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MAXIMAL REGULAR RIGHT IDEAL SPACE OF A PRIMITIVE RING. II

BY

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ABSTRACT. If R is a ring, let $X(R)$ be the set of maximal regular right ideals of R. For each nonempty subset E of R, define the *hull* of E to be the set $\{I \in X(R) \mid E \subseteq I\}$ and the *support of* E to be the complement of the hull of E. Topologize $X(R)$ by taking the supports of right ideals of R as a subbase. If R is a right primitive ring, then $X(R)$ is homeomorphic to an open subset of a compact space $X(R^{\#})$ of a right primitive ring $R^{\#}$, and $X(R)$ is a discrete space if and only if $X(R)$ is a compact Hausdorff space if and only if either R is a finite ring or a division ring. Call a closed subset F of $X(R)$ a *line* if F is the hull of $I \cap J$ for some two distinct elements I and J in $X(R)$. If R is a semisimple ring, then every line contains an infinite number of points if and only if either R is a division ring or R is a dense ring of linear transfomations of a vector space of dimension two or more over an infinite division ring such that every pair of simple (right) R-modules are isomorphic.

Introduction. Let R be a ring. A right ideal $I \subseteq R$ is called regular if there exists an $e \in R$ such that, for all $a \in R$, $a - ea \in I$. Let $X(R)$ be the set of maximal regular right ideals of R. If E is a nonempty subset of R, the set of maximal regular right ideals of R which contain E is called the *hull* of E and the complement of the hull of E with respect to $X(R)$ is called the *support* of E. We topologize $X(R)$ by taking the set of supports of right ideals of R as a subbase. Then the topological space $X(R)$ is called the *maximal regular right ideal space* of R [3]. If $1 \in R$, $X(R)$ is a compact space [3, 1.7]; however, in general, it is not a compact space. Recall that a topological space X is *irreducible* [4] if $X \neq \emptyset$ and X is not the union of two proper closed subsets. X is *reducible* if it is not irreducible. In [3], we have shown that if R is a (right) primitive ring, then $X(R)$ is reducible if and only if R is a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field, and $X(R)$ is a Hausdorff space if and only if either R is a division ring or $X(R)$ is reducible and R modulo its socle is a radical ring. In this paper we continue our previous work [3]. Our main results are as follows:

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If R is a ring and S is an ideal of R then $X(S)$ is homeomorphic to an open subset of $X(R)$. If R is a primitive ring then $X(R)$ is embeddable as an open subset of a compact space $X(R^{\#})$ for some primitive ring $R^{\#}$. As a consequence of this fact, we will be able to show that if R is a primitive ring then $X(R)$ is a compact Hausdorff space if and only if $X(R)$ is a discrete space if and only if either R is a division ring or a finite ring. For any subset Y of $X(R)$, define $j(y) = \bigcap \{I \in X(R) \mid I \in Y\}$.

Let us call a closed subset F of $X(R)$ a *line* if F is the hull of $j(\{x\}) \cap j(\{y\})$ for some distinct points x, y. Call a closed subset F of $X(R)$ a *hyperplane* provided that $j(F) \neq J(R)$, the Jacobson radical, and if F' is a closed subset such that $F' \supseteq F$ and $F' \neq F$ then $j(F') = J(R)$. If R is a primitive ring, then R has nonzero socle if and only if $X(R)$ has a *hyperplane*, and R is simple artinian if and only if (i) $X(R)$ is compact, (ii) $X(R)$ has no line which contains exactly two points, and (iii) $X(R)$ has a hyperplane. For a semisimple ring R, $X(R)$ contains at least two points and every line of $X(R)$ contains an infinite number of points if and only if R is a dense ring of linear transformations of a vector space of dimension two or more over an infinite division ring such that every pair of simple (right) R-modules are isomorphic. Throughout this paper, by a *primitive ring* we always mean a *right* primitive ring and every module is a *right* module.

1. Preliminaries.

1.1. **Definition.** If E is a subset of R, we define the *support* of E to be the set of maximal regular right ideals of R which do not contain the set E. It will be denoted by $\operatorname{supp}(E)$.

1.2. **Definition.** If R is a ring, let $X(R)$ be the set of maximal regular right ideals of R. We give a topology to $X(R)$ which is generated by the subbasis consisting of all supports of the right ideals of R. We will call $X(R)$ together with this topology the *maximal regular right ideal space* of R. It will simply be denoted by $X(R)$.

1.3. **Definition.** If E is a subset of R, the *hull* of E is the set $X(R) \setminus \operatorname{supp}(E)$. It will be denoted by $h(E)$. If $E = \{e\}$ is a singleton set, we let $h(e) = h(E)$.

1.4. **Definition.** A nonempty topological space X is *irreducible* if X is not the union of two proper closed subsets. X is *reducible* if it is not irreducible. A subset Y of a space X is irreducible if Y is irreducible in the induced topology.

1.5. **Definition.** If x is an element of $X(R)$ for some ring R, then x is also a right ideal of R. To make a distinction, we write $j(x)$ for the right ideal x. If Y is a subset of $X(R)$, we define $j(Y) = \bigcap \{j(x) \mid x \in Y\}$. If $Y = \{y\}$ a singleton set, we let $j(y) = j(Y)$.

1.6. **Definition.** Let R be a ring and let V be a right R-module. For each $v \in V$, we define $v^{\perp} = \{r \in R \mid vr = 0\}$. If E is a subset of V, we define $E^{\perp} = \bigcap \{v^{\perp} \mid v \in E\}$.

1.7. **Proposition.** *Let R be a ring and I be a maximal regular right ideal of R. If A, B are right ideals of R such that $AB \subseteq I$, then either $A \subseteq I$ or $B \subseteq I$.*

Proof. Suppose that $A \not\subseteq I$ and $B \not\subseteq I$. Then $A + I = R$ and $RB = (A + I)B \subseteq I$. Let e be a left identity modulo I. Since $eb \in I$, for every $b \in B$, $B \subseteq I$. This is a contradiction.

1.8. **Corollary.** *Let R be a ring and I be a maximal regular right ideal of R. If $a \in R$ such that $aR \subseteq I$, then $a \in I$.*

Proof. Let $A = \{a \in R \mid aR \subseteq I\}$. Then A is a right ideal of R and $AR \subseteq I$. Hence by 1.7, $A \subseteq I$ and $a \in I$.

1.9. **Corollary.** *Let R be a ring and S be an ideal of R. If m is a maximal regular right ideal of S then m is a right ideal of R.*

Proof. Let a be an element of m and let r be an element of R. Then $(ar)S = a(rS) \subseteq m$. Hence by 1.8, $ar \in m$.

1.10. **Proposition.** *Let S be an ideal of a ring R. Let m be a maximal regular right ideal of S and let f be a left identity modulo m in S. Define $I(m, f) = \{r \in R \mid fr \in m\}$. Then $I(m, f)$ is a maximal regular right ideal of R such that $I(m, f) \cap S = m$. If e is another left identity modulo m in S, then $I(m, e) = I(m, f)$.*

Proof. By 1.9, m is a right ideal of R. Hence $I(m, f)$ is a right ideal of R. Since $f(fr - r) = f(fr) - fr \in m$ for every $r \in R$, f is a left identity modulo $I(m, f)$. Clearly $m \subseteq I(m, f)$. Since $fs - s \in m$ for every $s \in S$, $m = I(m, f) \cap S$. If $I(m, f)$ were not maximal, then there would exist a maximal right ideal, say M of R, such that $I(m, f) \subseteq M$ but $I(m, f) \neq M$. Let $a \in M$ such that $a \notin I(m, f)$. Then $fa \notin m$ and $fa \in M$. Hence $m \subseteq M \cap S$ but $m \neq M \cap S$. Since $M \cap S$ is a right ideal of S and m is maximal, $M \cap S = S$ and $f \in M$. Thus $R = M$. This is a contradiction. In order to prove that $I(m, f) = I(m, e)$, we observe that $fs - s \in m$ and $s - es \in m$ for every $s \in S$. This implies that $fs - es = (f - e)s \in m$ for every $s \in S$. Hence, by 1.8, $f - e \in m$ and $fr \in m$ if and only if $er \in m$ for every $r \in R$.

1.11. **Proposition.** *Let R be a ring and S be an ideal of R. If I_1, I_2 are maximal regular right ideals of R such that $S \not\subseteq I_i$, $i = 1, 2$, then $S \cap I_i$ is a maximal regular right ideal of S for each i, $i = 1, 2$, and if $S \cap I_1 = S \cap I_2$, then $I_1 = I_2$.*

Proof. Let e_i be a left identity modulo I_i for each i where $i = 1, 2$. Since $S \not\subseteq I_i$, $S + I_i = R$ and $e_i = s_i + a_i$ for some $s_i \in S$ and $a_i \in I_i$. Then s_i is a left

identity modulo $I_i \cap S$ in S. Let m be a maximal regular right ideal of S such that $I_i \cap S \subseteq m$. Clearly, s_i is a left identity modulo m. Hence by 1.10, $m = I(m, s_i) \cap S$ and $I_i \cap S \subseteq I(m, s_i) \cap S$. Since S is a two-sided ideal of R, $I_i S \subseteq I_i \cap S \subseteq I(m, s_i)$. Since $S \not\subseteq I(m, s_i)$, $I_i \subseteq I(m, s_i)$ by 1.7 and hence $I_i = I(m, s_i)$. Thus $I_i \cap S = m$ and if $S \cap I_1 = S \cap I_2$ then, by 1.10, $I_1 = I_2$.

1.12. **Theorem.** *Let R be a ring and S be an ideal of R. Then* $\operatorname{supp}(S)$ *is homeomorphic to* $X(S)$.

Proof. Define a function ϕ from $\operatorname{supp}(S)$ into $X(S)$ by $\phi(I) = I \cap S$. Then ϕ is a bijection by 1.10 and 1.11. Let A be a right ideal of S. Then $\phi^{-1}(\operatorname{supp}(A)) = \{I \in \operatorname{supp}(S) \mid A \not\subseteq I \cap S\}$. But this set is equal to $\operatorname{supp}(AR)$ in $X(R)$ which is contained in $\operatorname{supp}(S)$. Thus ϕ is a continuous mapping. Since S is a two-sided ideal, if B is a right ideal of R then $\operatorname{supp}(S) \cap \operatorname{supp}(B) = \operatorname{supp}(BS)$ (refer 1.7) and $\phi(\operatorname{supp}(BS)) = \{I \cap S \mid I \in X(R) \text{ and } BS \not\subseteq I\}$. This is clearly an open set in $X(S)$ and ϕ is an open mapping.

1.13. **Proposition.** *If R is a primitive ring, then there is a primitive ring R^* with 1 such that R is an ideal of R^* and $X(R)$ is homeomorphic to an open subset of $X(R^*)$.*

Proof. Let $E = \mathbf{Z} \times R$, the usual extension ring of R, Let I be a maximal right ideal of E such that $R \not\subseteq I$ and $R/(I \cap R)$ is a faithful simple R-module. If E/I were not faithful, let S be the largest ideal of E which is contained in I. Then $S \cap R = 0$ since $R/(I \cap R)$ is faithful. Let $R^* = E/S$. Then R^* is a primitive ring with 1 and R is isomorphic to an ideal of R^*. Hence by 1.12, $X(R)$ is homeomorphic to an open subset of $X(R^*)$.

1.14. **Definition.** A ring R is called *(von Neumann) regular* if and only if for every element a in R there is b in R such that $aba = a$ (refer [6]).

1.15. **Remark.** If R is (von Neumann) regular, then $a \in aR$ for every $a \in R$ even though R may not contain a unit element and $aR = eR$ for some idempotent $e \in R$. von Neumann proved that if R is a regular ring with 1, then for any idempotent $e \in R$ and an element $b \in R$, $eR + (1 - e)bR = gR$ for some idempotent $g \in R$ (refer [6]). Since $eR + (1 - e)bR = eR + bR$, this means that any finitely generated right ideal of R is principal. With a slight modification of von Neumann's proof for the above assertion, one can also conclude that if R is a regular ring (not necessarily with 1) then for any idempotent $e \in R$ and an element $b \in R$, $eR + bR = eR + \{br - ebr \mid r \in R\} = gR$ for some idempotent g in R.

1.16. **Theorem.** *If R is a regular ring, then $X(R)$ is compact if and only if $1 \in R$.*

Proof. If $1 \in R$ then $X(R)$ is compact for every ring R by [3, 1.7]. Assume that R is a regular ring and $X(R)$ is a compact space. Let E be the set of idempotents in R. Then $X(R) = \bigcup_{e_\alpha \in E} \operatorname{supp}(e_\alpha R)$. Since $X(R)$ is compact, there exist $e_{\alpha_1}, e_{\alpha_2}, \cdots, e_{\alpha_n}$, a finite number of idempotents in R, such that $X(R) = \bigcup_{i=1}^n \operatorname{supp}(e_{\alpha_i} R)$. Hence $\varnothing = \bigcap_{i=1}^n h(e_{\alpha_i} R) = h(\sum_{i=1}^n e_{\alpha_i} R)$. Let e be an idempotent such that $\sum_{i=1}^n e_{\alpha_i} R = eR$. If $eR \neq R$, then the right ideal $\{r - er \mid r \in R\}$ is not zero. Let $b \in R$ be such that $b - eb \neq 0$. Then by 1.15, there is an idempotent $g \in R$ such that $gR = eR + (b - eb)R$. Hence $ge = e$ and $g - e \neq 0$. Since R is a semisimple ring, there is a simple right R-module M such that $M(g - e) \neq 0$. Let $m \in M(g - e)$ such that $m \neq 0$. Then $meR = 0$ and $eR \subseteq m^\perp$, which is maximal regular right ideal. This means that $h(eR) \neq \varnothing$, which is a contradiction. Thus $eR = R$ and $ex = x$ for every $x \in R$. Since $(x - xe)R = 0$ for every $x \in R$, $e = 1$.

1.17. **Theorem.** *If R is a primitive ring, then $X(R)$ is a compact Hausdorff space if and only if R is either a division ring or a finite ring.*

Proof. If R is either a division ring or a finite ring, then certainly $X(R)$ is a compact Hausdorff space. Suppose that $X(R)$ is a compact Hausdorff space. By [3, 2.5], if S is the socle of R, then R/S is a radical ring. Since $X(S)$ is homeomorphic to $\operatorname{supp}(S)$ and $X(R) = \operatorname{supp}(S)$, $X(S)$ is a Hausdorff compact space. Since S is a regular ring, by 1.16, $1 \in S$ and $R = S$. Thus by [3, 2.7] either R is a division ring or a finite ring.

2. **Irreducible closed sets.** If R is a dense ring of linear transformations of a vector space, then an element $a \in R$ generates a minimal right ideal if and only if the rank of a is 1 (refer [1, p. 76]). It is interesting to note that the following assertion is also true:

2.1. **Proposition.** *If R is a dense ring of linear transformations of a left vector space V over a division ring D and A is a nonzero right ideal of R such that every nonzero element of A is of rank 1, then A is a minimal right ideal.*

Proof. Let a, b be two nonzero elements of A. Then $V = \ker a \oplus Du = \ker b \oplus Dw$ for some nonzero vectors u and w since the ranks of a and b are one. If either $\ker a \subseteq \ker b$ or $\ker b \subseteq \ker a$ then $\ker a = \ker b$. Hence $ua \neq 0$, $ub \neq 0$ and $uar = ub$ for some $r \in R$. Thus $V(ar - b) = 0$ and $aR = bR$ and, therefore, A is a minimal right ideal. So suppose now that there exist $v_1 \in (\ker a)\backslash(\ker b)$, $v_2 \in (\ker b)\backslash(\ker a)$. There is $r \in R$ such that $v_2 ar$ and $v_1 b$ are linearly independent if the dimension of the space is greater than 1. Since a, b are elements of A, $ar + b \in A$. However, the rank of $ar + b$ is greater than or equal to 2 since $v_1(ar + b) = v_1 b$ and $v_2(ar + b) = v_2 ar$. This is a contradiction.

2.2. **Proposition.** *Let R be a dense ring of linear transformations of a vector space V over a division ring D and let A be a nonzero right ideal of R. Then $h(a) = h(A)$ for every nonzero $a \in A$ if and only if A is a minimal right ideal.*

Proof. Since $h(a) = h(aR)$ for every $a \in R$, if A is a minimal right ideal, then certainly $h(a) = h(A)$. Assume now that $h(a) = h(A)$ for every nonzero $a \in A$. In order to prove that A is minimal, it suffices to show that the rank of nonzero $a \in A$ is 1. If the rank of a is not 1, then there exist vectors v_1, v_2 in V such that $v_1 a$, $v_2 a$ are linearly independent. Hence there is $r \in R$ such that $v_1 ar = 0$ and $v_2 ar \neq 0$. By hypothesis, $h(A) = h(a) = h(ar)$. Since $v_1^{\perp} \in X(R)$ and $v_1^{\perp} \in h(ar) = h(a)$, $v_1 a = 0$. This is a contradiction.

2.3. **Definition.** If R is a ring and A is a right ideal of R, the radical of A is $\sqrt{A} = \bigcap \{I \mid I \in X(R) \text{ and } I \supseteq A\}$.

2.4. **Proposition.** *Let R be a dense ring of linear transformations of a left vector space V over a division ring D. If A is a minimal right ideal of R, then $\sqrt{A} = A$.*

Proof. Suppose $A \neq \sqrt{A}$. Since $A \subseteq \sqrt{A}$ always, if $A \neq \sqrt{A}$ then there is $b \in \sqrt{A}$ such that $b \notin A$. Since R is semisimple, $X(R) \neq h(b)$ and $h(b) \supseteq h(\sqrt{A}) = h(A)$. Let a be a nonzero element of A. Then $h(A) = h(a)$ since $A = aR$. Hence $h(b) \supseteq h(a)$. Let $\ker a = \{v \in V \mid va = 0\}$ and $\ker b = \{v \in V \mid vb = 0\}$. Then $\ker a \subseteq \ker b$ since $h(b) \supseteq h(a)$. Since the rank of a is 1, $V = \ker a \oplus Dw$ for some nonzero vector w in V. Since $\ker a \subseteq \ker b$ and $b \neq 0$, $wb \neq 0$. Let $r \in R$ such that $war = wb$. Then $V(ar - b) = 0$ and $ar = b$. Hence $b \in A$ and this is a contradiction.

2.5. **Definition.** Let R be a ring. A closed subset F of $X(R)$ is said to be a hyperplane of $X(R)$ provided that $j(F) \neq J(R)$, the Jacobson radical, and if F' is a closed subset of $X(R)$ such that $F' \supseteq F$ and $F' \neq F$ then $j(F') = J(R)$.

2.6. **Theorem.** *Let R be a primitive ring. If A is a minimal right ideal of R, then $h(A)$ is a hyperplane. If F is a hyperplance of $X(R)$ then $j(F)$ is a minimal right ideal of R.*

Proof. Let A be a minimal right ideal of R and let $F = h(A)$. Let F' be a closed subset of $X(R)$ such that $F' \supseteq F$ and $j(F') \neq 0$. Let $B = j(F')$. Then $0 \neq B = j(F') \subseteq j(F) = \sqrt{A} = A$. By minimality of A, $B = A$, so $F' \subseteq h(A) = F$ and $F = F'$. Conversely, assume that F is a hyperplane. Let $A = j(F)$. If a is a nonzero element of A, then $j(h(a)) \neq 0$ and $h(a) \supseteq h(A) \supseteq F$. It follows that $h(a) = h(A) = F$. Hence by 2.2, A is a minimal right ideal of R.

2.7. **Corollary.** *Let R be a primitive ring. Let Σ_1 be the set of minimal right ideals of R. Let $\mathcal{F} = \{F \mid F$ is a closed subset of $X(R)$ such that $j(F) \neq 0\}$.*

Let Σ_2 be the set of maximal elements of $\mathcal{F}$. Then there is a bijection from Σ_1 onto Σ_2.

Proof. The mapping $A \to h(A)$ for $A \in \Sigma_1$ is a bijection from Σ_1 onto Σ_2.

2.8. **Theorem.** *If R is a primitive ring such that $X(R)$ is irreducible and A is a minimal right ideal of R, then either $h(A) = \varnothing$ or $h(A)$ is a maximal proper irreducible closed subset of $X(R)$.*

Proof. If A is a minimal right ideal of a primitive ring R then there is an idempotent $e \in R$ such that $A = eR$, and if $h(A) \neq \varnothing$ then $e^l = \{r \in R \mid re = 0\}$ is a nonzero left ideal of R. Hence $V_0 = e^l \cap A \neq 0$ and V_0 is a subspace of the vector space eR over the division ring eRe. Since R is a dense ring of linear transformations of the space eR and $X(R)$ is irreducible, by [3, 2.4] eRe is an infinite field. Since A is minimal, by 2.2 $h(a) = h(A)$ if a is a nonzero element of A. Let $\ker a = \{x \in A \mid xa = 0\}$. Then $V_0 = \ker a$. Since the rank of a is 1, there is a vector u such that $eR = V_0 \oplus eReu$. Hence $V_0^\perp \cap u^\perp = 0$ and $V_0^\perp$ is a minimal right ideal of R since $u^\perp$ is a maximal right ideal of R. Since $V_0^\perp \supseteq aR$ and $V_0^\perp$ is a minimal right ideal, $V_0^\perp = aR$. Now suppose that $h(A)$ is reducible. Then by [3, 1.11] there exist right ideals $A_1, A_2, \cdots, A_n$ such that $h(A) \neq h(A_i)$ for every i and $h(A) = \bigcup_{i=1}^n h(A_i)$. Let $W_i = \{v \in eR \mid vA_i = 0\}$ for every i. Then W_i is a subspace of V_0 and $V_0 = \bigcup_{i=1}^n W_i$. If $V_0 = W_i$ for some i, then $V_0^\perp = W_i^\perp = aR$, and $h(A) = h(aR) = h(W_i^\perp)$. Since $A_i \subseteq W_i^\perp$, $h(A_i) \supseteq h(W_i^\perp) = h(A)$. This is a contradiction.

2.9. **Lemma.** *Let R be a simple ring with nonzero socle. If $X(R)$ is a discrete space then either R is a division ring or R is a finite ring.*

Proof. Suppose that R is neither a division ring nor a finite ring. By [3, 2.6], R is isomorphic to a dense ring of linear transformations of finite rank of a vector space V over a finite field D and hence $1 \notin R$. Let p be the characteristic of D then $pr = 0$ for every $r \in R$. Let $\mathbf{Z}/(p)$ be the field of integers modulo p. Then R is an algebra over $\mathbf{Z}/(p)$. Let $R^* = \mathbf{Z}/(p) \times R$ be the usual extension ring of R with 1 in which R is an ideal. Let m be a maximal regular right ideal of R and let e be a left identity modulo m in R. Then $I(m, e)$ is a maximal right ideal of R^* by 1.10. Let S be an ideal of R^* such that $S \subseteq I(m, e)$. If S were a nonzero ideal then $S \cap R = 0$ since m contains no nonzero ideal of R. Hence $SR = 0$. Let (f, s_0) be a nonzero element of S for some $f \in \mathbf{Z}/(p)$ and $s_0 \in R$. Then $f \neq 0$ and $(f, s_0)(0, r) = (0, 0)$ for every $r \in R$. Hence $-(f^{-1}s_0)r = r$ for every $r \in R$. Since R is a simple ring, $-(f^{-1}s_0) = 1$ and R must be a finite ring. Therefore, $S = 0$ and $R^*/I(m, e)$ is a faithful simple R^*-module. Thus R^* is a primitive ring with 1. Observe $h(R) = \{R\}$ in $X(R^*)$ and

$X(R^*) = \operatorname{supp}(R) \cup \{R\}$. Since $X(R)$ is homeomorphic to $\operatorname{supp}(R)$ in $X(R^*)$ by 1.12 and $X(R)$ is a discrete space, every point of $\operatorname{supp}(R)$ is open in $X(R^*)$. Since the point R in $X(R^*)$ can be separated from any point of $\operatorname{supp}(R)$ by open sets, this means that $X(R^*)$ is a Hausdorff space. Since R^* is a primitive ring with 1, by [3, 2.7] R^* must be a finite ring and therefore R must be also a finite ring. This is a contradiction.

2.10. **Theorem.** *Let R be a primitive ring. Then $X(R)$ is a discrete space if and only if either R is a division ring or a finite ring.*

Proof. Since $X(R)$ is a T_1-space, if R is a finite ring, then certainly $X(R)$ is a discrete space. Suppose $X(R)$ is a discrete space such that R is not a division ring. Then $X(R)$ is a Hausdorff space and, by [3, 2.5], R is a dense ring with nonzero socle of linear transformations of a vector space of dimension two or more over a finite field and if S is the socle of R then R/S is a radical ring. If S is a finite ring, then R must be also a finite ring since $X(R) = \operatorname{supp}(S) \cup b(S) = \operatorname{supp}(S)$ and since a primitive ring which is not a division ring with a finite number of maximal regular right ideals is a finite ring. Now observe that S satisfies the hypothesis of 2.9. Hence if $X(R)$ is a discrete space, then so is $X(S)$ and, by 2.9, S is a finite ring.

2.11. **Example.** Let $\mathbf{Z}$ be the ring of integers and let R be the ring of all row-finite infinite matrices over $\mathbf{Z}/(2)$ of the following form:

$$\begin{pmatrix} A_n & & & & & \\ & 0 & & & & * \\ & & 0 & & & \\ & & & 0 & & \\ & 0 & & & \cdot & \\ & & & & & \cdot \\ & & & & & & \cdot \end{pmatrix}$$

where A_n is an $n \times n$ matrix for some positive integer n. Then R is a primitive ring with nonzero socle and if S is the socle of R then R/S is a radical ring. Hence $X(R)$ is a Hausdorff space by [3, 2.5] and it is not a discrete space by 2.10.

3. **Rings in which every proper right ideal is incompressible.** Let I, J be two distinct maximal regular right ideals of a ring R. If K, M are distinct maximal regular right ideals of R such that $\{K, M\} \subseteq b(I \cap J)$, then $(K \cap M)/(I \cap J)$ is a submodule of the right R-module $R/(I \cap J)$ which is isomorphic to a direct sum of two simple modules $I/(I \cap J)$ and $J/(I \cap J)$. Hence if $(K \cap M)/(I \cap J)$ is not the zero submodule then either $(K \cap M)/(I \cap J) = R/(I \cap J)$ or $(K \cap M)/(I \cap J)$ is isomorphic to $I/(I \cap J)$ or to $J/(I \cap J)$. In any case, this

means that either $R/(K \cap M)$ is simple or $R/(K \cap M) = 0$. This, of course, is absurd. Thus $K \cap M = I \cap J$ and $h(K \cap M) = h(I \cap J)$.

3.1. **Definition.** If x, y are two distinct points in $X(R)$, a *line* containing x and y is $l(x, y) = h(j(x) \cap j(y))$.

3.2. **Proposition.** *Given two distinct points in $X(R)$, there is one and only one line containing these two points.*

Proof. Let x, y be two distinct points in $X(R)$. If z, w are distinct points in $X(R)$ such that $\{z, w\} \subseteq l(x, y)$, then $l(z, w) = l(x, y)$ since $j(z) \cap j(w) = j(x) \cap j(y)$.

3.3. **Proposition.** *If x, y are two distinct points in $X(R)$ such that $R/j(x) \cong R/j(y)$ as R-modules, then for every $w \in l(x, y)$, $R/j(w) \cong R/j(x)$.*

Proof. If $w \in l(x, y)$, then $j(w)/(j(x) \cap j(y))$ is a nonzero submodule of $R/(j(x) \cap j(y))$. Hence $j(w)/(j(x) \cap j(y))$ is either isomorphic to $R/j(x)$ or to $R/j(y)$.

3.4. **Lemma.** *Let R be a ring and M be a simple R-module. Let $D = \mathrm{Hom}_R(M, M)$. If m_1, m_2 are nonzero elements of M such that $m_1^\perp \neq m_2^\perp$ then $h(j(m_1^\perp) \cap j(m_2^\perp))$ contains at least three elements.*

Proof. Since M is a left vector space over a division ring D, if m is a nonzero element of $Dm_1 \oplus Dm_2$ then $m^\perp \in h(j(m_1^\perp) \cap j(m_2^\perp))$, and if $m = m_1 + m_2$ then $m^\perp \neq m_1^\perp$ and $m^\perp \neq m_2^\perp$.

3.5. **Theorem.** *Let R be a ring. Then there is exactly one isomorphic class of simple R-modules if and only if $X(R)$ has no line which contains exactly two points.*

Proof. Assume that R has a unique isomorphic class of simple modules. If there is a two-point line, say $l(x, y)$, for some two distinct points x and y in $X(R)$, then $h(j(x) \cap j(y)) = \{x, y\}$. Let $M = R/j(x)$. Then $x = m_1^\perp$ and $y = m_2^\perp$ for some m_1, m_2 in M and $M^\perp = J(R)$ since all simple R-modules are isomorphic. By 3.4, $l(x, y)$ contains at least 3 points. This is a contradiction. Conversely, suppose that $R/I_1 \cong M_1 \ncong M_2 \cong R/I_2$ are simple modules where $I_1, I_2 \in X(R)$. If there exists maximal regular right ideal K such that $K \supseteq I_1 \cap I_2$ and $I_j \neq K$ for $j = 1, 2$, then by the remark at the beginning of §3, $R/I_1 = (K + I_1)/I_1 \cong K/(I_1 \cap K) = K/(I_2 \cap K) \cong (K + I_2)/I_2 = R/I_2$, a contradiction. Thus the line containing I_1 and I_2 contains exactly two points.

3.6. **Proposition.** *If R is a ring with 1 and if there is exactly one isomorphic class of simple R-modules then $R/J(R)$ is a simple ring. There is a primitive*

ring with nonzero socle in which every pair of simple modules are isomorphic, but the ring is not simple.

Proof. If $R/J(R)$ is not a simple ring, then there is an ideal S in R such that $S \supseteq J(R)$ but $S \neq J(R)$ and $R \neq S$. Let I be a maximal right ideal of R such that $I \supseteq S$. Then R/I is a simple R-module and $(R/I)^{\perp} \supseteq S$. Since every pair of simple R-modules are isomorphic, if M is a simple R-module, then $M^{\perp} \supseteq S$. Hence $J(R) \supseteq S$. This is a contradiction. Now, let R be the primitive ring in Example 2.11. If S is the socle of R then R/S is a radical ring. Hence if $x \in X(R)$, $R/j(x)$ is isomorphic to a minimal right ideal of R. Thus every pair of simple R-modules are isomorphic. However, R is not a simple ring.

3.7. **Theorem.** *Let R be a primitive ring. Then R is simple artinian if and only if* (i) *$X(R)$ is compact,* (ii) *$X(R)$ has no two-point line and* (iii) *$X(R)$ has a hyperplane.*

Proof. Assume (i), (ii) and (iii). Then R has nonzero socle by 2.6. Let S be the socle of R. Since $X(R)$ has no two-point line, by 3.5 all simple R-modules are isomorphic. Hence every simple module is isomorphic to a minimal right ideal. Therefore, R/S is a radical ring. Thus by 1.12, $X(S)$ is compact and by 1.16, $1 \in S$. Since the unity of the ring S is also the unity of R, $R = S$ and R is a simple artinian ring. Conversely, if R is a simple artinian ring then $1 \in R$. Hence $X(R)$ is compact by [3, 1.7]. Since every pair of R-modules in a simple artinian ring are isomorphic, by 3.5, (ii) is true. Finally (iii) is true by 2.6.

3.8. **Theorem.** *Let R be a semisimple ring. Then the following two statements are equivalent:*

(i) *There exist at least two points in $X(R)$ and every line contains an infinite number of points.*

(ii) *R is a dense ring of linear transformations of a vector space of dimension two or more over an infinite division ring such that every pair of simple R-modules are isomorphic.*

Proof. Assume (i). Then by 3.5 there is exactly one isomorphic class of simple R-modules. Hence if M_1, M_2 are two simple R-modules then $M_1^{\perp} = M_2^{\perp}$. Thus $M_1^{\perp} = J(R) = 0$. Therefore, every simple R-module is faithful and, in particular, R is a primitive ring. Let M be a simple R-module and let D be the endomorphism ring of R-module M. Since $X(R)$ contains at least two points, the dimension of the vector space M over D is at least two. Let m_1, m_2 be two linearly independent vectors in M. Then $m_1^{\perp} \neq m_2^{\perp}$ and the line $l(m_1^{\perp}, m_2^{\perp})$ contains an infinite number of points by hypothesis. If $x \in l(m_1^{\perp}, m_2^{\perp})$ then $x = m^{\perp}$ for some $m \in M$ since $M \cong R/(j(x))$ and hence $m \in Dm_1 \oplus Dm_2$ by [1, Lemma, p. 27].

Conversely, if m is a nonzero element of $Dm_1 \oplus Dm_2$, then $m^\perp \in l(m_1^\perp, m_2^\perp)$. Thus D must be an infinite division ring. Assume, now, (ii). Let R be a dense ring of linear transformations of a vector space M over an infinite division ring D of dimension two or more, then $X(R)$ contains at least two points. Let x and y be two distinct points in $X(R)$. Then $x = m_1^\perp$ and $y = m_2^\perp$ for some m_1, m_2 in M since $M \cong R/j(x) \cong R/j(y)$. Hence the line $l(x, y) = h(m_1^\perp \cap m_2^\perp)$. Let W be the two-dimensional subspace of M, which is spanned by m_1 and m_2. Observe that $z \in l(x, y)$ if and only if $z = w^\perp$ for some $0 \neq w \in W$. Since D is infinite and $(m_1 + \alpha m_2)^\perp \neq (m_1 + \beta m_2)^\perp$ for $\alpha \neq \beta$ in D, W contains infinitely many vectors. Thus $l(x, y)$ contains an infinite number of points.

3.9. **Theorem.** *Let R be an arbitrary ring. Then every line in $X(R)$ contains exactly two points if and only if every maximal regular right ideal is also a left ideal.*

Proof. Assume that every line in $X(R)$ contains exactly two points. Let I be a maximal regular right ideal of R. Let $N(I) = \{r \in R \mid rI \subseteq I\}$. Then by [1, Theorem 1, p. 25], $N(I)/I = \mathrm{Hom}_R(R/I, R/I)$ and R/I is a left vector space over the division ring $N(I)/I$. If the dimension of the vector space R/I over $N(I)/I$ is greater than one, then by 3.4 there will be a line which contains more than two points. Hence R/I is one dimensional. Let e be a left identity modulo I. Then for every $a \in R$ there is $b \in N(I)$ such that $be - a \in I$. Since $be \in N(I)$, $a \in N(I)$. Thus I is a left ideal. Conversely, if every maximal regular right ideal of R is also a left ideal, then $IJ \subseteq I \cap J$ for every pair of maximal regular right ideals I and J. Hence if $K \in h(I \cap J)$ then $IJ \subseteq K$ and either $I = K$ or $J = K$ by 1.7. Thus $h(I \cap J) = \{I, J\}$.

3.10. **Proposition.** *Let R be an arbitrary ring. Assume that every proper right ideal of R is an intersection of maximal regular right ideals. Then for every right ideal A, $AR = A$.*

Proof. If $R^2 \neq R$, then by hypothesis, R^2 must be an intersection of maximal regular right ideals. This is of course impossible in view of 1.7. Therefore, $R^2 = R$. If A is a proper right ideal of R then $AR = \sqrt{AR}$ and $A = \sqrt{A}$. If there is a maximal regular right ideal I, such that $AR \subseteq I$, then by 1.7, $A \subseteq I$ so $\sqrt{AR} = \sqrt{A}$. Hence $AR = A$.

3.11. **Definition.** Let R be an arbitrary ring. An R-module M is called *injective* if and only if for every pair A, B of R-modules with $A \subseteq B$, each homomorphism of A into M can be extended to one of B into M. If R is a ring with 1 and if M is a unital R-module, then it is well known that M has a minimal injective extension which is also a maximal essential extension of M. Such minimal injective extension is unique up to an isomorphism. It has been shown by

R. E. Johnson [2, 7.1] that these results are also true for any ring R and any R-module M. For each R-module M, we denote by $\hat{M}$ the minimal injective extension of M.

3.12. **Proposition.** *Let R be a ring. If every simple R-module is injective, then for any simple module M the R-homorphism ϕ: $M \to \mathrm{Hom}_R(R, M)$ where $\phi(m)(r) = mr$ for every $m \in M$ and $r \in R$ is an isomorphism.*

Proof. Clearly, ϕ is a monomorphism. Let $f \in \mathrm{Hom}_R(R, M)$. Let $R^{\#}$ be an extension ring of R with 1. Then there is $\bar{f} \in \mathrm{Hom}_R(R^{\#}, M)$ such that $\bar{f}|_R = f$ since M is injective. Let $\bar{f}(1) = m_0$. Then $f(r) = m_0 r$ for every $r \in R$. Therefore, $f = \phi(m_0)$ and ϕ is an isomorphism.

3.13. **Proposition.** *Let R be a ring such that $R^2 = R$. If every simple R-module is injective then every maximal right ideal of R is regular.*

Proof. Let I be a maximal right ideal of R. Since $R^2 = R$, R/I is a simple R-module. Hence, by hypothesis, R/I is injective and, by 3.12, $\phi(R/I) = \mathrm{Hom}_R(R, R/I)$ where $\phi(r + I)(r') = rr' + I$ for $r, r' \in R$. Define g: $R \to R/I$ such that $g(r) = r + I$ for each $r \in R$. Then $g \in \mathrm{Hom}_R(R, R/I)$ and $g = \phi(a + I)$ for some $a \in R$. Thus $g(r) = ar + I$ for every $r \in R$ and hence $ar - r \in I$ for every $r \in R$.

3.14. **Proposition.** *Let R be a ring such that every maximal right ideal of R is regular. If M is a simple R-module such that $\hat{M}R = M$, then $\hat{M} = M$.*

Proof. Consider ϕ: $\hat{M} \to \mathrm{Hom}_R(R, \hat{M}R)$ as in 3.12. If the ker ϕ were not zero then $\phi(M) = 0$ and $MR = 0$ since $\hat{M}$ is an essential extension of the simple module M. ϕ is a monomorphism. Let $f \in \mathrm{Hom}_R(R, \hat{M}R)$. Since $\hat{M}R \subseteq \hat{M}$, there is $\bar{f} \in \mathrm{Hom}_R(R^{\#}, \hat{M})$ such that $\bar{f}(r) = f(r)$ for every $r \in R$, where $R^{\#}$ is an extension ring of R with 1. Let $\bar{f}(1) = \hat{m}$ where $\hat{m} \in \hat{M}$. Then $f = \phi(\hat{m})$ and $\phi(\hat{M}) = \mathrm{Hom}_R(r, \hat{M}R)$. On the other hand, if $f \neq 0$ then $R/\ker f \cong \hat{M}R = M$. Hence ker f is a maximal right ideal of R and therefore, by hypothesis, it is regular. Let e be a left identity modulo ker f. Then $f(er) = f(r) = f(e)r$ for every $r \in R$ and $f = \phi(f(e))$. Hence $\phi(M) = \mathrm{Hom}_R(R, \hat{M}R)$. Thus $\phi(M) = \phi(\hat{M})$ and $M = \hat{M}$.

3.15. **Remark.** Villamayor [5] proved that if R is a ring with 1 then every simple R-module is injective if and only if every proper right ideal of R is the intersection of maximal right ideals.

3.16. **Theorem.** *Let R be an arbitrary ring. Then the following statements are equivalent*:

(i) *Every proper right ideal of R is an intersection of maximal regular right ideals.*

(ii) *If A is a right ideal of R then $AR = A$ and every simple R-module is injective.*

Proof. Assume (i). Then by 3.10, $AR = A$ for every right ideal A of R. Let M be a simple R-module and let $0 \neq \hat{m} \in \hat{M}$. Then $(\hat{M})^{\perp} \neq R$. For if $(\hat{m})^{\perp} = R$ then the submodule $N = \{\hat{m} \in \hat{M} \mid \hat{m}R = 0\}$ must contain M since $\hat{M}$ is an essential extension of M and M is simple. Hence $MR = 0$. This is, of course, impossible. Thus $(\hat{m})^{\perp} \neq R$ and $(\hat{m})^{\perp} = \bigcap_{\alpha \in \Lambda} \{I_\alpha \mid I_\alpha$ is a maximal regular right ideal$\}$ for some index set Λ. Let $M_\alpha = R/I_\alpha$. Let p_α be the αth projection of $\Pi_{\beta \in \Lambda} M_\beta \to M_\alpha$ and let μ: $\hat{m}R \to \Pi_{\alpha \in \Lambda} M_\alpha$ be a map such that $p_\alpha \circ \mu(\hat{m}r) = r + I_\alpha$ for every $r \in R$. Then μ is a monomorphism. If there exists $\alpha \in \Lambda$ such that $p_\alpha \circ \mu$ is a monomorphism then $\hat{m}R$ is simple and $\hat{m}R = M$ since M is the only simple submodule of $\hat{M}$. Now, if, for each $\alpha \in \Lambda$, $p_\alpha \circ \mu$ is not a monomorphism, then $p_\alpha \circ \mu(M) = 0$ for every $\alpha \in \Lambda$ and $\mu(M) = 0$. Since μ is a monomorphism, this means that $M = 0$. This is impossible. Thus $\hat{M}R = M$ and, by 3.14, $\hat{M} = M$ and (ii) holds. Now assume (ii). Let A be a right ideal of R such that $A \neq R$. Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be the family of all maximal right ideals of R which contain A. Suppose there exists $b \in \bigcap_{\alpha \in \Lambda} I_\alpha$ such that $b \notin A$. Let A_0 be a right ideal of R maximal with respect to the property that $A \subseteq A_0$ but $b \notin A_0$. Then the submodule of R/A_0 generated by $b + A_0$ is equal to J/A_0 for some right ideal J which contains A_0 properly. Since $JR = J$, J/A_0 must be a simple R-module. Hence it is injective by hypothesis. Let ϕ be a homomorphism from R/A_0 onto J/A_0 such that $\phi|_{J/A_0} = 1_{J/A_0}$, the identity map on J/A_0. Such ϕ exists since J/A_0 is injective. Then $R/A_0 = J/A_0 \oplus \ker \phi$. If $\ker \phi \neq 0$ then $\phi(b + A_0) = 0$. This is impossible. Thus $R/A_0 = J/A_0$ and A_0 is a maximal right ideal of R. Hence $b \in A_0$. This is a contradiction. Thus, A is an intersection of maximal right ideals. Since every maximal right ideal of R is regular by 3.13, A is an intersection of maximal regular right ideals.

3.17. **Definition.** A right ideal C of a ring R is said to be *incompressible* provided that if $\{A_1, A_2, \cdots, A_n\}$ is a finite set of right ideals such that $A_i \not\subseteq C$ for every i, then there is a maximal regular right ideal I such that $C \subseteq I$ and $A_i \not\subseteq I$ for every i.

3.18. **Remark.** If R is a commutative ring, then every prime ideal P such that R/P is semisimple is incompressible. For if $\{A_1, A_2, \cdots, A_n\}$ is a finite set of ideals such that $A_i \not\subseteq P$ for every i, then $A_1 A_2 \cdots A_n \not\subseteq P$. Hence there is a maximal ideal I such that $P \subseteq I$ but $A_1 A_2 \cdots A_n \not\subseteq I$. Obviously in any ring every maximal regular right ideal is incompressible.

3.19. **Proposition.** *If C is an incompressible right ideal of a ring R, then $\sqrt{C} = C$.*

Proof. If $\sqrt{C} \neq C$, then there exists a maximal regular right ideal I such that $C \subseteq I$ but $\sqrt{C} \not\subseteq I$. This is a contradiction.

3.20. **Proposition.** *Let R be a ring. Then a right ideal C is incompressible if and only if $C = \sqrt{C}$ and $h(C)$ is an irreducible closed subset of $X(R)$.*

Proof. Assume that C is incompressible. Then $C = \sqrt{C}$ by 3.19. If $h(C)$ were reducible, then there exist right ideals $A_1, A_2, \cdots, A_n$ such that $h(A_i) \neq h(C)$ and $h(C) = \bigcup_{i=1}^n h(A_i)$. Since $h(A_i) \subseteq h(C)$ and $h(A_i) \neq h(C)$, $A_i \not\subseteq C$ for every i. Hence there is $I \in X(R)$ such that $C \subseteq I$ and $A_i \not\subseteq I$ for every i. This means that $h(C) \not\subseteq \bigcup_{i=1}^n h(A_i)$, which is a contradiction. Conversely, assume that $C = \sqrt{C}$ and $h(C)$ is irreducible. Let $\{A_1, A_2, \cdots, A_n\}$ be a finite set of right ideals such that $A_i \not\subseteq C$ for every i. If there is no maximal regular right ideal I such that $C \subseteq I$ and $A_i \not\subseteq I$ for every i, then $h(C) \subseteq \bigcup_{i=1}^n h(A_i)$. Since $h(C)$ is irreducible, $h(C) \subseteq h(A_i)$ for some i. Hence $C \supseteq \sqrt{A_i} \supseteq A_i$. This is impossible.

3.21. **Corollary.** *Let R be a ring such that R is not a radical ring. Then $X(R)$ is irreducible if and only if $J(R)$, the Jacobson radical, is incompressible.*

3.22. **Example.** Let R be a primitive ring with a minimal right ideal A. If $A = eR$ for some idempotent $e \neq 1$ in R such that eRe is an infinite division ring then eR is incompressible. Because, in this case, $X(R)$ is irreducible by [3, 2.4] and $h(eR)$ is irreducible by 2.8.

3.23. **Theorem.** *Let R be a ring. Then every proper right ideal of R is incompressible if and only if* (i) *$A = AR$ for every right ideal A of R, and* (ii) *either R is a division ring or R is a dense ring of linear transformations of a left vector space V of dimension two or more over an infinite division ring D such that V is injective as an R-module and every simple R-module is isomorphic to V.*

Proof. Assume that every proper right ideal of R is incompressible. Then for every right ideal A of R, $A = AR$ by 3.19 and 3.10. Suppose that R is not a division ring. Let x and y be two distinct points in $X(R)$. Since $j(x) \cap j(y)$ is incompressible, $h(j(x) \cap j(y))$ is irreducible by 3.20. Hence $l(x, y)$ must contain an infinite number of points. Thus by 3.5, every pair of simple R-modules are isomorphic. Since zero ideal is incompressible, R is semisimple and, since every pair of simple R-modules are isomorphic, every simple R-module is faithful. Hence R must be a primitive ring and $X(R)$ is irreducible. Thus by [3, 2.4], R must be a dense ring of linear transformations of a vector space of dimension two or more over an infinite division ring. If every proper right ideal

of R is incompressible, then every proper right ideal of R is an intersection of maximal regular right ideals by 3.19. Thus by 3.16, every simple R-module is injective. Conversely, assume (i) and (ii). Then by 3.16, if A is a right ideal such that $A \neq R$ then $\sqrt{A} = A$. Since every simple R-module is isomorphic to V, if $x \in X(R)$ then $x = v^{\perp}$ for some $0 \neq v \in V$. Hence there is subspace W of V such that $A = W^{\perp}$. If $h(A)$ were not irreducible then there exist right ideals $A_1, A_2, \cdots, A_n$ such that $h(A) = \bigcup_{i=1}^{n} h(A_i)$ and $h(A) \neq h(A_i)$ for every i. Let $W_i = \{w \in W \mid wA_i = 0\}$ for each i. Then W_i is a subspace of W and $W = \bigcup_{i=1}^{n} W_i$. Note that W_i is a proper subspace of W since $h(A_i) \neq h(A)$. Hence by [3, 2.1] the division ring D must be a finite field. This is impossible. Thus A is incompressible by 3.20.

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REFERENCES

1. N. Jacobson, *Structure of rings*, 2nd rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1964. MR 36 #5158.

2. R. E. Johnson, *Structure theory of faithful rings*. II. *Restricted rings*, Trans. Amer. Math. Soc. 84 (1957), 523–544. MR 18, 869.

3. K. Koh and J. Luh, *Maximal regular right ideal space of a primitive ring*, Trans. Amer. Math. Soc. 170 (1972), 269–277.

4. I. G. MacDonald, *Algebraic geometry. Introduction to schemes*, Benjamin, New York, 1968. MR 39 #205.

5. G. O. Michler and O. E. Villamayor, *On rings whose simple modules are injective* (to appear).

6. J. von Neumann, *On regular rings*, Proc. Nat. Acad. Sci. U. S. A. 22 (1936), 707–713.

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