DIFFERENTIAL GEOMETRIC STRUCTURES ON PRINCIPAL TOROIDAL BUNDLES

BY

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ABSTRACT. Under an assumption of regularity a manifold with an $f$-structure satisfying certain conditions analogous to those of a Kähler structure admits a fibration as a principal toroidal bundle over a Kähler manifold. In some natural special cases, additional information about the bundle space is obtained. Finally, curvature relations between the bundle space and the base space are studied.

Let $M^{2n+s}$ be a $C^\infty$ manifold of dimension $2n + s$. If the structural group of $M^{2n+s}$ is reducible to $U(n) \times O(s)$, then $M^{2n+s}$ is said to have an $f$-structure of rank $2n$. If there exists a set of 1-forms $\{\eta^1, \cdots, \eta^s\}$ satisfying certain properties described in §1, then $M^{2n+s}$ is said to have an $f$-structure with complemented frames. In [1] it was shown that a principal toroidal bundle over a Kähler manifold with a certain connection has an $f$-structure with complemented frames and $d\eta^1 = \cdots = d\eta^s$ as the fundamental 2-form. On the other hand, the following theorem is proved in §2 of this paper.

Theorem 1. Let $M^{2n+s}$ be a compact connected manifold with a regular normal $f$-structure. Then $M^{2n+s}$ is the bundle space of a principal toroidal bundle over a complex manifold $N^{2n} (= M^{2n+s}/\mathbb{H})$. Moreover, if $M^{2n+s}$ is a K-manifold, then $N^{2n}$ is a Kähler manifold.

After developing a theory of submersions in §3, we discuss in §4 further properties of this fibration in the cases where $d\eta^x = 0, x = 1, \cdots, s$ and $d\eta^x = \alpha^xF, F$ being the fundamental 2-form of the $f$-structure.

Finally in §5 we study the relation between the curvature of $M^{2n+s}$ and $N^{2n}$.

Since $U(n) \times O(s) \subset O(2n + s)$, $M^{2n+s}$ is a new example of a space in the class provided by Chern in his generalization of Kähler geometry [4]. S. I. Goldberg's paper [5] also suggests the study of framed manifolds as bundle spaces over Kähler manifolds with parallelisable fibers.

1. Normal $f$-structures. Let $M^{2n+s}$ be a $2n + s$-dimensional manifold with an $f$-structure. Then there is a tensor field $f$ of type $(1, 1)$ on $M^{2n+s}$ that is of rank
If there exist vector fields \( \xi_x, x = 1, \ldots, s \) on \( M^{2n+s} \) such that
\[
\begin{align*}
\frac{\partial \xi_x}{\partial x} &= 0, & \eta^x(\xi_y) &= \delta^x_y, & \eta^x \circ f &= 0, & f^2 &= -1 + \eta^y \otimes \xi_y,
\end{align*}
\]
we say \( M^{2n+s} \) has an \( f \)-structure with complemented frames. Further we say that the \( f \)-structure is normal if
\[
\begin{align*}
[\xi, \eta] + df \otimes \xi &= 0,
\end{align*}
\]
where \([\xi, \eta]\) is the Nijenhuis torsion of \( f \). It is a consequence of normality that \([\xi_x, \xi_y] = 0\). Moreover it is known that there exists a Riemannian metric \( g \) on \( M^{2n+s} \) satisfying
\[
\begin{align*}
g(X, Y) &= g(fX, fY) + \sum_x \eta^x(X)\eta^x(Y),
\end{align*}
\]
where \( X \) and \( Y \) are arbitrary vector fields on \( M^{2n+s} \). Define a 2-form \( F \) on \( M^{2n+s} \) by
\[
\begin{align*}
F(X, Y) &= g(X, fY).
\end{align*}
\]
A normal \( f \)-structure for which \( F \) is closed will be called a \( K \)-structure and a \( K \)-structure for which there exist functions \( \alpha^1, \ldots, \alpha^s \) such that \( \alpha^x F = df^x \) for \( x = 1, \ldots, s \) will be called an \( S \)-structure.

Lemma 1. If \( M^{2n+s}, n > 1 \), has an \( S \)-structure, then the \( \alpha^x \) are all constant.

Proof. \( \alpha^x F = df^x \) so that \( d\alpha^x \wedge F = 0 \) since \( dF = 0 \). However \( F \neq 0 \) so \( d\alpha^x = 0 \) and hence \( \alpha^x \) is constant.

The special case where the \( \alpha^x \) are all 0 or all 1 has been studied in [1]. Also, the following were proved.

Lemma 2. If \( M^{2n+s} \) has a \( K \)-structure, the \( \xi_x \) are Killing vector fields and
\[
d\eta^x(X, Y) = -2(\nabla_X \eta^x)(Y).
\]
Here \( \nabla \) is the Riemannian connection of \( g \) on \( M^{2n+s} \).

From Lemma 2, we can see that in the case of an \( S \)-structure \( \alpha^x f^y = -2\nabla_X \eta^x \).

Lemma 3. If \( M^{2n+s} \) has a \( K \)-structure, then
\[
(\nabla_X f)(Y, Z) = \frac{1}{2} \sum_x (\eta^x(Y)d\eta^x(fZ, X) + \eta^x(Z)d\eta^x(X, fY)).
\]
to an involutive $m$-dimensional distribution if $\{\partial(m)/\partial u^x\}, x = 1, \ldots, m,$ is a basis of $\mathfrak{H}_m$ for every $m \in U$ and if each leaf of $\mathfrak{H}$ intersects $U$ in at most one $m$-dimensional slice of $\{U, (u^1, \ldots, u^n)\}$. We say $\mathfrak{H}$ is regular if every leaf of $\mathfrak{H}$ intersects the domain of a cubical coordinate system which is regular with respect to $\mathfrak{H}$.

In [9] it is proven that if $\mathfrak{H}$ is regular on a compact connected manifold $M$, then every leaf of $\mathfrak{H}$ is compact and that the quotient $M/\mathfrak{H}$ is a compact differentiable manifold. Moreover the leaves of $\mathfrak{H}$ are the fibers of a $C^\infty$ fibering of $M$ with base manifold $N/\mathfrak{H}$ and the leaves are all $C^\infty$ isomorphic.

We now note that the distribution $\mathfrak{H}$ spanned by the vector fields $\xi_1, \ldots, \xi_s$ of a normal $f$-structure is involutive. In fact we have by normality

$$0 = [f, f](\xi_y, \xi_z) + d\eta^x(\xi_y, \xi_z)\xi_x = f^2(\xi_y, \xi_z) - \eta^x((\xi_y, \xi_z))\xi_x = -[\xi_y, \xi_z]$$

from which it easily follows that $\mathfrak{H}$ is involutive. If $\mathfrak{H}$ is regular and the vector fields $\xi_x$ are regular we say that the normal $f$-structure is regular. Thus from the results of [9] we see that if $M^{2n+s}$ is compact and has a regular normal $f$-structure, then $M^{2n+s}$ admits a $C^\infty$ fibering over the $(2n)$-dimensional manifold $N^{2n} = M^{2n+s}/\mathfrak{H}$ with compact, $C^\infty$ isomorphic, fibers.

Since the distribution $\mathfrak{H}$ of a regular normal $f$-structure consists of $s$ 1-dimensional regular distributions each given by one of the $\xi_x$'s, if $M^{2n+s}$ is compact, the integral curves of $\xi_x$ are closed and hence homeomorphic to circles $S^1$. The $\xi_x$'s being independent and regular show that the fibers determined by the distribution $\mathfrak{H}$ are homeomorphic to tori $T^s$.

Now define the period function $\lambda_X$ of a regular closed vector field $X$ by

$$\lambda_X(m) = \inf\{t > 0 | (\exp tX)(m) = m\}.$$ 

For brevity we denote $\lambda_{\xi_x}$ by $\lambda_x$. W. M. Boothby and H. C. Wang [3] proved that $\lambda_X(m)$ is a differentiable function on $M^{2n+s}$. We now prove the following

**Lemma 4.** The functions $\lambda_x$ are constants.

The proof of the lemma makes use of the following theorem of A. Morimoto [7].

**Theorem (Morimoto [7]).** Let $M$ be a complex manifold with almost complex structure tensor $J$. Let $X$ be an analytic vector field on $M$ such that $X$ and $JX$ are closed regular vector fields. Set $p(m) = \lambda_X(m) + \sqrt{\lambda_JX(m)}$. Then $p$ is a holomorphic function on $M$.

**Proof of lemma.** For $s$ even,

$$\tilde{f} = f + \sum_{i=1}^{s/2} (\eta^i \otimes \xi^{i*} - \eta^{i*} \otimes \xi_i), \quad i = 1, \ldots, s/2, \quad i^* = i + s/2,$$
defines a complex structure on \( M = M^{2n+s} \) (cf. [6]). It is clear from the normality that \( \xi_x \) is a holomorphic vector field. For \( s \) odd, a normal almost contract structure \((\xi, \xi_0, \eta_0)\) is defined where \( \xi_0 \) and \( \eta_0 \) generically denote one of the \( \xi_x \)'s and \( \eta_x \)'s respectively [6]. It is well known that this structure induces a complex structure \( J \) on \( M = M^{2n+s} \times S^1 \). Moreover, by the normality, \( \xi_0 \) considered as a vector field on \( M \) is analytic. Then \( p(m) = \lambda_{x}(m) + \sqrt{-1} \lambda_{x}(m) \) or \( p((m, q)) = \lambda_{\xi_0}((m, q)) + \sqrt{-1}\lambda_{\xi_0}((m, q)), \) \( q \in S^1 \), for \( s \) odd, is a holomorphic function on \( M \) by the theorem of Morimoto. Since \( M \) is compact, \( p \) must be constant. Thus \( \lambda_{x} \) is constant on \( M \) and since \( \lambda_{\xi_0}(m, q) = \lambda_{x}(m), \lambda_{x} \) is constant on \( M^{2n+s} \).

Let \( C_x = \lambda_{x}(m) \), then the circle group \( S^1 \) of real numbers modulo \( C_x \) acts on \( M^{2n+s} \) by \( (t, m) \mapsto (\exp t \xi_x(m), t \in \mathbb{R} \). Now the only element in \( T^S = S^1 \times \ldots \times S^1 \) with a fixed point in \( M^{2n+s} \) is the identity and since \( M^{2n+s} \) is a fiber space over \( N^{2n} \), we need only show that \( M^{2n+s} \) is locally trivial [3]. Let \{\( U_a \)\} be a cover of \( N^{2n} \) such that each \( U_a \) is the projection of a regular neighborhood on \( M^{2n+s} \) and let \( s_a: U_a \rightarrow M^{2n+s} \) be the section corresponding to \( u^1 = \text{constant}, \ldots, u^s = \text{constant} \). Then the maps \( \Psi_a: U_a \times T^S \rightarrow M^{2n+s} \) defined by

\[
\Psi_a(p, t_1, \ldots, t_s) = (\exp (t_1 \xi_1 + \cdots + t_s \xi_s))(s_a(p))
\]

give coordinate maps for \( M^{2n+s} \).

Finally (cf. [1]) we note that \( \gamma = (\eta^1, \ldots, \eta^s) \) defines a Lie algebra valued connection form on \( M^{2n+s} \) and we denote by \( \tilde{\gamma} \) the horizontal lift with respect to \( \gamma \). Define a tensor field \( J \) of type \((1, 1)\) on \( N^{2n} \) by \( JX = \pi_* f^\gamma X \). Then, since the distribution \( \mathfrak{L} \) complementary to \( \mathfrak{H} \) is horizontal with respect to \( \gamma \),

\[
J^2X = \pi_* f^\gamma \pi_* f^\gamma X = \pi_* f^2 \tilde{\gamma} X = -X.
\]

Moreover

\[
[J, J](X, Y) = -[\gamma, Y] + [\pi_* f^\gamma X, \pi_* f^\gamma Y] - \pi_* f^\gamma [\pi_* f^\gamma X, Y] - \pi_* f^\gamma [\pi_* f^\gamma X, \pi_* f^\gamma Y]
\]

\[
= -\pi_* ([\gamma^\ast \pi_* \tilde{\gamma} X, \pi_* \tilde{\gamma} Y] + \pi_* [\gamma^\ast \pi_* \tilde{\gamma} X, \pi_* \tilde{\gamma} Y] - \pi_* f^\gamma [\pi_* f^\gamma X, \pi_* f^\gamma Y] - \pi_* f^\gamma [\pi_* f^\gamma X, \pi_* f^\gamma Y] - \pi_* f^\gamma [\pi_* f^\gamma X, \pi_* f^\gamma Y]
\]

\[
= \pi_* ([f, f] f^\gamma X, f^\gamma Y] - \eta^\ast f(\pi^\gamma X, f^\gamma Y), \xi_x) + \pi_* ([\pi^\gamma X, f^\gamma Y] - \pi_* f^\gamma [\pi^\gamma X, f^\gamma Y] - \pi_* f^\gamma [\pi^\gamma X, f^\gamma Y] - \pi_* f^\gamma [\pi^\gamma X, f^\gamma Y]
\]

\[
= \pi_* ([f, f] f^\gamma X, f^\gamma Y] + d\eta^\ast f(\pi^\gamma X, f^\gamma Y) f^\gamma X, \xi_x)
\]

\[
= 0.
\]

Thus we see that \( N^{2n} \) is a complex manifold.

We define an Hermitian metric \( G \) on \( N^{2n} \) by \( G(X, Y) = g(\pi^\gamma X, \pi^\gamma Y) \). Indeed

\[
G(JX, JY) = g(\pi^\gamma \pi_* f^\gamma X, \pi^\gamma \pi_* f^\gamma Y) = g(f^\gamma X, f^\gamma Y)
\]

\[
= g(\pi^\gamma X, \pi^\gamma Y) - \sum \eta^\ast(\pi^\gamma X) \eta^\ast(\pi^\gamma Y) = G(X, Y).
\]
Now define the fundamental 2-form \( \Omega \) by \( \Omega(X, Y) = G(X, JY) \). Then for vector fields \( \widetilde{X}, \widetilde{Y} \) on \( M^{2n+s} \) we have
\[
\pi^*\Omega(\widetilde{X}, \widetilde{Y}) = \Omega(\pi_*\widetilde{X}, \pi_*\widetilde{Y}) = G(\pi_*\widetilde{X}, J\pi_*\widetilde{Y}) \\
= g(\pi_*\widetilde{X}, \pi_*\widetilde{Y}) = g(-f^2\widetilde{X}, f\widetilde{Y}) = F(\widetilde{X}, \widetilde{Y}).
\]
Thus \( F = \pi^*\Omega \). If now \( dF = 0 \), then \( 0 = d\pi^*\Omega = \pi^*d\Omega \) and hence \( d\Omega = 0 \) since \( \pi^* \) is injective. Thus the manifold \( N^{2n} \) is Kählerian.

3. Submersions. Let \( \nabla \) denote the Riemannian connection of \( g \) on \( M^{2n+s} \).

Since the \( \xi^i \)'s are Killing, \( g \) is projectable to the metric \( G \) on \( N^{2n} \). Then following [8] the horizontal part of \( \nabla_{\pi X} \pi Y \) is \( \pi \nabla_X Y \) where as we shall see \( \nabla \) is the Riemannian connection of \( G \). Now for an \( S \)-structure we have seen that \( \nabla_{\pi X} \xi = \alpha^i \xi_i \) for any vector field \( \xi \) on \( M^{2n+s} \). By normality \( f \) is projectable (\( \xi, f = 0 \)) and the \( \alpha^i \)'s are constants; thus we can write
\[
\nabla_{\pi X} \xi = -\pi H(x),
\]
where \( H(x) \) is a tensor field of type \((1, 1)\) on \( N^{2n} \).

We can now find the vertical part of \( \nabla_{\pi X} \pi Y \).
\[
g(\nabla_{\pi X} \pi Y, \xi) = -g(\pi Y, \nabla_{\pi X} \xi) = g(\pi Y, \pi H(x) \xi).
\]
Thus we can write
\[
\nabla_{\pi X} \pi Y = \pi\nabla_X Y + b^x(X, Y)\xi_x
\]
where each \( b^x \) is a tensor field of type \((0, 2)\) and
\[
G(H_x, Y) = b^x(X, Y).
\]

Lemma 5. \( \xi, G(\pi X) = 0 \) for any vector field \( X \) on \( N^{2n} \), where \( \xi_x \) is the operator of Lie differentiation in the \( \xi_x \) direction.

Proof. We have that \( g(\xi_y, \pi X) = 0 \) for \( y = 1, \ldots, s \). By Lemma 2, the \( \xi_x \) are Killing, that is \( \xi_x g = 0 \). From the normality of \( f \), \( \xi_x \xi = 0 \). Hence, we have that
\[
g(\xi_y, \xi_x(\pi X)) = 0, \quad y = 1, \ldots, s,
\]
and so \( \xi_x(\pi X) \) is horizontal. However,
\[
\pi_*[\xi_x(\pi X)] = [\pi_*\xi_x, \pi_*\pi X] = 0
\]
and so \( \xi_x(\pi X) \) is vertical.
Using the lemma we see that \( \tilde{\nabla}_X \tilde{\xi}_x = \tilde{\nabla}_{\tilde{\xi}_x} \tilde{\xi}_x \) for any vector field \( X \) on \( N^{2n} \). Since \( \tilde{\xi}_x \) is Killing, we have

\[
0 = g(\tilde{\nabla}_{\tilde{\xi}_x} \xi_x, \tilde{\xi}_x) = -g(\xi_x, \tilde{\nabla}_{\xi_x} \xi_x) = -g(\xi_x, b^\xi(X, X)\xi_x) = -b^\xi(X, X)
\]

for all \( X \). That is to say \( b^\xi(X, Y) = -b^\xi(Y, X) \) for all \( X \) and \( Y \). Now we have that

\[
0 = \tilde{\nabla}_X (\tilde{\nabla}_Y \xi_x) - \tilde{\nabla}_Y (\tilde{\nabla}_X \xi_x) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] \xi_x
\]

(\( 6 \))

\[
= \tilde{\nabla} (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]) + (b^\xi(X, Y) - b^\xi(Y, X) + d\eta^\xi(\tilde{\tau}_X, \tilde{\tau}_Y))\xi_x
\]

where we have used the following lemma.

**Lemma 6.** \([\tilde{\nabla}_X, \tilde{\nabla}_Y] = \tilde{\nabla}([\tilde{\nabla}_X, \tilde{\nabla}_Y]) - d\eta^\xi(\tilde{\tau}_X, \tilde{\tau}_Y))\xi_x\).

**Proof.** Since \( \tilde{\nabla}_X, \tilde{\nabla}_Y \) = \([\tilde{\nabla}_X, \tilde{\nabla}_Y] = [X, Y] \) we see that \( \tilde{\nabla}([X, Y]) \) is the horizontal part of \([\tilde{\nabla}_X, \tilde{\nabla}_Y]\). By Lemma 2, we have

\[
d\eta^\xi(\tilde{\tau}_X, \tilde{\tau}_Y) = -2g(\tilde{\nabla}_{\tilde{\tau}_X} \eta^\xi, \tilde{\tau}_Y) = -2g(\tilde{\nabla}_{\tilde{\tau}_Y} \eta^\xi, \tilde{\tau}_X) = +2g(\xi_x, \tilde{\nabla}_{\tilde{\tau}_Y} \eta^\xi).
\]

Also \( d\eta^\xi(\tilde{\tau}_X, \tilde{\tau}_Y) = -d\eta^\xi(\tilde{\tau}_Y, \tilde{\tau}_X) = -2g(\xi_x, \tilde{\nabla}_{\tilde{\tau}_X} \eta^\xi) \). Thus

\[
2d\eta^\xi(\tilde{\tau}_X, \tilde{\tau}_Y) = 2g(\xi_x, \tilde{\nabla}_{\tilde{\tau}_X} \eta^\xi - \tilde{\nabla}_{\tilde{\tau}_Y} \eta^\xi)
\]

or

\[
d\eta^\xi(\tilde{\tau}_X, \tilde{\tau}_Y)\xi_x = \sum_x g(\xi_x, [\tilde{\tau}_X, \tilde{\tau}_Y])\xi_x = \text{vertical part of } [\tilde{\tau}_X, \tilde{\tau}_Y].
\]

From (6) we see \( \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = 0 \) and \( b^\xi(X, Y) = -\frac{1}{2}d\eta^\xi(\tilde{\tau}_X, \tilde{\tau}_Y) \). Furthermore,

\[
XG(Y, Z) = \tilde{\nabla}Xg(\tilde{\tau}_Y, \tilde{\tau}_Z) = g(\tilde{\nabla}_{\tilde{\tau}_X} \tilde{\tau}_Y, \tilde{\tau}_Z) + g(\tilde{\tau}_X, \tilde{\nabla}_{\tilde{\tau}_X} \tilde{\tau}_Z)
\]

\[
= g(\tilde{\nabla}_{\tilde{\tau}_X} Y, \tilde{\tau}_Z) + g(\tilde{\tau}_X, \tilde{\nabla}_{\tilde{\tau}_X} Z) = G(\nabla_X Y, Z) + G(Y, \nabla_X Z).
\]

Thus, we have the following proposition.

**Proposition.** \( \nabla \) is the Riemannian connection of \( G \) on \( N^{2n} \).

4. The \( S \)-structure case. Let \( M^{2n+s} \), \( n > 1 \), be a manifold with an \( S \)-structure. Then, as we have seen, there exist constants \( \alpha^x \), \( x = 1, \ldots, s \), such that \( \alpha^x P = d\eta^x \). We will consider two cases, namely \( \sum_x (\alpha^x)^2 = 0 \) and \( \sum_x (\alpha^x)^2 \neq 0 \).

In the first case each \( \alpha^x = 0 \) and by Lemma 2 each \( \xi_x \) is Killing, hence the
regular vector fields $\xi_1, \ldots, \xi_s$ are parallel on $M^{2n+s}$. Moreover the complementary distribution $\mathcal{Q}$ (projection map is $-f^2 = 1 - \eta_x \otimes \xi_x$) is parallel. If now the distribution $\mathcal{Q}$ is also regular, we have a second fibration of $M^{2n+s}$ with fibers the integral submanifolds $L^{2n}$ of $\mathcal{Q}$ and base space an $s$-dimensional manifold $N^s$. Thus by a result of A. G. Walker [10] we see that although $M^{2n+s}$ is not necessarily reducible (even though it is locally the product of $N^{2n}$ and $T^s$) it is a covering space of $N^{2n} \times N^s$ and is covered by $L^{2n} \times T^s$. In summary we have

**Theorem 2.** If $M^{2n+s}$ is as in Theorem 1 with $df = 0, x = 1, \ldots, s$, and $\mathcal{Q}$ regular, then $M^{2n+s}$ is a covering space of $N^{2n} \times N^s$, where $N^s$ is the base space of the fibration determined by $\mathcal{Q}$.

Now as in Theorem 1, since the $\xi_i$'s, $i = 1, \ldots, s$, are regular, we could fibrate by any $s - t$ of them to obtain a fibration of $M^{2n+s}$ as a principal $T^{s-t}$ bundle over a manifold $P^{2n+t}$. By normality the remaining $t$ vector fields are projectable to $P^{2n+t}$. Moreover they are regular on $P^{2n+t}$; for if not, their integral curves would be dense in a neighborhood $U$ over which $M^{2n+s}$ is trivial with compact fiber $T^{s-t}$ contradicting their regularity on $M^{2n+s}$. Thus $P^{2n+t}$ is a principal $T^t$ bundle over $N^s$.

**Theorem 3.** If $M^{2n+s}, n > 1$, is as in Theorem 1 with $d\eta^x = \alpha^xF$ and $\Sigma_x(\alpha^x)^2 \neq 0$, then $M^{2n+s}$ is a principal $T^{s-1}$ bundle over a principal circle bundle $p^{2n+1}$ over $N^{2n}$ and the induced structure on $p^{2n+1}$ is a normal contact metric (Sasakian) structure.

**Proof.** Without loss of generality we suppose $\alpha^s \neq 0$. Then fibrating as above by $\xi_1, \ldots, \xi_{s-1}$ we have that $M^{2n+s}$ is a principal $T^{s-1}$ bundle over a principal circle bundle $P^{2n+1}$ over $N^{2n}$. Let $p: M^{2n+s} \to p^{2n+1}$ denote the projection map. By normality $f, \xi_s, \eta^s$ are projectable, so we define $\phi, \xi, \eta$ on $p^{2n+1}$ by

$$\phi X = p_* \tilde{\phi} X, \quad \xi = p_* \xi_s, \quad \eta(X) = \eta^S(\tilde{p} X)$$

where $\tilde{\phi}$ denotes the horizontal lift with respect to the connection $(\eta^1, \ldots, \eta^{s-1})$ considered as a Lie algebra valued connection form as in the proof of Theorem 1. Then by a straightforward computation we have

$$\eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \phi^2 = -1 + \xi \otimes \eta, \quad [\phi, \phi] + \xi \otimes d\eta = 0,$$

that is, $(\phi, \xi, \eta)$ is a normal almost contact structure on $p^{2n+1}$. Defining a metric $\tilde{g}$ by $\tilde{g}(X, Y) = \tilde{g}(\tilde{\phi} X, \tilde{\phi} Y)$ we have $\tilde{g}(X, \xi) = \eta(X)$ and $\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y)$. Moreover setting $\Phi(X, Y) = \tilde{g}(X, \phi Y)$ we obtain $F = p^*\Phi$. Thus since
\[ d\eta^s = \alpha^s F, \quad \Phi = d\eta^s/\alpha^s \quad \text{and} \]
\[ \Phi(X, Y) = g(\widehat{\Phi}X, \widehat{\Phi}Y) = d\eta^s(\widehat{\Phi}X, \widehat{\Phi}Y)/\alpha^s = (\chi\eta(Y) - \eta(X) - \eta^s([\widehat{\Phi}X, \widehat{\Phi}Y]))/\alpha^s = d\eta(X, Y)/\alpha^s \]
since \( \eta^s \) is horizontal. Thus we have that \( \eta^s(d\eta)^n = \eta^s(\alpha^s\Phi)^n \neq 0 \) and hence that \( P^{2n+1} \) has a normal contact metric structure with \( \xi \) regular.

**Remark 1.** While it is already clear that \( P^{2n+1} \) is a principal circle bundle over \( N^{2n} \), it now also follows from the well-known Boothby-Wang and Morimoto fibrations.

**Remark 2.** Under the hypotheses of Theorem 3, it is possible to assume without loss of generality that \( \alpha^x \) equals 0 or \( 1/\sqrt{t} \) where \( t \) is the number of non-zero \( \alpha^x \) and hence there exist constants \( \beta^x_q, q = 1, \ldots, s - 1 \), such that \( \eta^q = \sum \alpha^x \eta^x \) and \( \bar{\eta}^q = \sum \alpha^x \eta^x \) are 1-forms with \( d\eta^q = 0 \) and \( d\bar{\eta}^q = F \). Then \( f, \bar{\eta}^x \) and the dual vector fields \( \xi \) again define a \( K \)-structure on \( M^{2n+s} \). If now this \( K \)-structure is regular, then, since the distribution spanned by \( \xi_1, \ldots, \xi_{s-1} \) and its complement are parallel, \( M^{2n+s} \) is a covering of the product of \( P^{2n+1} \) and a manifold \( P^{s-1} \) as in the proof of Theorem 2.

**Remark 3.** In [1] one of the authors gave the following example of an \( S \)-manifold as a generalization of the Hopf-fibration of the odd-dimensional sphere over complex projective space, \( \eta' : S^{2n+1} \to PC^n \). Let \( \Delta \) denote the diagonal map and define a space \( H^{2n+s} \) by the diagram

\[ H^{2n+s} \xrightarrow{\hat{\Delta}} S^{2n+1} \times \cdots \times S^{2n+1} \]
\[ PC^n \xrightarrow{\Delta} PC^n \times \cdots \times PC^n \]

that is \( H^{2n+s} = \{(P_1, \ldots, P_s) \in S^{2n+1} \times \cdots \times S^{2n+1} | \eta'(P_1) = \cdots = \eta'(P_s)\} \) and thus \( H^{2n+s} \) is diffeomorphic to \( S^{2n+1} \times T^{s-1} \). Further properties of the space \( H^{2n+s} \) are given in [1], [2].

If however the \( d\eta^x \)'s are independent then there can be no intermediate bundle \( P^{2n+t} \) over \( N^{2n} \) such that \( M^{2n+s} \) is trivial over \( P^{2n+t} \).

**Remark 4.** If \( M^{2n+s} \) is as in Theorem 1 with the \( d\eta^x \)'s independent, then there is no fibration by \( s - t \) of the \( \xi_x \)'s yielding a principal toroidal bundle \( P^{2n+t} \) over \( N^{2n} \) such that \( M^{2n+s} = P^{2n+t} \times T^{s-t} \). For suppose \( P^{2n+t} \) is such an intermediate bundle, then it is necessary that \( \nabla_{\xi_x} \xi_x = 0 \) (see e.g. [8]) and thus the \( \eta^x \)'s are parallel contradicting the independence of the \( d\eta^x \)'s.

5. Curvature. Let \( \nabla \) and \( R \) denote the curvature tensors of \( \nabla \) and \( \nabla \) respectively. Then
In [1], one of the present authors developed a theory of manifolds with an /-structure of constant /-sectional curvature. This is the analogue of a complex manifold of constant holomorphic curvature. A plane section of $M^{2n+s}$ is called an /-section if there is a vector $X$ orthogonal to the distribution spanned by the $\xi_i$'s such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of this section is called an /-sectional curvature and is of course given by $(\xi_X f X / X f X)_{\xi_X f X}$ to De 0 of constant /-sectional curvature if the /-sectional curvatures are constant for all /-sections. This is an absolute constant. We then have the following theorem.

**Theorem 5.** If $M^{2n+s}$ is a compact, connected manifold with a regular $S$-structure of constant /-sectional curvature $c$, then $N^{2n}$ is a Kähler manifold of constant holomorphic curvature.

**Proof.** That $N^{2n}$ is Kähler follows from Theorem 1. By definition there exist $\alpha^1, \ldots, \alpha^s$, necessarily constant such that $\alpha^s F = d\eta^X$. If $X$ is a unit vector on $N^{2n}$, then we have

$$G(R_{XJ}^J X, X) = g(\tilde{\eta}_{\pi X} \tilde{\eta}_{\pi J} \tilde{\eta}_{\pi J} \tilde{\eta}_{\pi X}, \tilde{\eta}_{\pi X})$$

$$+ \sum_x \left( \frac{1}{2} \alpha^s F(\tilde{\eta}_{\pi J} X, \tilde{\eta}_{\pi J} X) \frac{1}{2} \alpha^s F(\tilde{\eta}_{\pi X}, \tilde{\eta}_{\pi X}) \right)$$

$$- \frac{1}{2} \alpha^s F(\tilde{\eta}_{\pi X}, \tilde{\eta}_{\pi J} X) \frac{1}{2} \alpha^s F(\tilde{\eta}_{\pi J} X, \tilde{\eta}_{\pi X})$$

$$- 2(\frac{1}{2} \alpha^s F(\tilde{\eta}_{\pi X}, \tilde{\eta}_{\pi J} X) \frac{1}{2} \alpha^s F(\tilde{\eta}_{\pi J} X, \tilde{\eta}_{\pi X}))$$

$$= c + \frac{3}{4} \sum_x (\alpha^x)^2 (F(\tilde{\eta}_{\pi X}, \tilde{\eta}_{\pi X}))^2$$

$$= c + \frac{3}{4} \sum_x (\alpha^x)^2, \text{ which is constant.}$$
Remark. This agrees with the results in [1] on $H^{2n+s}$. $H^{2n+s}$ is a principal toroidal bundle over $PC^n$ and $PC^n$ is of constant holomorphic curvature equal to 1. Also, $\alpha^x = 1$ for $x = 1, \ldots, s$ and $H^{2n+s}$ was found to be of constant $f$-sectional curvature equal to $1 - 3s/4$.

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