ADDENDUM TO "MODULAR REPRESENTATIONS OF METABELIAN GROUPS"

BY

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In this note the principal indecomposable modules of \( \Omega G \) are determined where \( G \) is a finite metabelian group and \( \Omega \) is an algebraically closed field with characteristic \( p \) dividing \(|G|\). The notations are the same as of [1].

Let \( P \) be a \( p \)-Sylow subgroup of \( K(H) \). Since \( K(H)/K(H)' \) is abelian, there exist subgroups \( V_1 \supseteq K(H)' \) and \( V_2 \supseteq K(H)' \) such that \( K(H)/K(H)' \cong V_1/K(H)' \times V_2/K(H)' \), \( V_1/K(H)' \) is a \( p \)-group and \( p \nmid |V_2/K(H)'| \). Let \( P_1 \) be a \( p \)-sylow subgroup of \( V_2 \), then \( P_1 \subseteq K(H)' \) and thus \( P_1 \) is normal in \( V_2 \). Hence there exists a subgroup \( V \) of \( V_2 \) such that \( V = P_1 \circ V \), the semidirect product, and \( p \nmid |V| \).

Clearly \( K(H) = \langle P, V \rangle \), \( P \cap V = 1 \), and \(|V| = |K(H)/P| \).

For each \( K(H) \), \( A/H \) cyclic and \( p \nmid |A/H| \), fix a subgroup \( V \) with the above properties. Let \( \tau' \) be a linear representation of \( K(H) \) with \( \ker \tau' \cap A = H \) such that \( \tau_K' \) is conjugate to \( \sigma \) where \( K = K(A) \). Then \( \tau'^{G} \) is irreducible and \( \tau'^{G} \in B(\sigma, H) \). Let \( x \in G \) and define

\[
e_x(\tau') = \frac{1}{|V|} \sum_{a \in V} \tau'(x^{-1}a^{-1})a
\]

and \( e_x(\tau') = e(\tau') \). We prove

**Theorem 4.** All the principal indecomposable modules of \( \Omega G \) are given by the collection of the ideals \( \Omega G e(\tau') \) with \( \tau' \in \bigcup M(H, K(H)) \) where the union is over all subgroups \( H \) of \( A \) such that \( A/H \) is cyclic and \( p \nmid |A/H| \).

**Proof.** Let \( T' \) be an ordinary representation of \( K(H) \) such that \( \ker \tau' = \ker T' \supseteq P \) and for all \( a \in K(H) \), \( T'(a) = \tau'(a) \), and \( T'_V \) be the restriction of \( T' \) to \( V \). Define \( T''(x)(a) = T'(x^{-1}ax) \) where \( x \in G \). Since \( \ker T' \supseteq P \), it follows that \( T'_V \neq T''_V(x) \) if \( x \notin K(H) \). Define

\[
e_x(T') = \frac{1}{|V|} \sum_{a \in V} T'(x^{-1}a^{-1})a;
\]

then \( e_x(T') \) are minimal indempotents of \( \Omega V \) and \( e_x(T') \cdot e_y(T') = 0 \) if and only if \( xK(H) \neq yK(H) \). Similarly, if \( \tau'_1 \) is another linear representation of \( K(H) \) not conjugate to \( \tau' \) and \( \ker \tau'_1 \cap A = H \), and if \( T'_1 \) is similarly defined then

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\[ e_x(T') \cdot e_z(T) = 0 \text{ for any } x \text{ and } z \text{ in } G. \]

Now \( e(T') = e_1(T') \) is also an idempotent of \( G \). We have \( \bar{Q}G \otimes_{QV} \bar{Q}Ve(T') \cong \bar{Q}Ge(T') \), since from the definitions of tensor products and balanced maps there is a \( \bar{Q}G \)-homomorphism of \( \bar{Q}G \otimes_{QV} \bar{Q}Ve(T') \) onto \( \bar{Q}Ge(T') \), and both modules are of \( \bar{Q}-\text{dimension } |G|/|V| = p^a|G/K(H)|, p^a||K(H)||, \) with \( b \) runs over a set of coset representatives of \( V \) in \( G \) a \( \bar{Q} \)-basis for \( \bar{Q}Ge(T') \). Hence \( \bar{Q}Ge(T') \) affords \( (\tau')^G \). Thus we have \( e_x(r') \cdot e_y(r') = 0 \) if and only if \( xK(H) \neq yK(H) \), \( e_x(r') \cdot e_z(r') = 0 \) if \( \ker r' \cap A = H \) and \( r' \) is not conjugate to \( r' \), and \( \bar{Q}Ge(r') \), a direct summand of \( \bar{Q}G \), affording \( (\tau')^G \) of degree \( p^a|G/K(H)| \). If \( \chi \) is the character of \( T' \), then from the Frobenius reciprocity theorem, \( 1 = (\chi_V, (\chi^G)_V) = (\chi^G_V, \chi^G) \), or \( \tau^G \) is a composition factor of \( (\tau')^G \). Assume \( \bar{Q}Ge(r') = U_1 \oplus \cdots \oplus U_r, U_i \) some indecomposable components of \( \bar{Q}G \), then \( \tau^G \) is afforded by a composition factor of some \( U_i \) or \( U_i \) belongs to \( B(\sigma, H) \). But from Theorem 3 of [1], \( U_i \) is of degree \( p^a|G/K(H)| \) and hence \( U_i = \bar{Q}Ge(r') \) or \( \bar{Q}Ge(r') \), and \( (\tau')^G \), indecomposable.

Each \( \tau^G \in B(\sigma, H) \) is associated with \( |G/K(H)| \) (\( = \text{degree of } \tau^G \)) distinct indecomposable components of \( \bar{Q}G \), namely \( \bar{Q}Ge(x), x \in G/K(H) \). Moreover, if \( \bar{Q}Ge(r') \) belongs to \( B(\sigma_1, H_1) \), where \( B(\sigma_1, H_1) \) is a block different from \( B(\sigma, H) \), then \( e(r') \cdot e(r') = 0 \). Now the result follows by applying Theorems 1 and 2, which completes the proof.

Define
\[ e(\sigma, H) = \sum_{x \in G/K(H)} e_x(T') \]
where the summation \( \Sigma' \) is over all distinct \( \tau^G \in B(\sigma, H) \). We have

**Corollary.** All the indecomposable two-sided ideals (blocks) of \( \bar{Q}G \) are given by the collection of the ideals \( \bar{Q}Ge(\sigma, H) \) where \( H \) runs over all nonconjugate subgroups of \( A, A/H \) cyclic, \( p^a|A/H| \), and \( \sigma \) runs over the elements of \( C(H, K(A)) \).

**BIBLIOGRAPHY**