies that not all P-points are minimal and their RK ordering is nontrivial (if Martin's

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axiom holds). In this paper, we shall show that this ordering can be very rich.

Before stating our results, we must confront an unpleasant fact. We shall surely want to know that P-points exist, for otherwise we would be talking about nothing. But, as we mentioned above, Rudin proved this existence theorem only under the assumption that the continuum hypothesis holds. In Booth [3], the result is strengthened by assuming only Martin's axiom (which will be stated below), but it appears that some assumption beyond ordinary set theory (Zermelo-Fränkel set theory including the axiom of choice) is needed. It therefore seems reasonable, in investigations of P-points, to assume Martin's axiom, and we shall do so.

Our main results are the following, all assuming Martin's axiom. In the Rudin-Keisler ordering of P-points, there are 2^c minimal elements (where c is the cardinal of the continuum) but no maximal elements. Every decreasing ω-sequence is bounded below, and every increasing ω-sequence is bounded above. Thus, $\aleph_1$ with its usual ordering (as an ordinal) can be order-isomorphically embedded into the set of P-points. The same is true of the real line with its usual ordering. Some, but not all, pairs of incomparable P-points have upper bounds, and these also have lower bounds.

2. **Preliminaries.** In this section, we collect the basic facts about P-points and the RK ordering which we will need. We also state Martin's axiom and deduce from it the existence of 2^c minimal P-points. Proofs that are omitted here may be found in [1, §§2–6] or in [3].

If D is an ultrafilter on ω and $f : \omega \to \omega$, then

$$f(D) = \{A \subseteq \omega \mid f^{-1}(A) \in D\}$$

is also an ultrafilter, the image of D by f. If f and g agree on a set of D, then $f(D) = g(D)$. The *Rudin-Keisler* (or RK) *ordering* is defined by

$$D \leq E \text{ iff } D = f(E) \quad \text{for some } f : \omega \to \omega.$$

Isomorphism of ultrafilters is defined by

$$D \cong E \text{ iff } D = f(E) \quad \text{for some permutation } f \text{ of } \omega.$$

Clearly, the RK ordering is reflexive and transitive, and isomorphism is an equivalence relation. If $D = f(E)$ and f is one-to-one on a set of E, then $D \cong E$, because there is a permutation which agrees with f on a set of E. If two ultrafilters are isomorphic, then each is $\leq$ the other. All principal ultrafilters are isomorphic and are $\leq$ all ultrafilters. The following very important result was discovered independently by many people; for a proof see [1, Theorem 2.5] or [3, Theorem 3.3].

Theorem 1. *If $f(D) = D$, then $\{x \in \omega \mid f(x) = x\} \in D$.*

Corollary 1. *If $f(D) \cong D$, then f is one-to-one on a set of D.*

Corollary 2. *If $D \leq E$ and $E \leq D$, then $D \cong E$.*

By Corollary 2, the RK ordering is (or, more precisely, induces) a partial ordering of the set of isomorphism classes of ultrafilters.

Select, once and for all, a bijective pairing function $J : \omega \times \omega \to \omega$ with inverse (π_1, π_2); thus

$$J(\pi_1(x), \pi_2(x)) = x \quad \text{and} \quad \pi_i J(x_1, x_2) = x_i.$$

It will be convenient to identify $\omega \times \omega$ with ω via J; this convention amounts to omitting all J's from our formulas. If D_1 and D_2 are ultrafilters on ω, then the family

$$\{\pi_1^{-1}(A) \mid A \in D_1\} \cup \{\pi_2^{-1}(A) \mid A \in D_2\}$$

has the finite intersection property, and if E is any ultrafilter containing it, then $\pi_i(E) = D_i$. This shows that the RK ordering is directed upward. In fact, it is shown in [1, Proposition 5.10] that any countable subset has an upper bound, and it can be shown that the same is true for any set of cardinality at most c.

Obviously, the isomorphism class consisting of the principal ultrafilters is the least element of the RK ordering. If we restrict our attention to nonprincipal ultrafilters, then D is minimal among these if and only if every function $f : \omega \to \omega$ is either constant or one-to-one on some set of D (so that $f(D)$ is principal or isomorphic to D, respectively). We shall call such a nonprincipal D a *minimal ultrafilter*. These ultrafilters are studied in [5], where they are called *ultrafiltres absolus*, and in [3], where they are called Ramsey ultrafilters and shown to be characterized by several other properties. (Note that in [3] the word "minimal" refers to the Rudin-Frolik ordering rather than the Rudin-Keisler ordering.)

A P-point is a nonprincipal ultrafilter D such that every function $f : \omega \to \omega$ is either constant or finite-to-one on a set of D. (To say that a function is finite-to-one means, of course, that the inverse image of any point is finite.) Obviously, every minimal ultrafilter is a P-point. The converse fails if the continuum hypothesis holds [5, Theorem 19] or even if Martin's axiom holds [3, Theorem 4.12]. Any nonprincipal ultrafilter that is $\leq$ a P-point is itself a P-point.

Let $\mathscr{P}$ denote the set of isomorphism classes of P-points, partially ordered by the RK-ordering. Thus $\mathscr{P}$ is an initial segment of the RK ordering of all (isomorphism classes of) nonprincipal ultrafilters on ω.

We use the usual convention that an ordinal is the set of smaller ordinals and a cardinal is an initial ordinal. It will be convenient to have a fixed well-ordering of the set of all functions $f : \omega \to \omega$. Since this set has the cardinality of the continuum, it can be put in one-to-one correspondence with the cardinal c. Choose, once and for all, such a correspondence, and let f_α be the function corresponding to the ordinal $\alpha < c$.

Martin's axiom is a consequence of the continuum hypothesis but is strictly weaker than that hypothesis, and therefore seems to have a better chance of being true. Most of the theorems in this paper in which Martin's axiom is assumed were originally (and somewhat more easily) proved using the continuum hypothesis. We shall state Martin's axiom below and refer the reader to [6] for more information about it.

Let $(P, \leq)$ be a partially ordered set. A subset C of P is an *antichain* iff no two distinct elements of C have an upper bound in P. A subset D of P is *dense* iff

(1) $p \in D, p \leq q \rightarrow q \in D$, and

(2) $(\forall p \in P)(\exists q \in D) p \leq q$.

A subset G of P is generic for a family Δ of dense subsets of P iff

(1) $p \in G, q \leq p \rightarrow q \in G$,

(2) $p, q \in G \rightarrow (\exists r \in G) p, q \leq r$,

(3) $D \in \Delta \rightarrow G \cap D$ is nonempty.

Martin's axiom is the assertion that if P is a partially ordered set all of whose antichains are countable, and if Δ is a set of fewer than c dense subsets of P, then there is a subset of P generic for Δ.

It is shown in [6, §3.1] that Martin's axiom implies that c is a regular cardinal.

Lemma. *Martin's axiom implies that every nonprincipal ultrafilter-base on ω has cardinality c.*

Proof. Let $\mathscr{B}$ be a filterbase on ω, containing no finite set, and having cardinality $\aleph < c$. By [6, §2.2], Martin's axiom implies the existence of an infinite subset T of ω such that $T - B$ is finite for all $B \in \mathscr{B}$. Let S_1 and S_2 be two disjoint infinite subsets of T. Thus, both S_1 and S_2 intersect every set of $\mathscr{B}$. If $\mathscr{B}$ generated an ultrafilter D, both of the S_i would have to be in D, which is impossible as they are disjoint. □

Theorem 2. *Martin's axiom implies the existence of 2^c minimal ultrafilters.*

With the continuum hypothesis in place of Martin's axiom, this result is due to H. J. Keisler. With 1 in place of 2^c, it is due to Booth [3, Theorem 4.14]. The following proof is essentially the union of Keisler's and Booth's proofs.

Proof. For any map $\varphi : c \rightarrow \{0, 1\}$ and any ordinal $\alpha < c$, we shall define a filterbase $\mathscr{Y}_\alpha^\varphi$ on ω such that

(1) $\mathscr{Y}_\alpha^\varphi$ contains all cofinite subsets of ω.

(2) If $\alpha < \beta$, then $\mathscr{Y}_\alpha^\varphi \subseteq \mathscr{Y}_\beta^\varphi$.

(3) $\mathscr{Y}_\alpha^\varphi$ has cardinality less than c.

(4) f_α is one-to-one or bounded on some set of $\mathscr{Y}_{\alpha+1}^\varphi$.

(5) If φ and ψ first differ at α, then $\mathscr{Y}_\alpha^\varphi = \mathscr{Y}_\alpha^\psi$, but $\mathscr{Y}_{\alpha+1}^\varphi$ contains a set disjoint from a set in $\mathscr{Y}_{\alpha+1}^\psi$.

The definition of $\mathcal{Y}_\alpha^\varphi$ is by induction on α. Let $\mathcal{Y}_0^\varphi$ consist of the cofinite subsets of ω. If λ is a limit ordinal $< c$, then let $\mathcal{Y}_\lambda^\varphi = \bigcup_{\alpha<\lambda} \mathcal{Y}_\alpha^\varphi$; here we need the regularity of c to show that property (3) is preserved.

Now let $\mathcal{Y}_\alpha^\varphi$ be given; we wish to define $\mathcal{Y}_{\alpha+1}^\varphi$. As an intermediate step, we will define a filterbase $\mathcal{B} \supseteq \mathcal{Y}_\alpha^\varphi$, of cardinality $< c$, and such that f_α is bounded or one-to-one on a set of $\mathcal{B}$. (Thus, $\mathcal{B}$ would work as $\mathcal{Y}_{\alpha+1}^\varphi$ except that property (5) might fail.) If f_α is bounded on a set of $\mathcal{Y}_\alpha^\varphi$, then we may set $\mathcal{B} = \mathcal{Y}_\alpha^\varphi$. So suppose that f_α is unbounded on every set of $\mathcal{Y}_\alpha^\varphi$. Let P be the set of finite subsets of ω on which f_α is one-to-one, and partially order P by inclusion. All antichains of P are countable because P is countable. For each $Y \in \mathcal{Y}_\alpha^\varphi$, let D_Y be the set of those $p \in P$ which meet Y. Because f_α is unbounded on Y, D_Y is dense in P. The family Δ of all these D_Y's has cardinality $< c$ by (3). Applying Martin's axiom, we obtain a subset G of P generic for Δ. Let T be the union of the elements of G. It follows from clause (2) of the definition of generic that f_α is one-to-one on T, and it follows from clause (3) of that definition that T meets every set of $\mathcal{Y}_\alpha^\varphi$. Let $\mathcal{B}$ be the filterbase obtained by adjoining T to $\mathcal{Y}$ and closing the resulting class under finite intersections.

As $\mathcal{B}$ contains all cofinite sets and has cardinality $< c$, it cannot be an ultrafilter-base by the lemma. So there are sets S such that both S and $\omega - S$ meet every set of $\mathcal{B}$. Choose such an S in some canonical (i.e. independent of φ) manner; for definiteness, take the one whose characteristic function occurs earliest in the enumeration $\{f_\beta\}$. Form $\mathcal{Y}_{\alpha+1}^\varphi$ by adjoining to $\mathcal{B}$ the set S if $\varphi(\alpha) = 0$ and the set $\omega - S$ if $\varphi(\alpha) = 1$, and closing under finite intersection.

It is clear that the $\mathcal{Y}_\alpha^\varphi$'s we have defined have all the properties (1) through (5). Let D^φ be any ultrafilter containing $\bigcup_{\alpha<c} \mathcal{Y}_\alpha^\varphi$ (which is a filterbase by (2)). D^φ is nonprincipal by (1) and minimal by (4) because a function bounded on a set of an ultrafilter D must be constant on a set of D. Finally, (5) implies that $D^\varphi \neq D^\psi$ when $\varphi \neq \psi$. □

We remark that the appeal to the regularity of c could have been avoided by changing condition (3) to read: The cardinality of $\mathcal{Y}_\alpha^\varphi$ is at most $\max(\alpha, \omega)$.

Corollary. *Martin's axiom implies the existence of 2^c isomorphism classes of minimal ultrafilters (hence also of P-points).*

Proof. There being only c permutations of ω, no isomorphism class can contain more than c ultrafilters. □

3. **Model theory.** Let $\mathfrak{A}$ be a model for some first-order language. If D and E are ultrafilters on ω and $D = f(E)$ for a certain map $f : \omega \to \omega$, then we define a map f^* of the ultrapower D-prod $\mathfrak{A}$ into E-prod $\mathfrak{A}$ as follows. A typical element of D-prod $\mathfrak{A}$ is g/D where $g : \omega \to |\mathfrak{A}|$. Let $f^*(g/D) = (g \circ f)/E$. It is easy to verify that f^* is well defined and is an elementary embedding. It is an isomorphism for all $\mathfrak{A}$ if and only if f is one-to-one on a set of D, i.e. iff $D \cong E$. (The details of this and the following discussion are worked out in [1, §§11, 12].)

Now consider the specific model $\mathscr{N}$ whose universe is ω and whose relations and functions are all the relations and functions on ω. It is called the *complete model* on ω; clearly its associated language has cardinality c. Suppose D and E are ultrafilters on ω and suppose e is an elementary embedding of D-prod $\mathscr{N}$ into E-prod $\mathscr{N}$. The identity function id : $\omega \to \omega$ determines an element id$/D$ of D-prod $\mathscr{N}$; let its image under e be f/E. Then $f(E) = D$ and $f^* = e$.

This discussion proves the following result.

Theorem 3. *A necessary and sufficient condition for $D \leq E$ (resp. $D \cong E$) is that, for every model $\mathfrak{A}$, D-prod $\mathfrak{A}$ can be elementarily embedded in (resp. is isomorphic to) E-prod $\mathfrak{A}$. This statement remains true if we delete "for every model $\mathfrak{A}$" and replace the two remaining occurrences of "$\mathfrak{A}$" by "$\mathscr{N}$".*

In [2], we defined an elementary extension $\mathfrak{A}$ of $\mathscr{N}$ to be *principal* iff it is generated by a single element a of $\mathfrak{A}$ in the following sense: Given any other element a' of $\mathfrak{A}$, there is a map $f : \omega \to \omega$ such that $\mathfrak{A} \models a' = \underline{f}(a)$, where $\underline{f}$ is the function symbol of the formal language corresponding to the function f of $\mathscr{N}$. It is trivial that if D is an ultrafilter on ω, then D-prod $\mathscr{N}$ is principal, being generated by id$/D$. Conversely, if $\mathfrak{A}$ is principal, generated by a, then

$$D = \{S \subseteq \omega \mid \mathfrak{A} \models \underline{S}(a)\}$$

is an ultrafilter, and D-prod $\mathscr{N}$ is isomorphic to $\mathfrak{A}$ via the map taking f/D to the unique a' such that $\mathfrak{A} \models a' = \underline{f}(a)$.

If $D = f(E)$, then f^* maps the generator id$/D$ of D-prod $\mathscr{N}$ to f/E which, therefore, generates $f^*(D\text{-prod } \mathscr{N})$. The following is thus an immediate consequence of the lemma of [2].

Theorem 4. *Let E and $f(E) = D$ be ultrafilters on ω. The image of D-prod $\mathscr{N}$ under f^* is cofinal in E-prod $\mathscr{N}$ if and only if f is finite-to-one on a set of E.*

Corollary. *Let E be a nonprincipal ultrafilter on ω. The following are equivalent.*

(1) *E is a P-point.*

(2) *Every nonstandard principal submodel of E-prod $\mathscr{N}$ is cofinal with E-prod $\mathscr{N}$.*

(3) *Every nonstandard submodel of E-prod $\mathscr{N}$ is cofinal with E-prod $\mathscr{N}$.*

Using the model-theoretic interpretations of the RK ordering and P-point given by Theorem 3 and the corollary to Theorem 4, we can deduce the following theorem from the model-theoretic results of [2].

Theorem 5. *If countably many elements of $\mathscr{R}$ have an upper bound in $\mathscr{R}$ then they also have a lower bound in $\mathscr{R}$.*

Proof. Let countably many P-points D_i ($i < \omega$) have an upper bound E which is also a P-point. Then, by Theorem 3, D_i-prod $\mathscr{N}$ is isomorphic to an elementary submodel $\mathfrak{A}_i$ of $\mathscr{M} = E$-prod $\mathscr{N}$. As the D_i are nonprincipal, the $\mathfrak{A}_i$ are nonstandard. By the corollary to Theorem 4, the $\mathfrak{A}_i$ are cofinal with $\mathscr{M}$, hence with each other. As all of the $\mathfrak{A}_i$ are principal models, we may invoke Theorem 3 of [2] to obtain a principal model $\mathfrak{B}$, contained in all of the $\mathfrak{A}_i$ and cofinal with $\mathscr{M}$. Since $\mathfrak{B}$ is principal, there is an ultrafilter F on ω such that $\mathfrak{B} \cong F$-prod $\mathscr{N}$. Being cofinal with $\mathscr{M}$, $\mathfrak{B}$ cannot be the standard model, so F is nonprincipal. For each i,

$$F\text{-}\operatorname{prod} \mathscr{N} \cong \mathfrak{B} \prec \mathfrak{A}_i \cong D_i\text{-}\operatorname{prod} \mathscr{N},$$

so, by Theorem 3, $F \leq D_i$. Being below a P-point, F must itself be a P-point, and the theorem is proved. $\square$

Two of the results announced in the introduction are immediate consequences of this theorem.

Corollary 1. *Any two elements of $\mathscr{R}$ that have an upper bound also have a lower bound.*

Corollary 2. *Any decreasing ω-sequence in $\mathscr{R}$ has a lower bound.*

The following is a special case of Corollary 1.

Corollary 3. *No two distinct minimal elements of $\mathscr{R}$ have an upper bound. Hence, if Martin's axiom holds, $\mathscr{R}$ is not directed upward.*

Note that, except for the last sentence of Corollary 3 where we needed the existence of nonisomorphic minimal ultrafilters, none of the results of this section depend on Martin's axiom.

4. **Chains of P-points.** According to Theorem 2, the partially ordered set $\mathscr{R}$ is very wide: its lowest level has cardinality 2^c. In this section we shall show that $\mathscr{R}$ is also quite high. The theorem of Choquet and Booth that not all P-points are minimal is a step in this direction, for it says that $\mathscr{R}$ has height at least 2. Our next theorem will imply that $\mathscr{R}$ has height at least ω. With the continuum hypothesis in place of Martin's axiom, this result is easier and was proved by M. E. Rudin [8] and independently by myself [1].

Theorem 6. *Assume Martin's axiom. $\mathscr{R}$ has no maximal element.*

Proof. Given any P-point D, we must find a P-point $E > D$ (i.e. $E \geq D$ but $E \not\cong D$). E will be an ultrafilter on $\omega \times \omega$ (which, we remind the reader, has been identified with ω) such that $\pi_1(E) = D$; that is, if $A \in D$, then $\pi_1^{-1}(A) \in E$. Then $E \geq D$. To ensure that $E \not\cong D$, it will suffice (by Corollary 1 of Theorem 1) that π_1 is not one-to-one on any set of E. This means that E must contain the complement of the graph of every function $\omega \to \omega$; then E must also contain the complement of every finite union of such graphs.

Let us define, for any $Y \subseteq \omega \times \omega$, its cardinality function c_Y to be the function from ω into $\omega + 1$ given by

$$c_Y(x) = \text{cardinality of } \{y \mid (x,y) \in Y\}.$$

We shall call a subset Y of $\omega \times \omega$ *small* iff c_Y is bounded by some $n < \omega$ on some set of D; otherwise Y is *large*. (The terminology "large" and "small" is to remain fixed only for this proof; in later proofs, we will want to use the same words with different meanings.)

Now if Y is the graph of a function, then Y is small, for c_Y is bounded by 1 on all of ω. Also, if $A \in D$ and $Y = (\omega \times \omega) - \pi_1^{-1}(A)$, then Y is small, for c_Y is bounded by 0 on A. Therefore, if E is an ultrafilter on $\omega \times \omega$ containing no small sets, then $E > D$.

It is easy to prove the existence of such an E, because $\omega \times \omega$ is not a finite union of small sets. However, a good deal of work will be needed to find such an E which is a P-point. We will construct E by a transfinite induction of length c. At stage α of the construction, we will put into E a set on which f_α is finite-to-one or bounded. We must, of course, be careful to put only large sets into E.

Lemma 1. *The union of two small sets is small. If $Y \cup Z$ is large and Y is small, then Z is large. Finite sets are small. $\omega \times \omega$ is large. Any superset of a large set is large.*

Proof. If c_Y is bounded by n on $A \in D$ and c_Z is bounded by m on $B \in D$, then $c_{Y \cup Z}$ is bounded by $n + m$ on $A \cap B \in D$. The rest of the lemma is obvious.

Lemma 2. *Let Y be large, and let $f : \omega \times \omega \to \omega$. Then either f is finite-to-one on a large subset of Y, or f is bounded on $Y - S$ for some small set S.*

Proof. For each $x \in \omega$, let

$$h(x) = \text{the least } n \leq x \text{ such that } \{y \mid (x,y) \in Y \text{ and } f(x,y) > n\}$$
$$\text{has at most } n \text{ elements, or } x + 1 \text{ if no such } n \text{ exists.}$$

Thus, for each x, the set

$$L(x) = \{y \mid (x,y) \in Y \text{ and } f(x,y) \geq h(x)\}$$

has at least $h(x)$ elements. (This is trivial if $h(x) = 0$ and follows from the leastness of $h(x)$ otherwise.)

Case 1. h is constant on a set $A \in D$, say $h(A) = \{a\}$. Then, for any k,

$$k \in A \text{ and } k \geq a \to h(k) = a \leq k$$
$$\to \{y \mid (k,y) \in Y \text{ and } f(k,y) > a\} \text{ has at most } a \text{ elements.}$$

Let $S = \{p \in Y \mid f(p) > a\}$. Then, as we have just shown, c_S is bounded by a on the set $A \cap \{k \mid k \geq a\} \in D$. So S is small, and f is bounded by a on $Y - S$.

Case 2. h is finite-to-one on a set $A \in D$. For $x \in A$, let $V(x)$ consist of the first $h(x)$ elements of $\{x\} \times L(x)$. Define Z to be $\bigcup_{x \in A} V(x)$. Then clearly $Z \subseteq Y$, and Z is large because, on A, $c_Z = h$ and h is finite-to-one. Finally,

$$\begin{aligned}(x,y) \in Z \text{ and } f(x,y) = n &\to (x,y) \in V(x) \text{ and } x \in A \\ &\qquad \text{and } n = f(x,y) \geq h(x) \\ &\to (x,y) \in \bigcup_{x \in A; h(x) \leq n} V(x).\end{aligned}$$

As h is finite-to-one on A, this is a finite union of finite sets. Therefore, f is finite-to-one on Z.

As D is a P-point, one of the two cases occurs, so the lemma is proved.

Lemma 3. *Assume Martin's axiom. Let a filterbase $\mathcal{Y}$ of fewer than c large subsets of $\omega \times \omega$ be given, and let $f : \omega \to \omega$. There is a set $T \subseteq \omega \times \omega$ on which f is finite-to-one or bounded, and such that $T \cap Y$ is large for every $Y \in \mathcal{Y}$.*

Proof. Let P be the set of all pairs (R, n) where R is a subset of $\omega \times \omega$ on which f is finite-to-one and where $n \in \omega$. Partially order P by

$$(R, n) \leq (R', n') \leftrightarrow R \subseteq R',\ n \leq n', \text{ and } (\forall x \in R' - R) f(x) \geq n.$$

(Intuitively, we think of (R, n) as the following partial description of the set T we want. R is a subset of T and, on $T - R$, f is $\geq n$. The ordering of P corresponds to implication between the descriptions of T.)

We shall show first that all antichains of P are countable. Let any uncountable subset of P be given; we must find two elements of it which have an upper bound in P. Begin by extracting an uncountable subset $\{(R_\alpha, n) \mid \alpha < \aleph_1\}$ all of whose elements have the same second component n. As f is finite-to-one on each R_α, the sets

$$B_\alpha = \{q \in R_\alpha \mid f(q) < n\}$$

are finite subsets of $\omega \times \omega$. Thus, there must be two α's, say 0 and 1, such that $B_0 = B_1$. Consider $(R_0 \cup R_1, n)$. Clearly, it lies in P and we claim that it is $\geq$ both (R_0, n) and (R_1, n). The only nontrivial thing to check is that f is $\geq n$ on

$$(R_0 \cup R_1) - R_0 = R_1 - R_0 \subseteq R_1 - B_0 = R_1 - B_1$$

and on

$$(R_0 \cup R_1) - R_1 = R_0 - R_1 \subseteq R_0 - B_1 = R_0 - B_0,$$

and this is clear from the definition of B_α. Thus, in any uncountable subset of P, there are two elements with an upper bound.

For each $Y \in \mathcal{Y}$, let

$$D_Y = \{(R, n) \in P \mid R \cap Y \text{ is large}\}.$$

Case 1. There is a $Y \in \mathcal{Y}$ for which D_Y is not dense. This means that, for a certain $(R,n) \in P$,

$$(R,n) \leq (R',n') \in P \rightarrow R' \cap Y \text{ is small.}$$

Suppose f were finite-to-one on a large subset A of Y. Replacing A by $A \cap \{x \mid f(x) \geq n\}$, we may suppose, without loss of generality, that f is $\geq n$ on A. (Note that $A \cap \{x \mid f(x) < n\}$ is finite, hence small, so $A \cap \{x \mid f(x) \geq n\}$ is large by Lemma 1.) But then $(R \cup A, n)$ is an element of P which is $\geq (R,n)$, and $(R \cap A) \cap Y \supseteq A$ is large. This contradicts the choice of (R,n), so f is not finite-to-one on any large $A \subseteq Y$. By Lemma 2, there is a small subset S of Y such that f is bounded on $T = Y - S$. For any $Y' \in \mathcal{Y}$, $Y \cap Y'$ is large, as $\mathcal{Y}$ is a filterbase. But

$$Y \cap Y' \subseteq (T \cap Y') \cup (Y - T) = (T \cap Y') \cup S.$$

As S is small, $T \cap Y'$ is large, and the lemma is verified in this case.

Case 2. All the D_Y's are dense. Furthermore, for $k \in \omega$,

$$D_k = \{(R,n) \in P \mid n \geq k\}$$

is dense, because $(R,n) \leq (R, n + k)$. As $\mathcal{Y}$ has fewer than c elements, we can apply Martin's axiom to get a set G generic for

$$\Delta = \{D_Y \mid Y \in \mathcal{Y}\} \cup \{D_k \mid k \in \omega\}.$$

Let $T = \bigcup_{(R,n) \in G} R$. (Intuitively, we construct T so that all the descriptions corresponding to $(R,n) \in G$ are correct.)

For any $Y \in \mathcal{Y}$, let $(R,n) \in G \cap D_Y$ (by clause (3) of the definition of generic). Then $R \subseteq T$ and $R \cap Y$ is large, so $T \cap Y$ is large.

Finally, we show that f is finite-to-one on T. Let any $k \in \omega$ be given, and let $(R_0, n_0) \in G \cap D_k$, so $n_0 \geq k$. Now let (R,n) be any element of G. By clause (2) of the definition of generic, there exists $(R',n') \geq$ both (R,n) and (R_0,n_0). By definition of the ordering of P, f is $\geq n_0 \geq k$ on $R' - R_0 \supseteq R - R_0$. So,

$$\{q \in R \mid f(q) < k\} \subseteq \{q \in R_0 \mid f(q) < k\} = C,$$

where C is independent of (R,n). Therefore, taking the union over all $(R,n) \in G$, we obtain

$$\{q \in T \mid f(q) < k\} \subseteq C.$$

As C is finite (for f is finite-to-one on R_0), f is finite-to-one on T. This completes the proof of the lemma.

Armed with these lemmas, we return to the construction of the required ultrafilter E. We shall define filterbases $\mathcal{Y}_\alpha$ for $\alpha < c$ with the following properties.

(1) Every set in $\mathcal{Y}_\alpha$ is large.

(2) If $\alpha < \beta$ then $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$.

(3) $\mathcal{Y}_\alpha$ has cardinality less than c.

(4) f_α is finite-to-one or bounded on some set of $\mathcal{Y}_{\alpha+1}$.

The inductive definition of $\mathcal{Y}_\alpha$ begins with $\mathcal{Y}_0 = \{\omega \times \omega\}$. At limit ordinals λ, set $\mathcal{Y}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{Y}_\alpha$. (As in the proof of Theorem 2, property (3) is preserved at limit ordinals because Martin's axiom implies that c is regular.) If $\mathcal{Y}_\alpha$ is given, use Lemma 3 to find a set T on which f_α is finite-to-one or bounded and such that $T \cap Y$ is large for all $Y \in \mathcal{Y}_\alpha$. Obtain $\mathcal{Y}_{\alpha+1}$ by adjoining T to $\mathcal{Y}_\alpha$ and closing under finite intersection.

Let $\mathcal{Y} = \bigcup_{\alpha<c} \mathcal{Y}_\alpha$, and let $\mathcal{B}$ be the filter of all sets whose complements are small. Every set of $\mathcal{Y}$, being large, meets every set of $\mathcal{B}$, so there is an ultrafilter $E \supseteq \mathcal{Y} \cup \mathcal{B}$. Since E contains no small sets (for their complements are in $\mathcal{B} \subseteq E$), we know that $E > D$. Because every $f: \omega \times \omega \to \omega$ is bounded or finite-to-one on a set of $\mathcal{Y} \subseteq E$, E is a P-point. $\square$

Corollary. *Assume Martin's axiom. There are increasing ω-sequences in $\mathcal{R}$. In fact, every element of $\mathcal{R}$ is the first element of such a sequence.*

Thus, $\mathcal{R}$ has height at least ω, but we can obtain a stronger result.

Theorem 7. *Assume Martin's axiom. In $\mathcal{R}$, any increasing ω-sequence has an upper bound.*

Proof. Let $D_0 \leq D_1 \leq \dots$ be an increasing ω-sequence of P-points. For each i, let $D_i = f_i(D_{i+1})$. For $i \leq j$, define

$$g_{ij} = f_i \circ f_{i+1} \circ \dots \circ f_{j-1};$$

$$g_{ii} = \text{identity}; \qquad g_{i,j+1} = g_{ij} \circ f_j.$$

Thus $g_{ij}(D_j) = D_i$. Also define maps $p_i : \omega \times \omega \to \omega$ by

$$\begin{aligned} p_i(x,k) &= g_{ik}(x) && \text{if } i \leq k, \\ &= 0 && \text{if } i > k. \end{aligned}$$

Thus, if $i \leq j$ then

$$\begin{aligned} g_{ij} p_j(x,k) &= g_{ij} g_{jk}(x) = g_{ik}(x) = p_i(x,k) && \text{when } k \geq j, \\ &= g_{ij}(0) && \text{when } k < j. \end{aligned}$$

Therefore, for any set Y, $g_{ij} p_j(Y) - p_i(Y)$ has at most one element, namely $g_{ij}(0)$.

We shall construct a P-point E such that $p_i(E) = D_i$ for all i; this will, of course, suffice to prove the theorem.

Call a set $Y \subseteq \omega \times \omega$ *large* if $p_i(Y) \in D_i$ for all i; otherwise call Y *small.*

Lemma 1. *The union of two small sets is small. If $Y \cup Z$ is large and Y is small, then Z is large. Finite sets are small. $\omega \times \omega$ is large. Any superset of a large set is large.*

Proof. The last three statements are obvious. (To see that $\omega \times \omega$ is large, simply note that $p_i(x, i) = x$, so $p_i(\omega \times \omega) = \omega$.) The first two assertions are clearly equivalent.

Suppose Y and Z are both small but $Y \cup Z$ is large. Thus, there exist i and j such that $p_i(Y) \notin D_i$ and $p_j(Z) \notin D_j$. Without loss of generality, we may suppose $i \leq j$. Since $Y \cup Z$ is large,

$$p_j(Y) \cup p_j(Z) = p_j(Y \cup Z) \in D_j.$$

Because D_j is an ultrafilter, it follows that $p_j(Y) \in D_j$. Hence $g_{ij}p_j(Y) \in g_{ij}(D_j) = D_i$. From this and $p_i(Y) \notin D_i$, it follows that the singleton set $g_{ij}p_j(Y) - p_i(Y) \in D_i$, which contradicts the fact that D_i is nonprincipal. This completes the proof of the lemma.

Lemma 2. *Let Y be large and let $f : \omega \times \omega \to \omega$. Then either f is finite-to-one on a large subset of Y, or f is bounded on $Y - S$ for some small set S.*

Proof. For i, $x \in \omega$, let

$$h_i(x) = \text{the least } n \leq x \text{ such that } (\forall y \in Y)(p_i(y) = x \to f(y) \leq n)$$
$$\text{if such an } n \text{ exists, and } x + 1 \text{ otherwise.}$$

Case 1. For some i, h_i is constant on a set $A \in D_i$, say $h_i(A) = \{a\}$. Thus

$$(*) \qquad x \in A \text{ and } x \geq a \to (\forall y \in Y)(p_i(y) = x \to f(y) \leq a).$$

If we let $S = \{y \in Y \mid f(y) > a\}$, then (*) tells us that $p_i(S)$ is disjoint from the set $A \cap \{x \mid x \geq a\} \in D$. Therefore, $p_i(S) \notin D_i$, so S is small. As f is obviously bounded by a on $Y - S$, the lemma is verified in this case.

Case 2. For each i, h_i is finite-to-one on some $B_i \in D_i$. By subtracting a finite set from B_i, we may suppose without loss of generality that $h_i(x) > i$ for all $x \in B_i$. Let $i \in \omega$, $x \in B_i$. By the leastness of $h_i(x)$, we can choose $y = y(i, x) \in Y$ such that $p_i(y) = x$ and $f(y) \geq h_i(x)$. Let

$$Z = \{y(i, x) \mid i \in \omega,\ x \in B_i\}.$$

We claim that Z is large and f is finite-to-one on Z.

For any $i \in \omega, p_i(Y) \supseteq B_i \in D_i$, because $p_i(y(i, x)) = x$. Therefore, Z is large.

For any n,

$$f(y(i,x)) = n \text{ and } x \in B_i \to n = f(y(i,x)) \geq h_i(x) > i.$$

The condition $i < n$ is satisfied by only finitely many i, and, for each such i, the condition $h_i(x) \leq n$ is satisfied by only finitely many $x \in B_i$, as h_i is finite-to-one on B_i. Therefore, there are only finitely many pairs (i, x) such that $i \in \omega$, $x \in B_i$, and $f(y(i,x)) = n$. So f is finite-to-one on Z.

As all of the D_i are P-points, one of the two cases occurs, and the lemma is therefore proved.

Notice that the two lemmas we have just proved are word for word the same as the first two lemmas in the proof of Theorem 6. Of course, the meanings are different, because the words "large" and "small" were defined differently in the two proofs. In the proof of Lemma 3 of Theorem 6, in the construction of E following that lemma, and in the verification that E is a P-point containing no small sets, the only facts we needed about large and small sets were Lemmas 1 and 2. Since these lemmas are also true in the present context, we see that there is a P-point E containing no small sets.

If $A \in D_i$, then $(\omega \times \omega) - p_i^{-1}(A)$ is small, hence is not in E. Therefore, $p_i^{-1}(A) \in E$, which implies that $p_i(E) = D_i$. $\square$

Corollary. *Assume Martin's axiom. Every element of $\mathscr{R}$ is the first element of an increasing sequence of order type $\aleph_1$ in $\mathscr{R}$.*

5. **An embedding of the reals into $\mathscr{R}$.** We have seen that $\mathscr{R}$ contains antichains of cardinality 2^c and chains of order type $\aleph_1$. In this section, we shall see that $\mathscr{R}$ contains chains of cardinality c and that $\mathscr{R}$ is not well founded.

Theorem 8. *Assume Martin's axiom. There is an order-isomorphic embedding of the real line $\mathbf{R}$ into $\mathscr{R}$.*

Proof. Let X be the set of all functions $x : \mathbf{Q} \to \omega$ such that $x(r) = 0$ for all but finitely many $r \in \mathbf{Q}$; here $\mathbf{Q}$ is the set of rational numbers. As X is denumerable, we may identify it with ω via some bijection. For each $\xi \in \mathbf{R}$, we define $h_\xi : X \to X$ by

$$\begin{aligned} h_\xi(x)(r) &= x(r) \qquad \text{if } r < \xi, \\ &= 0 \qquad\quad\ \text{if } r \geq \xi. \end{aligned}$$

Clearly, if $\xi < \eta$, then $h_\xi \circ h_\eta = h_\eta \circ h_\xi = h_\xi$. The embedding of $\mathbf{R}$ into $\mathscr{R}$ will be defined by $\xi \longmapsto D_\xi = h_\xi(D)$ for a certain ultrafilter D on X. If $\xi < \eta$, then

$$D_\xi = h_\xi(D) = h_\xi \circ h_\eta(D) = h_\xi(D_\eta) \leq D_\eta.$$

We wish to choose D in such a way that

(a) $D_\xi \not\geq D_\eta$ (therefore, $D_\xi < D_\eta$) when $\xi < \eta$, and

(b) D_ξ is a P-point.

Observe that it will be sufficient to choose D so that

(a′) $D_\xi \not\geq D_\eta$ when $\xi < \eta$ and both ξ and η are rational, and

(b′) D is a P-point.

Indeed, (a′) implies (a) because $\mathbf{Q}$ is dense in $\mathbf{R}$. If (a) holds, then $D_{\xi-1} < D_\xi$, so D_ξ is a nonprincipal ultrafilter $\leq D$; hence (b′) implies (b).

Condition (a′) means that, for all $\xi < \eta \in \mathbf{Q}$ and all $g : X \to X$, $D_\eta \neq g(D_\xi) = gh_\xi(D_\eta)$. By Theorem 1, this is equivalent to $\{x \mid gh_\xi(x) = x\} \notin D_\eta$, or by definition of D_η,

$$\text{(a}''\text{)} \qquad \{x \mid gh_\xi(x) = h_\eta(x)\} = h_\eta^{-1}\{x \mid gh_\xi(x) = x\} \notin D.$$

We now proceed to construct a P-point D satisfying (a″) for all $\xi < \eta \in \mathbf{Q}$ and all $g : X \to X$; this will suffice to establish the theorem.

Let σ be a finite sequence of closed intervals with rational endpoints,

$$\sigma = ([p_0, q_0], \ldots, [p_{\lambda-1}, q_{\lambda-1}]),$$

where $p_i < q_i$ and $p_0 \leq p_1 \leq \cdots \leq p_{\lambda-1}$. Another such sequence, of the same length λ,

$$\sigma' = ([p_0', q_0'], \ldots, [p_{\lambda-1}', q_{\lambda-1}']),$$

refines σ iff $[p_i', q_i'] \subseteq [p_i, q_i]$ for all $i < \lambda$. It is easy to see that every σ is refined by a σ' whose terms are pairwise disjoint.

A set $\{x_J \mid J : \lambda \to 2\}$, consisting of 2^λ elements of X indexed by λ-sequences $J = j_0 \ldots j_{\lambda-1}$ of zeros and ones, will be called a *σ-tree* iff, whenever two sequences J, J' first differ at the argument i, then $h_{p_i}(x_J) = h_{p_i}(x_{J'})$ but $h_{q_i}(x_J) \neq h_{q_i}(x_{J'})$, i.e. x_J and $x_{J'}$ first differ at a point of $[p_i, q_i)$. Thus, $h_{p_i}(x_J)$ depends only on the first i terms of J. Observe that if σ' refines σ, then any σ'-tree is also a σ-tree.

Call a subset A of X *large* iff, for every σ as above (of arbitrary length λ), there is a σ-tree included in A; otherwise call A *small*. Note that the sets

$$\{x \mid gh_\xi(x) = h_\eta(x)\}, \qquad \xi < \eta \in \mathbf{Q}, g : X \to X,$$

which (a″) says should be outside D, are small. Indeed, such a set contains no σ-tree where σ is the sequence of length one $([\xi, \eta])$. Thus, to prove the theorem, it suffices to find a P-point D which contains no small sets.

Lemma 1. *The union of two small sets is small. If $Y \cup Z$ is large and Y is small, then Z is large. Finite sets are small. X is large. Any superset of a large set is large.*

Proof. It is obvious that supersets of large sets are large. A large set contains σ-trees, of cardinality 2^λ, for sequences σ of arbitrary length $\lambda < \omega$; hence it must be infinite. To show that X is large, we must exhibit a σ-tree for every σ. In view of the remarks above, it will suffice to do this when the components $[p_i, q_i]$ of σ are pairwise disjoint. Let such a σ be given, and let λ be its length. For $J = j_0 \ldots j_{\lambda-1}$, let

$$\begin{aligned} x_J(r) &= j_i \qquad \text{if } r = p_i, \\ &= 0 \qquad \text{if } r \text{ is not one of the } p_i\text{'s}. \end{aligned}$$

Then $\{x_J \mid J : \lambda \to 2\}$ is clearly a σ-tree.

Finally, let us suppose that Y and Z are small but $Y \cup Z$ is large. Replacing Z by $Z - Y$, we may suppose Y and Z are disjoint. Let

$$\sigma^Y = ([p_0^Y, q_0^Y], \ldots, [p_{\mu-1}^Y, q_{\mu-1}^Y]) \qquad \text{and}$$

$$\sigma^Z = ([p_0^Z, q_0^Z], \ldots, [p_{\nu-1}^Z, q_{\nu-1}^Z])$$

be such that Y includes no σ^Y-tree and Z includes no σ^Z-tree. Let

$$\sigma = ([p_0, q_0], \ldots, [p_{\lambda-1}, q_{\lambda-1}]) \qquad (\lambda = \mu + \nu)$$

be the sequence consisting of all the intervals in σ^Y and all those in σ^Z, listed in order of increasing left endpoints. Say that a number $i < \lambda$ is of type Y (resp. type Z) if the interval $[p_i, q_i]$ came from σ^Y (resp. σ^Z). (If the same interval appears m times in σ^Y and n times in σ^Z, list it $m + n$ times in σ, and let its first m occurrences be of type Y, the rest of type Z.) As $Y \cup Z$ is large, let $\{x_J \mid J : \lambda \to 2\}$ be a fixed σ-tree in $Y \cup Z$.

Consider the following two-person game. The game consists of λ moves numbered 0 through $\lambda - 1$. At move i, one of the numbers 0 or 1 is written by player I if i is of type Z and by player II if i is of type Y. After λ moves, the players have produced a λ-sequence $J : \lambda \to 2$. Player I wins if $x_J \in Y$; player II wins if $x_J \in Z$. Since all the x_J are in $Y \cup Z$, and since Y and Z are disjoint, exactly one player wins. Because the game is finite, one of the players has a winning strategy.

Suppose player I has such a strategy. Given a sequence of μ zeros and ones, $J : \mu \to 2$, let $J^* : \lambda \to 2$ be the sequence obtained when player II writes the terms of J, in order, at his μ moves, while player I uses his winning strategy. Then, as I wins, $x_{J^*} \in Y$.

Further, if J and J' first differ at position i, then J^* and J'^* first differ at the $i + 1$st position of type Y, which is a position i' such that $p_{i'} = p_i^Y$ and $q_{i'} = q_i^Y$.

It follows that $\{x_{J^*} \mid J : \mu \to 2\}$ is a σ^Y-tree included in Y, which contradicts the choice of σ^Y. A similar contradiction results if II has a winning strategy. Thus, the lemma is proved.

The reader has surely noticed that this lemma is entirely analogous to the first lemmas in the proofs of Theorems 6 and 7. We could obtain Theorem 8 by continuing the analogy. The required Lemma 2 is essentially proved in [1, Theorem 9.8]. Lemma 3 and the construction of the required P-point then proceed exactly as before. However, we shall complete the proof of Theorem 8 in a different manner, analogous to the proof of Theorem 9 in the next section. This means that we shall do a bit more work than we must in the present proof, but we shall then find Theorem 9 somewhat simpler.

Lemma 2. *Assume Martin's axiom. Let $\mathcal{Y}$ be a filterbase on X consisting of fewer than c large sets. There is a large set T such that $T - Y$ is finite for all $Y \in \mathcal{Y}$.*

Proof. Let P be the set of all pairs (F, Y) where F is a finite subset of X and where $Y \in \mathcal{Y}$. Partially order P by defining

$$(F, Y) \leq (F', Y') \leftrightarrow F \subseteq F' \subseteq F \cup Y \text{ and } Y' \subseteq Y.$$

(Intuitively, we think of (F, Y) as the following partial description of the required $T: F \subseteq T \subseteq F \cup Y$.)

Let (F, Y_1) and (F, Y_2) be elements of P with the same first component. As $\mathcal{Y}$ is a filterbase, it contains a set $Y \subseteq Y_1 \cap Y_2$. Then (F, Y) is an upper bound for the (F, Y_i) in P. Hence, in any antichain of P, the first components of all the elements must be distinct, so the antichain must be countable.

For any sequence σ as above, let $D_\sigma = \{(F, Y) \in P \mid F \text{ includes a } \sigma\text{-tree}\}$. Because each $Y \in \mathcal{Y}$ is large, each D_σ is dense. For any $Y_0 \in \mathcal{Y}$, let

$$D(Y_0) = \{(F, Y) \in P \mid Y \subseteq Y_0\}.$$

$D(Y_0)$ is dense because $\mathcal{Y}$ is a filterbase. Let

$$\Delta = \{D_\sigma \mid \sigma \text{ a sequence as above}\} \cup \{D(Y_0) \mid Y_0 \in \mathcal{Y}\}.$$

As $\mathcal{Y}$ has cardinality $< c$, so does Δ. By Martin's axiom, let $G \subseteq P$ be generic for Δ, and let $T = \bigcup_{(F,Y)\in G} F$.

For each σ, there is an $(F, Y) \in G \cap D_\sigma$. So some σ-tree is $\subseteq F \subseteq T$. Therefore T is large.

Let Y_0 be any element of $\mathcal{Y}$. Let $(F_1, Y_1) \in G \cap D(Y_0)$, so $Y_1 \subseteq Y_0$. Let (F, Y) be any element of G. As G is generic, we can find $(F_2, Y_2) \geq$ both (F_1, Y_1) and (F, Y). Then, by definition of the ordering on P,

$$F \subseteq F_2 \subseteq F_1 \cup Y_1 \subseteq F_1 \cup Y_0,$$

so $F - Y_0 \subseteq F_1$. Since F_1 is independent of (F, Y), it follows that $T - Y_0 \subseteq F_1$, so $T - Y_0$ is finite. This completes the proof of the lemma.

We shall define inductively an increasing sequence $\{\mathcal{Y}_\alpha \mid \alpha < c\}$ of filterbases on X such that each $\mathcal{Y}_\alpha$ consists of large sets and has cardinality $< c$. We begin with $\mathcal{Y}_0 =$ the family of all cofinite sets. (Cofinite sets are large by Lemma 1.) At limit ordinals λ, we set $\mathcal{Y}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{Y}_\alpha$. The construction at successor ordinals splits into two cases.

Case 1. There is a set $Y \in \mathcal{Y}_\alpha$ and an $n \in \omega$ such that $\{x \in Y \mid f_\alpha(x) \geq n\}$ is small. Let T be $\{x \mid f_\alpha(x) < n\}$. For any $Y' \in \mathcal{Y}_\alpha$,

$$Y \cap Y' \subseteq \{x \in Y \mid f_\alpha(x) \geq n\} \cup (Y' \cap T).$$

As $\mathcal{Y}_\alpha$ is a filterbase of large sets, $Y \cap Y'$ is large, and Lemma 1 shows that $Y' \cap T$ is large. Thus, if we adjoin T to $\mathcal{Y}_\alpha$ and close under finite intersection, we obtain a filterbase of large sets, which we take as $\mathcal{Y}_{\alpha+1}$. Thus, f_α is bounded on a set $T \in \mathcal{Y}_{\alpha+1}$.

Case 2. The hypothesis of Case 1 fails. Let $\mathcal{B}$ be the filterbase obtained from $\mathcal{Y}_\alpha$ by adjoining the sets $\{x \mid f_\alpha(x) \geq n\}$ and closing under finite intersection. All the sets in $\mathcal{B}$ are large, as otherwise we would be in Case 1. By Lemma 2, let T be a large set such that $T - B$ is finite for all $B \in \mathcal{B}$. In particular, for any $Y \in \mathcal{Y}$, $T - Y$ is small, but Y is large so $T \cap Y$ is large by Lemma 1. Let $\mathcal{Y}_{\alpha+1}$ be obtained from $\mathcal{Y}_\alpha$ by adding T and closing under finite intersection. As

$$f_\alpha^{-1}\{n\} \cap T \subseteq T - \{x \mid f_\alpha(x) \geq n + 1\}$$

is finite, we see that, in this case, f is finite-to-one on a set of $\mathcal{Y}_{\alpha+1}$.

Let $\mathcal{Y} = \bigcup_{\alpha<c} \mathcal{Y}_\alpha$, and let $\mathcal{B}$ be the filter of sets whose complements are small. As in §4, there is an ultrafilter $D \supseteq \mathcal{Y} \cup \mathcal{B}$. D is a P-point because every f_α is finite-to-one or bounded on a set $T \in \mathcal{Y}_{\alpha+1} \subseteq \mathcal{Y} \subseteq D$. Furthermore, D contains no small sets. This completes the proof of Theorem 8. □

By suitably combining the constructions used in the proofs of Theorems 6 and 8, one can show that, for each element of $\mathcal{R}$, there is an embedding of **R** into $\mathcal{R}$ whose range is above that element. It follows (using Theorem 7) that the long line can be embedded in $\mathcal{R}$. We leave the details to the reader.

The existence of an embedding of **R** into the RK ordering (or even into the Rudin-Frolik ordering) of *all* ultrafilters on ω has been proved without the use of Martin's axiom (or any other special hypothesis) by Booth [3, Corollary 2.11]. However, the ranges of the embeddings he constructs are all disjoint from $\mathcal{R}$. A weaker form of Theorem 8, in which the continuum hypothesis is assumed, was proved in [1, Theorem 9.8].

6. ***P*-points with incomparable predecessors.** Although we have shown the existence of large chains and antichains in $\mathcal{R}$, it still seems possible for $\mathcal{R}$ to have a fairly simple structure. For example, the disjoint union of 2^c copies of the long line has all the properties which we have proved for $\mathcal{R}$ until now. In this section, we shall show that $\mathcal{R}$ is somewhat more complicated. There are pairs of incomparable elements of $\mathcal{R}$ which have upper and lower bounds in $\mathcal{R}$. The existence of such pairs with upper bounds is the content of the next theorem; the existence of lower bounds for such pairs follows by Corollary 1 of Theorem 5.

Theorem 9. *Assume Martin's axiom. There is a P-point with two incomparable predecessors* (*both of which must therefore also be P-points*).

Proof. We shall construct a P-point D on $\omega \times \omega$ such that $\pi_1(D)$ and $\pi_2(D)$ are incomparable.

A subset of $\omega \times \omega$ of the form $P \times Q$, where P and Q are subsets of ω of cardinality $n < \omega$, will be called an *n-square*. A subset of $\omega \times \omega$ will be called *large* if it includes an n-square for every n, and *small* otherwise.

Lemma 1. *The union of two small sets is small. If $Y \cup Z$ is large and Y is small, then Z is large. Finite sets are small. $\omega \times \omega$ is large. Any superset of a large set is large.*

Proof. It will suffice to prove the first assertion, as the second follows and the remaining three are obvious. So let $Y \cup Z$ be large. We shall show that Y or Z is large.

Let $n < \omega$ be given. As $Y \cup Z$ is large, it includes a k-square $P \times Q$, where $k = (n-1)\binom{2n}{n} + 1$. As $k \geq 2n - 1$, we can define Q_1 to be the set of the first $2n - 1$ elements of Q. For each $p \in P$, let $Y_p = \{q \in Q_1 \mid (p,q) \in Y\}$, $Z_p = \{q \in Q_1 \mid (p,q) \in Z\}$. As $Y \cup Z \supseteq P \times Q_1$, $Y_p \cup Z_p$ must be all of Q_1, so either Y_p or Z_p has at least n elements. Say that p is of type 1 if Y_p has at least n elements and of type 2 if Z_p does. At least $\frac{1}{2}(k-1) + 1 = (n-1)\binom{2n-1}{n} + 1$ elements of P must be of the same type, say type 1. Let P_1 be the set of these elements of P. For each $p \in P_1$, Y_p has at least n elements; let C_p be the set of the first n elements of Y_p. The function $p \longmapsto C_p$ maps the set P_1 of cardinality at least $(n-1)\binom{2n-1}{n} + 1$ into the set of n-element subsets of Q_1, i.e. into a set of cardinality only $\binom{2n-1}{n}$. Therefore, this function takes some value $Q_2 \subseteq Q_1$ at least n times. Let P_2 consist of the first n elements p of P_1 such that $C_p = Q_2$. Then

$$\begin{aligned} p \in P_2, q \in Q_2 &\rightarrow p \in P_1, q \in C_p \subseteq Y_p \\ &\rightarrow (p,q) \in Y. \end{aligned}$$

Thus, $P_2 \times Q_2$ is an n-square in Y.

We have shown that, for each n, either Y or Z includes an n-square. Hence one of them, say Y, includes n-squares for arbitrarily large n. But then it includes n-squares for all n, so it is large. This proves the lemma.

Lemma 2. *Assume Martin's axiom. Let $\mathcal{Y}$ be a filterbase on $\omega \times \omega$ consisting of fewer than c large sets. There is a large set T such that $T - Y$ is finite for all $Y \in \mathcal{Y}$.*

Proof. This proof is entirely analogous to the proof of Lemma 2 of Theorem 8. We define the partially ordered set P and the dense subsets $D(Y_0)$, and we verify that all antichains are countable exactly as in that proof (except that we have $\omega \times \omega$ in place of X). Instead of the D_σ's, we now have the sets

$$D_n = \{(F, Y) \in P \mid F \text{ includes an } n\text{-square}\},$$

which are dense because every $Y \in \mathcal{Y}$ is large. We let G be generic for

$$\Delta = \{D(Y_0) \mid Y_0 \in \mathcal{Y}\} \cup \{D_n \mid n \in \omega\},$$

and we let

$$T = \bigcup_{(F,Y)\in G} F.$$

For each n, there is an $(F, Y) \in G \cap D_n$, so some n-square is $\subseteq F \subseteq T$. Thus T is large. The proof that $T - Y_0$ is finite for all $Y_0 \in \mathcal{Y}$ proceeds exactly as before, so the lemma is proved.

Proceeding as in the proof of Theorem 8, we shall inductively define an increasing sequence $\{\mathcal{Y}_\alpha \mid \alpha < c\}$ of filterbases on $\omega \times \omega$ such that each $\mathcal{Y}_\alpha$ consists of large sets and has cardinality $< c$. We begin with $\mathcal{Y}_0$ = the family of cofinite sets, and we take unions at limit ordinals as before. The construction at successor ordinals will be split into *three* cases.

Case 1. There is a set $Y \in \mathcal{Y}_\alpha$ and an $n \in \omega$ such that $\{x \in Y \mid f_\alpha(x) \geq n\}$ is small. This case is handled exactly like Case 1 in the proof of Theorem 8. We obtain $\mathcal{Y}_{\alpha+1}$ such that f_α is bounded on a set of $\mathcal{Y}_{\alpha+1}$.

Case 2. The hypothesis of Case 1 fails, and f is not finite-to-one. This case is treated exactly like Case 2 in the proof of Theorem 8. We obtain $\mathcal{Y}_{\alpha+1}$ such that f_α is finite-to-one on a set of it.

Case 3. f_α is finite-to-one. (Note that the three cases are exclusive and exhaustive.) By Lemma 2, let T be a large set such that $T - Y$ is finite for all $Y \in \mathcal{Y}_\alpha$. We shall first construct a large subset T' of T such that $\pi_2(T')$ and $f_\alpha\pi_1(T')$ are disjoint. (Eventually, T' will be in the ultrafilter D, and we will be able to conclude that $\pi_2(D) \neq f_\alpha\pi_1(D)$.)

Inductively, define finite subsets S_n, X_n, $Y_n \subseteq \omega$ with

$$X_n \cap \pi_1(S_n) = Y_n \cap \pi_2(S_n) = \varnothing$$

and $X_n \subseteq X_{n+1}$, $Y_n \subseteq Y_{n+1}$ as follows. $S_0 = X_0 = Y_0 = \varnothing$. Let S_n, X_n, Y_n be given. Let k be the largest element of $X_n \cup Y_n \cup \pi_1(S_n) \cup \pi_2(S_n)$. (If this set is empty, set $k = 0$.) Let $P \times Q$ be a $2n + k + 1$-square included in T. Then P and Q each contain at least $2n$ elements $> k$, so we can find a $2n$-square,

$$P_1 \times Q_1 \subseteq P \times Q \subseteq T,$$

which is disjoint from $\pi_1^{-1}(X_n)$, $\pi_2^{-1}(Y_n)$, and S_n. As P_1 has only $2n$ elements, $\{y \in Q_1 \mid P_1 \cap f_\alpha^{-1}\{y\}$ has two distinct elements$\}$ has at most n elements. Thus, there is an n-element subset Q_2 of Q_1 such that

$$y \in Q_2 \rightarrow P_1 \cap f_\alpha^{-1}\{y\} \text{ has at most one element.}$$

Then $\bigcup_{y\in Q_2}(P_1 \cap f_\alpha^{-1}\{y\})$ has at most n elements, so let $P_2 \subseteq P_1$ be an n-element set disjoint from it. Let S_{n+1} be the n-square $P_2 \times Q_2$. Thus,

$$(*) \qquad x \in P_2, y \in Q_2 \to f_\alpha(x) \neq y.$$

Let

$$X_{n+1} = X_n \cup \{x \mid f_\alpha(x) \in Q_2\}, \qquad \text{and}$$
$$Y_{n+1} = Y_n \cup \{f_\alpha(x) \mid x \in P_2\}.$$

Note that X_{n+1} is finite because f_α is finite-to-one. We must check that the induction hypothesis remains true.

$$X_{n+1} \cap \pi_1(S_{n+1}) = (X_n \cup \{x \mid f_\alpha(x) \in Q_2\}) \cap P_2$$
$$= \varnothing$$

because P_2 is a subset of P_1 which is disjoint from X_n by construction, and because $x \in P_2 \to f_\alpha(x) \notin Q_2$ by (*). Also,

$$Y_{n+1} \cap \pi_2(S_n) = (Y_n \cup \{f_\alpha(x) \mid x \in P_2\}) \cap Q_2 = \varnothing$$

because Q_2 is a subset of Q_1 which is disjoint from Y_n, and because of (*).

Now let $T' = \bigcup_{n<\omega} S_n$. As S_{n+1} is an n-square, T' is large. We claim that $\pi_2(T')$ and $f_\alpha\pi_1(T')$ are disjoint. Let $(a,b) \in S_{n+1}$ and $(c,d) \in S_{m+1}$; we must prove that $f_\alpha(a) \neq d$.

If $n < m$, then, in the construction at stage $n+1$, we have $(a,b) \in S_{n+1} = P_2 \times Q_2$ so $f_\alpha(a) \in Y_{n+1} \subseteq Y_m$. At stage $m+1$, $(c,d) \in P_2 \times Q_2$ so d is in Q_2 which is disjoint from Y_m. Thus $f_\alpha(a) \in Y_m$ while $d \notin Y_m$.

If $m < n$, then, at stage $n+1$, we have $a \in P_1$, so $a \notin X_n$, $a \notin X_{m+1}$. By definition of X_{m+1}, we see $f_\alpha(a) \notin Q_2$ at stage $m+1$. But $d \in Q_2$ at stage $m+1$.

If $m = n$, then $f_\alpha(a) \neq d$ by (*).

Therefore, $\pi_2(T')$ is disjoint from $f_\alpha\pi_1(T')$. Let T'' be a large subset of T' such that $\pi_1(T'')$ is disjoint from $f_\alpha\pi_2(T'')$. We can obtain T'' from T' exactly as we obtained T' from T, except that π_1 and π_2 are interchanged. If $Y \in \mathcal{Y}_\alpha$, then $T'' - Y$ is finite (because $T - Y$ is), hence small. But

$$T'' = (T'' - Y) \cup (T'' \cap Y)$$

is large, so Lemma 1 implies that $T'' \cap Y$ is large.

Therefore, we may define $\mathcal{Y}_{\alpha+1}$ to be the filterbase obtained from $\mathcal{Y}_\alpha$ by adjoining T'' and closing under finite intersection. Thus, in Case 3, $\mathcal{Y}_{\alpha+1}$ contains a set T'' such that each of $\pi_1(T'')$ and $\pi_2(T'')$ is disjoint from the image of the other under f_α. As in previous proofs, there is an ultrafilter D which extends the filter $\mathcal{Y} = \bigcup_{\alpha<c} \mathcal{Y}_\alpha$ and which contains no small sets.

Neither $\pi_1(D)$ nor $\pi_2(D)$ is principal, for if $n \in \omega$, then $\pi_i^{-1}\{n\}$ is small, hence is not in D, so $\{n\} \notin \pi_i(D)$. It follows trivially that D is nonprincipal.

Let $f : \omega \to \omega$ be any map; so $f = f_\alpha$ for a certain $\alpha < c$. Consider the construction of $\mathcal{Y}_{\alpha+1}$. If Case 1 occurred, then f_α is bounded on a set in $\mathcal{Y}_{\alpha+1} \subseteq \mathcal{Y} \subseteq D$. If Case 2 occurred, then f_α is finite-to-one on a set in $\mathcal{Y}_{\alpha+1} \subseteq D$. If Case 3 occurred, then f_α is finite-to-one on the set $\omega \in D$. Therefore, D is a P-point. It follows, of course, that $\pi_1(D)$ and $\pi_2(D)$ are also P-points.

Now suppose $\pi_1(D)$ and $\pi_2(D)$ were comparable, say $\pi_2(D) = f\pi_1(D)$. As $\pi_1(D)$ is a P-point, f is finite-to-one or constant on a set $A \in \pi_1(D)$. If it were constant, $f\pi_1(D)$ would be principal, which we know it is not, so $f \upharpoonright A$ is finite-to-one. Choose $\alpha < c$ so that

$$\begin{aligned} f_\alpha(n) &= f(n) \qquad && \text{if } n \in A, \\ &= n && \text{if } n \notin A. \end{aligned}$$

Then f and f_α agree on the set $A \in \pi_1(D)$, so $f_\alpha\pi_1(D) = f\pi_1(D) = \pi_2(D)$.

As f_α is finite-to-one, Case 3 occurred in the construction of $\mathcal{Y}_{\alpha+1}$. Therefore, $\mathcal{Y}_{\alpha+1}$ (and hence also D) contains a set T'' such that $\pi_2(T'')$ is disjoint from $f_\alpha\pi_1(T'')$. But $\pi_2(T'') \in \pi_2(D)$ and $f_\alpha\pi_1(T'') \in f_\alpha\pi_1(D)$, which contradicts the fact that no two disjoint sets can be in the same ultrafilter.

Therefore, we have two incomparable P-points, $\pi_i(D)$, bounded above by the P-point D. $\square$

7. **Questions.** For minimal elements D of $\mathcal{R}$, let

$$\mathcal{R}_D = \{E \in \mathcal{R} \mid D \leq E\}.$$

According to Corollary 3 of Theorem 5, the various $\mathcal{R}_D$'s are disjoint, and elements of distinct $\mathcal{R}_D$'s are incomparable. Thus, we may think of $\mathcal{R}$ as decomposed into the disjoint pieces $\mathcal{R}_D$, plus possibly other pieces.

Question 1. Are there no other pieces? In other words, is every P-point $\geq$ some minimal ultrafilter? More generally, is every nonprincipal ultrafilter $\geq$ some minimal one? Choquet [4] has raised the related question whether every nonprincipal ultrafilter is $\geq$ some P-point.

Assuming the continuum hypothesis, R. A. Pitt and M. E. Rudin have (independently) answered Question 1 negatively. From this result and Corollary 2 of Theorem 5, it follows that there are decreasing sequences of length $\aleph_1$ in $\mathcal{R}$.

Question 2. Are any (or all) of the $\mathcal{R}_D$'s directed upward? More generally, is the converse of Corollary 1 of Theorem 5 true?

Question 3. Are any (or all) of the $\mathcal{R}_D$'s isomorphic?

Question 4. What ordinals can be embedded in $\mathcal{R}$? By the corollary to Theorem 7, $\aleph_1$ can be embedded. On the other hand, it is easy to see that an ultrafilter has at most c predecessors in the RK ordering, so $c^+ + 1$ cannot be embedded.

Question 5. What can be proved about $\mathscr{R}$ without using Martin's axiom? Can one even prove that there is a P-point?

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