THE LATTICE TRIPLE PACKING OF SPHERES IN EUCLIDEAN SPACE

BY

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ABSTRACT. We say that a lattice \( \Lambda \) in \( n \)-dimensional Euclidean space \( E_n \) provides a \( k \)-fold packing for spheres of radius 1 if, when open spheres of radius 1 are centered at the points of \( \Lambda \), no point of space lies in more than \( k \) spheres. The multiple packing constant \( \Delta_k^{(n)} \) is the smallest determinant of any lattice with this property. In the plane, the first three multiple packing constants \( \Delta_2^{(2)} \), \( \Delta_3^{(2)} \), and \( \Delta_4^{(2)} \) are known, due to the work of Blundon, Few, and Heppes. In \( E_3 \), \( \Delta_2^{(3)} \) is known, because of work by Few and Kanagasabapathy, but no other multiple packing constants are known. We show that \( \Delta_3^{(3)} \leq 8\sqrt{38}/27 \) and give evidence that \( \Delta_3^{(3)} = 8\sqrt{38}/27 \). We show, in fact, that a lattice with determinant \( 8\sqrt{38}/27 \) gives a local minimum of the determinant among lattices providing a 3-fold packing for the unit sphere in \( E_3 \).

1. Introduction. Let \( \Lambda \) be an \( n \)-dimensional lattice in \( n \)-dimensional Euclidean space \( E_n \) such that, if open spheres of radius 1 are centered at the points of \( \Lambda \), then no point of space is covered more than \( k \) times. That is, for any point \( X \) in \( E_n \) there do not exist distinct points \( L_1, L_2, \ldots, L_{k+1} \) of \( \Lambda \) such that \( |X - L_1|, \ldots, |X - L_{k+1}| < 1 \). Then we say that \( \Lambda \) provides a \( k \)-fold packing for spheres of radius 1. The terms single, double and triple are synonymous with \( k \)-fold for \( k = 1, 2 \) and 3.

Let \( d(\Lambda) \) denote the determinant of \( \Lambda \), and let \( \Delta_k^{(n)} \) denote the lower bound of \( d(\Lambda) \), taken over all lattices \( \Lambda \) that provide a \( k \)-fold packing for spheres of radius 1. (Thus \( \Delta_1^{(n)} \) is the critical determinant of a sphere of radius 2.) It is well known and easy to see (e.g., divide one generator of the lattice by \( k \)) that \( \Delta_k^{(n)} \leq \Delta_1^{(n)}/k \).

It has been shown by Few [1] that \( \Delta_2^{(2)} = (\frac{\sqrt{2}}{2})\Delta_1^{(2)} \), and Heppes [5] showed that \( \Delta_2^{(2)} = \Delta_1^{(2)}/k \) if and only if \( k \leq 4 \).

In [4] Few and Kanagasabapathy determined the exact value of \( \Delta_2^{(3)} \), namely \( 3\sqrt{3}/2 \), which is less than \( \Delta_1^{(3)}/2 = 2\sqrt{5} \). By constructing particular lattices they also showed that \( \Delta_2^{(n)} < \Delta_1^{(n)}/2 \) for every \( n \geq 3 \).

Few remarks in [2] that \( \Delta_2^{(3)} \) is the only multiple packing constant known exactly in three dimensions or more, and in this note I shall prove that \( \Delta_3^{(3)} \leq \)
$8\sqrt{38}/27 < \Delta_1^{(3)}/3 = 4\sqrt{2}/3$ and give evidence suggesting that $\Delta_3^{(3)} = 8\sqrt{38}/27$.

In fact, I prove

Theorem 1. A certain lattice $\Lambda_0$ of determinant $d_0 = 8\sqrt{38}/27$ provides a triple packing for the unit sphere $S$. Also $\Lambda_0$ has generators $P$, $Q$, $R$ with $|P| = 2/3$.

Theorem 2. Any lattice $\Lambda$ having generators $P'$, $Q'$, $R'$ with $|P'| < 0.95$ providing a triple packing for $S$ must have determinant $d(\Lambda) \geq d_0$ with equality only when $\Lambda = \Lambda_0$. Hence $\Lambda_0$ gives a local minimum of $d(\Lambda)$ for triple packing of unit spheres.

Remark. There is extensive numerical evidence that $d(\Lambda)$ does not fall below $d_0$ for any triple packing with $S$.

2. An economical lattice $\Lambda_0$.

Theorem 1. The best lattice triple packing for spheres in $E^3$ has determinant $d(\Lambda) \leq 8\sqrt{38}/27 = \sqrt{2432}/729 = 1.82649\ldots$, since indeed the lattice $\Lambda_0$ generated by $P$, $Q$ and $R$ where $P = (a, 0, 0) = (2/3, 0, 0)$, $Q = (b, b, 0) = (1/3, \sqrt{3}, 0)$, and $R = (g, f, c)$, where $g = 1/3$, $f = (11\sqrt{3})/27$ and $c^2 = 3 - f^2$, provides a triple packing for the unit sphere $S$.

Proof. Convention: The letters $\lambda$, $\mu$ and $\nu$ will denote integers. $S(A, r)$ will be the open sphere of radius $r$ centered at $A$; $S(A)$ will denote $S(A, 1)$; thus $S = S(0, 1)$. Suppose that the point $X = (x, y, z)$ is covered four times. Translating $X$ by a lattice point, we may suppose that $X \in S$, and replacing $X$ by $-X$ if necessary we may suppose $z > 0$. The three other spheres covering $X$ can be written $S(\lambda P + \mu Q + \nu R) = S + \lambda P + \mu Q + \nu R$ where $(\lambda, \mu, \nu) \neq (0, 0, 0)$. We must have $|\lambda P + \mu Q + \nu R| < 2$ since they must intersect $S$. Therefore

\begin{equation}
(\lambda a + \mu b + \nu g) + (\mu b + \nu f)^2 + \nu^2 c^2 < 4
\end{equation}

and $|\nu| < 2/c$. Since $c > 1$, we have $\nu \in \{-1, 0, 1\}$. Now $\nu$ cannot be $-1$, since otherwise $|X - (\lambda P + \mu Q - R)| \geq |c + z| > 1$. Hence $\nu \in \{0, 1\}$. From (\ref{equation:*}) we also get $|\mu b + \nu f| < 2$. Since $0 \leq \nu \leq 1$ and $0 \leq f \leq b/2$ and $b = \sqrt{3} > 4/3$ this gives $-2 < \mu < 2$, $\mu \in \{-1, 0, 1\}$. We divide the proof into two parts.

Part 1. $\gamma \geq 0$. Then $\mu \in \{0, 1\}$; in fact if $\mu = -1$, then for $X \in S(\lambda P + \mu Q + \nu R)$ we would have $|X - (\lambda P + Q + R)|^2 \geq (b + \gamma - \nu f)^2 > (16b/27)^2 = 256/243$, since $\nu \in \{0, 1\}$. Also $(\mu, \nu) \neq (1, 1)$, since $|\lambda P + Q + R|^2 \geq b^2 + c^2 > 4$. Hence $(\mu, \nu) \in \{(0, 0), (1, 0), (0, 1)\}$.

Type 1 spheres. Suppose $(\mu, \nu) = (0, 0)$. Then $\lambda P + \mu Q + \nu R = \lambda P$, and $S(\lambda P) \cap S = \emptyset$ if $|\lambda| > 2$, since $|3P| = 3a = 2$. The $S(\lambda P)$ such that $0 < |\lambda| \leq 2$ are called Type 1 spheres.
Type 2 spheres. Suppose $(\mu, \nu) = (1, 0)$. Then $\lambda P + \mu Q + \nu R = \lambda P + Q$, and $S(\lambda P + Q) \cap S = \emptyset$ if $\lambda \notin \{0, -1\}$, since then $|\lambda P + Q|^2 = b^2 + (\lambda a + b)^2 = 3 + |2\lambda/3 + 1/3|^2 > 4$. The $S(Q - P)$ and $S(Q)$ are called Type 2 spheres.

Type 3 spheres. Suppose $(\mu, \nu) = (0, 1)$. Then $\lambda P + \mu Q + \nu R = R + \lambda P$, $S(R + \lambda P) \cap S = \emptyset$ if $\lambda \notin \{0, -1\}$. To see this, observe that if $S(R + \lambda P) \cap S \neq \emptyset$ then $(\lambda a + b)^2 + f^2 + c^2 < 4$; since $f^2 + c^2 = 3$, $|2\lambda/3 + 1/3| < 1$ and $\lambda \in \{0, -1\}$. We call $S(R)$ and $S(R - P)$ Type 3 spheres.

It follows from the discussion of Type 2 and Type 3 spheres that $S \cap S(\lambda P + E) = \emptyset$ if $\lambda \notin \{-1, 0\}$ where $E \notin \{R, Q\}$. In particular,

\[
\emptyset = S \cap S(E + P) = S(-P) \cap S(E) = S(-2P) \cap S(E - P),
\]
\[
\emptyset = S \cap S(E + 2P) = S(-2P) \cap S(E),
\]
\[
\emptyset = S \cap S(E - 2P) = S(P) \cap S(E - P) = S(2P) \cap S(E),
\]
\[
\emptyset = S \cap S(E - 3P) = S(2P) \cap S(E - P).
\]

From the discussion of Type 1 spheres $S \cap S(\lambda P) = \emptyset$ for $|\lambda| > 2$ so that

\[
\emptyset = S \cap S(3P) = S(-P) \cap S(2P) = S(-2P) \cap S(P),
\]
\[
\emptyset = S \cap S(4P) = S(-2P) \cap S(2P).
\]

We now draw a graph $G$ where edges $A$ and $B$ are joined only if we know that $S(A) \cap S(B) = \emptyset$.

We next observe that $\emptyset = S(Q + \lambda P) \cap S \cap S(R + \lambda' P + \nu Q)$. For $|Q + \lambda P| \geq |Q| = \sqrt{b^2 + b^2} > \sqrt{3}$. Therefore the height (maximal value of the $z$ coordinate of the closure) of $S(Q + \lambda P) \cap S$ is less than $\sqrt{1 - 3/4} = \sqrt{1/2} < c - 1$, since $c = 1.5 \cdots > 3/2$. Since $R + \lambda' P$ has $z$ component $c$, the above intersection is void. Hence we cannot have $X$ simultaneously inside a sphere of Type 2 and a sphere of Type 3, so
\[ X \in S(\lambda_1 P) \cap S(\lambda_2 P) \cap S(E + \lambda_3 P) \] with \( 0 < |\lambda_1|, |\lambda_2| \leq 2, -1 \leq \lambda_3 \leq 0, \)
or
\[ X \in S(\lambda_1 P) \cap S(E + \lambda_2 P) \cap S(E + \lambda_3 P) \] with \( 0 < |\lambda_1| \leq 2, -1 \leq \lambda_2, \lambda_3 \leq 0, \)
where \( E \in \mathbb{R}, Q^1 \). Both of these contradict the graph \( G \), and Part 1 follows.

**Part 2.** We now suppose that \( y < 0 \). Recall that if \( S \cap S(\lambda P + \mu Q + \nu R) \neq \emptyset \),
then \(-1 \leq \mu \leq 1 \) and \( 0 \leq \nu \leq 1 \). For those \( S(\lambda P + \mu Q + \nu R) \) containing \( X \) we must have \(-1 \leq \mu \leq 0 \). For suppose that \( \mu = 1 \); since \( X = (x, y, z), y < 0 \), we would have \( |\lambda P + \mu Q + \nu R - X| \geq |b + v - y| > b > 1 \). The spheres \( S(\lambda P + \mu Q + \nu R) \) containing \( X \) other than \( S \) may therefore be divided into four types, as follows:

**Type 1 spheres,** when \((\mu, \nu) = (0, 0)\). As before the only spheres \( S(\lambda P) \) intersecting \( S \) satisfy \( 0 < |\lambda| \leq 2 \), i.e., the Type 1 spheres are \( S(2P), S(P), S(-P) \) and \( S(-2P) \).

**Type 2 spheres,** when \((\mu, \nu) = (-1, 0)\). If \( S(\lambda P + \mu Q + \nu R) \) is to intersect \( S \) we must have \( 4 > |\lambda P + \mu Q + \nu R|^2 = (\lambda a - b)^2 + b^2 = (2\lambda/3 - 1/3)^2 + 3 \). Hence \( 0 \leq \lambda \leq 1 \), i.e., the Type 2 spheres are \( S(-Q) \) and \( S(-P) \).

**Type 3 spheres,** when \((\mu, \nu) = (0, 1)\). As in Part 1, \(-1 \leq \lambda \leq 0 \) if \( S(\lambda P + \mu Q + \nu R) \) intersects \( S \), i.e., the Type 3 spheres are \( S(R) \) and \( S(-R) \).

**Type 4 spheres,** when \((\mu, \nu) = (-1, 1)\). If \( S(\lambda P + \mu Q + \nu R) \) intersects \( S \), then \( (\lambda a + g - b)^2 + (f - b)^2 + c^2 < 4, 4\lambda^2/9 < 4 - (b - f)^2 - c^2 = 4/9, \lambda^2 < 1, \lambda = 0, \)
and \( S(R - Q) \) is the only sphere of Type 4.

As we did in Part 1, we deduce several new disjoint pairs of spheres. From the discussion of Type 4 spheres, \( S \cap S(-Q + R + \lambda P) = \emptyset \) if \( \lambda \neq 0 \), so we have \( S(\lambda P) \cap S(-Q - R) = \emptyset \) if \( \lambda \neq 0 \). From the Type 2 spheres we have
\[
\emptyset = S \cap S(-Q + 2P) = S(-P) \cap S(-Q + P) = S(-2P) \cap S(-Q),
\]
\[
\emptyset = S \cap S(-Q + 3P) = S(-2P) \cap S(-Q + P),
\]
\[
\emptyset = S \cap S(-Q - P) = S(2P) \cap S(-Q + P),
\]
\[
\emptyset = S \cap S(-Q - 2P) = S(2P) \cap S(-Q).
\]

If we combine these with some of the disjoint pairs that we already know from Part 1 and draw a graph, \( G' \), in which \( A \) is joined to \( B \) only if we know \( S(A) \cap S(B) = \emptyset \), we obtain the following graph.

In addition to these disjoint spheres, we observe that, for any \( \lambda \) and \( \lambda' \), \( S(-Q + \lambda P) \cap S(R + \lambda' P) \cap S = \emptyset \), since \( |-Q + \lambda P| \geq |Q| \), so that the height of \( S(-Q + P) \cap S \) is not greater than the height of \( S \cap S(Q) \), which is less than \( \frac{1}{2} < c - 1 \), and \( c \) is the height of \( R + \lambda' P \).

Also, for any \( \lambda, \emptyset = S \cap S(Q + \lambda P) \cap S(R) = S \cap S(-Q - \lambda P) \cap S(-R) = S(R) \cap S(R - Q - \lambda P) \cap S \).

In particular,
\[
S(R) \cap S(R - Q) \cap S = \emptyset.
\]
The following enumeration of possibilities shows that $X$ cannot be contained in the necessary spheres, and Theorem 1 follows:

Clearly Types 2 and 3 cannot both occur by the fourth paragraph above, and two spheres of Type 1 cannot occur with anything else by the graph $G'$. Again by the graph $G'$, if two spheres of Type 2 (or two spheres of Type 3) occur, the remaining sphere cannot have Type 1. Hence the only remaining possibilities are

(2) $X \in S(\lambda_1 P) \cap S(\lambda_2 P - Q) \cap S(R - Q), \quad 0 < |\lambda_1| \leq 2, \quad 0 \leq \lambda_2 \leq 1$,

(3) $X \in S(\lambda_1 P) \cap S(\lambda_2 P + R) \cap S(R - Q), \quad 0 < |\lambda_1| \leq 2, \quad -1 \leq \lambda_2 \leq 0$,

(4) $X \in S(P - Q) \cap S(-Q) \cap S(R - Q), \quad \text{or}$

(5) $X \in S(R) \cap S(-P + R) \cap S(R - Q)$.

Now (2) and (3) contradict $G'$, and (1) excludes (5). To eliminate (4) we observe that, since $S(P) \cap S \cap S(R) \cap S(Q) = \emptyset$, we must have $S(P - Q) \cap S(-Q) \cap S(R - Q) \cap S = \emptyset$.

3. The lattice $\Lambda_0$ is locally optimal.

Remark 1. An arbitrary lattice $\Lambda$ in $E_3$ has a basis $P, Q, R$ where $|P| \leq |Q| \leq |R|$ are the successive minima of the unit sphere, $P = (a, 0, 0), \quad Q = (b, b, 0), \quad R = (g, f, c), \quad a, b, c > 0, \quad 0 \leq b \leq a/2, \quad 0 \leq f \leq b/2, \quad \text{and} \quad -a/2 < g \leq a/2$. Such a basis is said to be reduced in the sense of Gauss or simply reduced. For a proof, see [6, p. 163 et seq., "Seebers inequality"].

Remark 2. If $\Lambda$ has a reduced basis $P, Q, R$ with $P = (a, 0, 0), \quad Q = (b, b, 0), \quad R = (g, f, c)$ and if $d(\Lambda) < d_0$, then $b^2 < b_m^2$, where $b_m^2 = a^2/6 + (2/3)\sqrt[4]{a^4 + 3d_0^2/a^2}$.

Proof. Using $|R|^2 = g^2 + f^2 + c^2 \geq |Q|^2 = b^2 + b^2$, and the other inequalities of reduction, we have $d^2 \leq d^2(\Lambda) = a^2b^2c^2 \geq a^2b^2(b^2 + b^2 - g^2 - f^2) \geq$
a^2 b^2 \left( 3b^2/4 - a^2/4 \right). Putting t = b^2, we get 3a^2 t^2 - a^4 t - 4d_0^2 \leq 0. Hence b^2 must lie between the roots \( a^2/6 - (2/3)\sqrt{a^4/16 + 3d_0^2/a^2} \) and \( a^2/6 + (2/3)\sqrt{a^4/16 + 3d_0^2/a^2} \) of the quadratic.

Let \( p^+ \) denote \( \max \{0, p\} \).

**Theorem 2.** If \((\Lambda, S)\) is a triple packing and \( P = (a, 0, 0), Q = (b, b, 0) \) and \( R = (g, f, c)\) gives a basis for \( \Lambda \) reduced in the sense of Gauss, and \( a \leq 1 \), then

\[
(6) \quad d(\Lambda) = abc \geq ab \sqrt{14 - (a + b)^2 - ((b^2 - 2ab - b^2)/2b)^2} \geq f_1(a, b, b)
\]

when \( 0 \leq g \leq b \), and

\[
(7) \quad d(\Lambda) = abc \geq ab \sqrt{14 - 9a^2/4 - (2b^2 + 3ab)^2/4b^2} \geq f_2(a, b, b)
\]

when \(-a/2 \leq g \leq 0\) and when \( b \leq g \leq a/2\). Furthermore \( f_1(a, b, b) \geq f_2(a, b, b) \), so that in fact

\[
(8) \quad d(\Lambda) \geq f_2(a, b, b)
\]

in all cases. Also, if \( d(\Lambda) \leq d_0 \) and \( 2/3 \leq a \leq 0.9508 \), we have

\[
(9) \quad f_2(a, b, b) \geq \min \{p(a), d_0^2 + 1/100\},
\]

where \( p(a) = 2a^6 - 11a^4 + 12a^2 \), \( p(2/3) = d_0^2 \), and

\[
(10) \quad p(a) > d_0^2 \quad \text{for} \quad 2/3 < a \leq 0.9508.
\]

Hence \( d(\Lambda) \geq d_0 \) for \( 2/3 \leq a \leq 0.9508 \), and with equality only if \( a = 2/3 \).

**Proof.** Suppose that \((\Lambda, S)\) gives a triple packing and that \( P, Q, R \) form a reduced basis of \( \Lambda \). From reduction, we have

\[
(11) \quad |P| \leq |Q| \leq |R|, \quad 0 \leq b \leq a/2, \quad 0 \leq f \leq b/2, \quad \text{and} \quad |g| \leq a/2.
\]

We also have \( a \geq 2/3 \), since otherwise the point \((2/3)P\) would be covered by \( S(-P), S, S(P) \) and \( S(2P) \). Observe that the center of the parallelogram with vertices \( P, Q, Q + P \) and the origin will be covered by the four spheres \( S, S(P), S(Q) \), and \( S(Q + P) \) unless one of the diagonals \( |Q + P|, |Q - P| \) is at least 2. Since \( |Q + P| \geq |Q - P| \) by (11), it follows that \( 4 \leq |Q + P|^2 = b^2 + (a + b)^2 \); hence

\[
(12) \quad b^2 \geq 4 - (a + b)^2 \quad \text{and} \quad a \geq 2/3.
\]

**Case 1.** In this case we assume

\[
(13) \quad 0 \leq g \leq b.
\]

A consequence of (13) is that \( |R - Q - P| \) is not less than \( |R - Q + P| \). They
cannot both be less than 2, since then the center of the parallelogram having
vertices \( R, Q, Q + P, R + P \) would be covered four times. Hence we have
\[
(14) \quad |R - Q - P| \geq 2.
\]
Another consequence of (13) is that \(|R + P|\) is not less than \(|R - P|\). Considering
the parallelogram with vertices \( P, R, R + P \) and the origin shows that
\[
(15) \quad |R + P| \geq 2.
\]
With a view to proving (6) we imagine \( a, b, b \) to be fixed and find the point
\( R = (g, f, c) \) having least nonnegative \( c \) such that (13), (14) and (15) hold and also
\[
(16) \quad 0 \leq f \leq b/2.
\]
We are in fact looking for the lowest point \( X = (x, y, z) \) inside the rectangular
prism given by
\[
(17) \quad Q < x < ¿, \ 0 < y < b/2, \ z > 0,
\]
subject to the additional constraint
\[
(18) \quad |X - P - Q| \geq 2, \ |X + P| \geq 2.
\]
The problem is somewhat simplified by the fact that the centers \( P + Q, - P \) of
the spheres lies on the plane \( z = 0 \), outside the prism.

![Diagram](https://via.placeholder.com/150)

Case 1

If the right-hand side of (6) is zero, there is nothing to prove. Let us suppose,
therefore, that it is positive. We shall show that the point \( X^* \) that lies on the
intersection of the boundary of \( S(- P, 2) \) and \( S(P + Q, 2) \) and the plane \( x = b \) is
the lowest point satisfying (17) and (18). We start by finding \( X^* \). Let \( X^* =
(x^*, y^*, z^*) \). Let \( \pi \) be the radical plane of \( S(- P, 2) \) and \( S(P + Q, 2) \) (the plane
obtained by subtracting the equations of the two spheres). Then \( \pi \) passes through
\((\frac{b}{2})Q = (b/2, b/2, 0)\), which is halfway between the center of the two spheres, and
has the equation \( y - b/2 = (- (2a + b)/b)(x - b/2) \). Putting \( x^* = b \), we obtain \( y^* =
b/2 - (2ab + b^2)/(2b) \). We must show that \( 0 \leq y^* \) so that (17) is satisfied. By (12)
we have \( b^2 \geq 4 - (a + b)^2 \geq 2ab + b^2 + \frac{a}{2} \), since \( b < a/2 \); hence \( 2by^* = b^2 - (2ab + b^2) > 0 \)
and (17) follows. We see that \( z^* = \sqrt{4 - (a + b)^2 - ((b^2 - 2ab - b^2)/(2b))^2} > 0 \) by a pre-
vious assumption.
The first step in showing that $X^*$ is optimal is to show that the bottom of the prism is covered by $S(-P, 2)$ and $S(P + Q, 2)$. This means that there is no $X$ satisfying (17) and (18) with $z = 0$. Let $A^* = (x^*, y^*, 0)$, $E = (0, b/2, 0)$, $F = (b, b/2, 0)$ and $G = (b, 0, 0)$. Then $2 = |P + Q - X^*| > |P + Q - A^*| > |P + Q - F|$, and also $|P + Q - (\frac{\sqrt{2}}{2})Q| < 2$, since the spheres $S(-P, 2)$ and $S(P + Q, 2)$ intersect and $(\frac{\sqrt{2}}{2})Q$ is halfway between their centers. Hence the triangle with vertices $(\frac{\sqrt{2}}{2})Q$, $A^*$, $F$ lies in the interior of $S(P + Q, 2)$.

Similarly, $2 = |-P - X^*| > |-P - A^*| > |-P - G|$ and $2 > |-P - (\frac{\sqrt{2}}{2})Q| > |-P - E|$ so that the convex pentagon with vertices $G, A^*, (\frac{\sqrt{2}}{2})Q, E$ and the origin lies in the interior of $S(-P, 2)$. Hence the bottom of the prism is covered.

We now let $X_1 = (x_1, y_1, z_1)$ be a lowest point satisfying (17) and (18). We know that $X_1$ exists, because the set of solutions is nonempty and closed. The point $X_1$ must be on the boundary of $S(-P, 2)$ or $S(P + Q, 2)$ since otherwise it could be lowered and still satisfy (18).

Let us suppose first that $X_1$ is on the boundary of $S(-P, 2)$. We shall deduce that $X_1$ is on the boundary of $S(P + Q, 2)$. Suppose not. Then $(x_1, y_1)$ must be the point satisfying (17) that is farthest from $-P$, namely $(b, b/2)$. But then the point $X_1 = (b, b/2, \sqrt{4 - (b + a)^2 - b^2/4})$ is easily seen to be inside $S(P + Q, 2)$, contrary to (18).

Suppose that $X_1$ is not on the boundary of $S(-P, 2)$. Then $|X_1 - P - Q| = 2$, and $X_1 = (0, 0, \sqrt{4 - (a + b)^2 - b^2})$ lies inside $S(-P, 2)$ contrary to (18).

Hence $X_1$ lies on the arc of the intersection of $S(-P, 2)$ and $S(P + Q, 2)$ with $z \geq 0$. The highest point of the arc is the point directly above $(\frac{\sqrt{2}}{2})Q$, which is on the boundary of the prism, and the lowest point in the prism is $X^*$, where the arc cuts the $x = b$ plane. Hence $X_1 = X^*$ and (6) follows.

Case 2. Assume
$$(19) - a/2 \leq b \leq 0.$$

Since the center of the parallelogram with vertices $R, Q - P, R + P, Q$ must not be covered four times, we know that one of its two diagonals $|R - Q|, |R - Q + 2P|$ must be at least 2. Assuming Gauss reduction, we always have $|R - Q + 2P| \geq |R - Q|$, since the vectors $R - Q + 2P$ and $R - Q$ differ only in the first component, and $|g - b + 2a| \geq 2a - |g| - b \geq a \geq |g - b|$. Hence $|R - Q + 2P| \geq 2$.

As in Case 1, one of $|R + P|, |R - P|$ must be at least 2, and from (19) we know that $|R - P| \geq |R + P|$ so that $|R - P| \geq 2$.

In a manner similar to Case 1, we are looking for the lowest point $X = (x, y, z)$ inside the rectangular prism given by
$$(20) - a/2 \leq x \leq 0, 0 \leq y \leq b/2, z \geq 0,$$
such that
$$(21) |X - P| \geq 2, \text{ and } |X - Q + 2P| \geq 2.$$
The procedure is the same as in Case 1. We may suppose that the right-hand side of (7) is positive. Let \( X^* = (x^*, y^*, z^*) \) be the point on the two spheres and the plane \( x = -a/2 \), with \( z^* \geq 0 \).

The radical plane \( \pi \) of the two spheres passes through \( (\frac{1}{2})(Q - P) \). The equation of \( \pi \) is \( 2(b - 3a)x + 2by = b^2 + (b - 3a)(b - a) \). Putting \( x = x^* = -a/2 \) yields \( y^* = (b^2 + b^2 - 3ab)/(2b) \). We must show that \( 0 \leq y \leq b/2 \). Now \( b^2 \geq 4 - (a + b)^2 \) for a triple packing; hence \( 2by^* = b^2 + b^2 - 3ab \geq 4 - a^2 - 5ab \geq \frac{1}{2} \). On the other hand \( b^2 < \frac{ab}{2} \leq 3ab, 2by^* = b^2 + b^2 - 3ab \leq b^2, y^* \leq b/2 \). We see that \( z^* = \frac{\sqrt{4 - (3a/2)^2} - ((b^2 + b^2 - 3ab)/(2b))^2}{2} > 0 \) by assumption.

We now show that the bottom of the prism is covered by the two spheres \( S(P, 2) \) and \( S(Q - 2P, 2) \). Let \( A^* = (x^*, y^*, 0), E = (-a/2, b/2, 0), F = (0, b/2, 0) \) and \( G = (-a/2, 0, 0) \). Then \( 2 = |Q - 2P - A| > |Q - 2P - A^*| \geq |Q - 2P - E| \), and also \( |Q - 2P| - (\frac{1}{2})(Q - P) \) is halfway between their centers, Hence the triangle with vertices \( (\frac{1}{2})(Q - P), E, A^* \) lies in the interior of \( S(Q - 2P, 2) \). Similarly \( 2 > |P - (\frac{1}{2})(Q - P)| \geq |P - F| \geq |P - G|, \) so that the convex pentagon with vertices \( G, A^*, (\frac{1}{2})(Q - P), F, \) and the origin lies in the interior of \( S(P, 2) \). Hence the bottom of the prism is covered.

We now let \( X_1 = (x_1, y_1, z_1) \) be a lowest point satisfying (20) and (21); \( X_1 \) exists because the set of points satisfying (20) and (21) is closed and nonempty. Since \( z_1 \geq 0, X_1 \) must be on the boundary of one of the spheres. We suppose first that \( |X_1 - P| = 2, \) and \( |X_1 - Q + 2P| > 2 \). Then \( (x_1, y_1) \) must be as far from \( (a, 0) \) as possible still satisfying (20). Hence \( X_1 = (-a/2, b/2, \sqrt{4 - b^2}, \sqrt{4 - (3a/2)^2}) \), and calculation shows that \( |X_1 - Q + 2P| < 2 \), which is a contradiction.

Suppose now that \( |X_1 - P| > 2 \), so that \( |X_1 - Q + 2P| = 2 \). Then \( (x_1, y_1) \) must be as far as possible from \( (b - 2a, b) \) and still satisfy (20). Now \( -2a \leq b - 2a \leq -3/2a \leq -a/2 \). Hence \( X_1 = (0, 0, \sqrt{4 - (b - 2a)^2}, b^2) \). Hence \( |X_1 - P|^2 = a^2 + 4 - (b - 2a)^2 - b^2 < a^2 + 4 - 9a^2/4 < 4 \), which is a contradiction.
Hence $X_1$ must be on the boundary of both spheres. In a manner similar to Case 1, $X_1$ lies on a circular arc whose highest point $(\frac{a}{2}(Q - P)$ is on the boundary of the prism and whose lowest point inside the prism is $X^*$. Hence $X_1 = X^*$ and (7) is proved for this case.

Case 3. Assume

(22) $b \leq g \leq a/2$.

The vectors $R - Q + P, R - Q - P$ differ only in the first component and, by (22), $|g - b + a| \geq |g - b - a|$. Since the center of the parallelogram with vertices $P, R - Q, R - Q + P$ and the origin must not be covered four times, we conclude that $|R - Q + P| \geq 2$. On the other hand, as in Case 1, $|R + P| \geq 2$.

We are seeking the lowest point $X_1$ of the prism

(23) $b \leq x \leq a/2, 0 \leq y \leq b/2, z \geq 0$,

such that

(24) $|X + P| \geq 2, |X - Q + P| \geq 2$.

Case 3

If the right-hand side of (7) is zero, there is nothing to prove. Let us suppose that it is positive. Let $X$ be the point on the two spheres and the plane $x = a/2$. To find $X = (x, y, z)$ we solve $9a^2/4 + y^2 + z^2 = 4, (3a/2 - b)^2 + (y - b)^2 + z^2 = 4$, and obtain $x^* = a/2, y^* = (b^2 + b^2 - 3ab)/(2b)$, and $z^* = \sqrt{4 - 9a^2/4 - (y^*)^2}$.

This is the same $y^*$ that appears in Case 2, so $0 \leq y^* \leq b/2$ and $X^*$ satisfies (23) and (24).

Now let $X_1$ be a lowest point satisfying (23) and (24), and we shall show that $X_1 = X^*$.

We must show that the bottom of the prism is covered. Since $x = a/2$ maximizes the horizontal distance from both $-P$ and $Q - P$, it is sufficient to show that the line segment $[(a/2, t, 0): 0 \leq t \leq b/2]$ is covered. The spheres $S(-P, 2)$ and $S(Q - P, 2)$ intersect the plane $x = a/2$ in circles which intersect at $(a/2, y^*, z^*)$. Since $z^* > 0$, the segment is covered.

For a lowest point $X_1$ we must have $x_1 = a/2$. Since the spheres intersect
the \( x = a/2 \) plane in circles whose centers have \( y \) components 0 and \( b \) respectively, it is clear that \( X_1 = X^* \) and inequality (7) holds. This finishes Case 3.

Hence \( d(\Lambda) \geq \min \{ f_1, f_2 \} \). We observe immediately that \( f_1 \geq f_2 \). This would follow if \( \psi(b) \geq 0 \) where

\[
\psi(b) = \frac{9a^2}{4} + \left( b/2 - (3a - b)h/(2b) \right)^2 - \left( b^2 - (b/2)(b + 2a)/b \right)^2.
\]

Now \( \psi(0) = 9a^2/4 - a^2 = (5/4)a^2 > 0 \) and \( \psi(a/2) = 0 \). Differentiating, we have

\[
\psi'(b) = - \frac{(5a)(2b^2 + 3b^2 - ab)}{2} < 0 \quad \text{since} \quad b^2 \geq a^2 - b^2 \geq (3/4)a^2.
\]

We now prove

\[
f_2^2 \geq \min \{ d_0^2 + 1/100, 2a^6 - 11a^4 + 12a^2 \}
\]

under the hypothesis that \( d(\Lambda) \leq d_0 \) and \( 2/3 \leq a \leq 0.9508 \). Write \( F(a, b, h) = f_2^2 = a^2b^2/4 - 9a^2/4 - (b/2 + b(b - 3a)/(2b))^2 \) and put \( t = b^2 \). Clearly \( \partial^2 F/\partial t^2 = -a^2/2 < 0 \). Hence

\[
F(a, b, h) \geq \min \{ \phi(a, h), \psi(a, h) \}, \quad \text{where} \quad \phi(a, h) = F(a, b, \sqrt{4 - (a + b)^2}), \quad \psi(a, h) = F(a, b, b_m), \quad \text{where} \quad b_m \text{ was defined in Remark 2}.\]

\[
\phi(a, h) = a^2/4 - (a + b)^2/4 - (\psi(a, h) - (a + b)^2) - (\psi(b(h - 3a))/2 - 2b^2)(b - 3a)^2
\]

and calculation shows that \( \partial^2 \phi/\partial b^2 = -8a^2 - 8a^4 < 0 \). Since \( \phi(a, 0) = \phi(a, a/2) = p(a) \) where \( p(x) = 2x^6 - 11x^4 + 12x^2 \), we have \( \phi \geq p(a) \).

To complete the proof of (9), we will show that \( \psi(a, h) \geq d_0 + 1/100 \). Now

\[
\psi(a, b) = F(a, b, b_m) = a^2b^2/4 - 9a^2/4 - (b_m/2 + b(b - 3a)/(2b_m))^2 \}
\]

and calculation shows that \( \partial^2 \psi/\partial b^2 = -a^2b_m^2 + 3a^2b_m^2 - 9a^3b_m + 9a^4/4 \). We shall show that \( \partial^2 \psi/\partial b^2 < 0 \), so that \( \psi(a, h) \) is a concave function of \( b \).

We digress for a moment to show that \( h \geq h_m(a) \), where \( h_m = \sqrt{4 - b_m^2} - a \). To see this, recall that for triple packing we must have \( b^2 \geq 4 - (a + b)^2 \), whereas \( b^2 \leq b_m^2 \), since \( d(\Lambda) \leq d_0 \). The juxtaposition \( b_m^2 \geq 4 - (a + b)^2 \) yields \( b \geq b_m \).

Putting \( b = a/2 \) we see that \( b_m \leq a/2 \). It is conceivable that \( b_m \) is negative, even though \( b \) never is. For what follows, it is useful to know that \( b_m \geq -a/2 \). To see this, note that

\[
b_m^2 = a^2/6 + (2/3)a^2/16 + 3d_0^2/a^2 < 1/6 + (2/3)sqrt{1/16 + 27 < 3.64},
\]

since \( 2/3 \leq a \leq 1 \) and \( 3 < d_0^2 < 4 \), so that \( 4 - b_m^2 \geq 1/4 \geq a^2/4 \), \( b_m + a = \sqrt{4 - b_m^2} > a/2 \), and \( b_m > -a/2 \).

We now return to showing that \( \psi(a, h) \) is a concave function of \( b \) for \( a \leq b \leq a/2 \). Since \( \partial^2 \psi/\partial b^2 = 6a^2b - 9a^3 \) \( \leq 3a^3 - 9a^3 = -6a^3 < 0 \), it is enough to show that

\[
\psi(a, h) = (1/a^2)\partial^2 \psi/\partial b^2|_{b=b_m} < 0.
\]

Since \( f(a) \) is a rather complicated function of the single variable \( a \), we shall simply find an upper bound for its derivative as a function of \( a \) and use a computer to evaluate it on a fine grid.
Let $F(x, y, z) = -z + 3y^2 - 9xy + 9x^2/4$ so that $F(a, b_m, b_m^2) = f(a)$. Then

$$|f'(a)| \leq \left| \frac{\partial F}{\partial x} \right|_0 + \left| \frac{\partial F}{\partial y} \right|_0 + \frac{d b_m}{d a} \left| \frac{d F}{\partial z} \right|_0 + \frac{d b_m^2}{d a},$$

where the subscript 0 indicates that the partial derivatives are evaluated at $(x, y, z) = (a, b_m, b_m^2)$.

Then

$$\left| \frac{\partial F}{\partial x} \right|_0 = \left| -9b_m + 9a/2 \right| \leq 9a \leq 9, \quad \left| \frac{\partial F}{\partial y} \right|_0 \leq \left| 6b_m \right| + 9a \leq 12, \quad \text{and} \quad \left| \frac{\partial F}{\partial z} \right|_0 = 1.$$

Let $u = a^2$, $g(u) = b_m^2 = u/6 + (2/3)\sqrt{u^2/16 + 3d_0^2/u}$. Then

$$g'(u) = 1/6 + (1/3)(u/8 - 3d_0^2/u^2)/\sqrt{u^2/16 + 3d_0^2/u},$$

and

$$\left| g'(u) \right| \leq 1/6 + (1/3)(1/8 + 3 \times 4/(4/9)^2)/\sqrt{4/9}^2/16 + 9 = 1/6 + (1/3)(1/8 + 243/4)/\sqrt{9} < 7.$$

Therefore $|dg(a^2)/da| = |2ag'(a^2)| < 14$. That is, $|db_m^2/da| < 14$.

Finally, $b_m = \sqrt{4 - b_m^2 - a}$, $db_m/da = -1 - (1/2)(db_m^2/da)/\sqrt{4 - b_m^2}$, and

$$|db_m/da| \leq 1 + \frac{7|4 - b_m^2|}{16} \leq 1 + 7|0.36| \leq 13, \quad \text{since} \quad b_m^2 < 3.64.$$

Hence

$$|f'(a)| \leq 9 + 12 \times 13 + 1 \times 14 = 179.$$

Using a computer we verified that (allowing for roundoff error) $f(a_i) < -0.2$ at the points $2/3 = a_0 < a_1 < \cdots < a_n = 1$, where $n = 500$, and $|a_i - a_i^1| < 1/1200$ for $1 \leq i \leq n$.

Let $a$ be an arbitrary number in the interval $[2/3, 1]$. Then $a \in [a_i^1, a_i]$ for some $i$, and therefore

$$f(a) = f(a_i - 1) + \int_{a_i - 1}^a f'(t) \, dt \leq -0.2 + \frac{179}{1200} < -0.05.$$

Hence $\psi(a, b)$ is a concave function of $b$ as claimed, so $\psi(a, b) \geq \min \{ \psi(a, b_m), \psi(a, a/2) \}$.

The functions $\psi(a, b_m)$ and $\psi(a, a/2)$ are also rather complicated functions of the single variable $a$, and they are both above $d_0^2 + 1/100$. We shall simply find an upper bound for their derivatives as functions of $a^2$ or $a$ and use a computer to evaluate them on a fine mesh.

Let $u = a^2$ and $f(u) = \psi(a, a/2)$. Then $f(u) = ug(u)\Omega(u, g(u)) = 25u^3/64$, where $g(u) = b_m^2 = u/6 + (2/3)\sqrt{u^2/16 + 3d_0^2/u}$, and $\Omega(u, v) = 4 - (9/4)u - v/4 + (5/8)u$. Then $f'(u) = g(u)\Omega(u, g(u)) + ug'(u)\Omega(u, g(u)) + ug(u)^3\Omega'\, du + g'(u)\Omega^3/\partial u - 75u^2/64$.

To estimate $|f'(u)|$, we must estimate $g$, $\Omega$, and their derivatives. We have $|g(u)| \leq 3.64 < 4$ from before and $|g(u)| \geq u/3$, trivially. Hence $|\Omega(u, g(u))| \leq$
$4 + 13u/8 + |g(u)/4| \leq 4 + 13/8 + 1 \leq 7$. We also have $|g'(u)| \leq 7$ from before. On the other hand $|\partial \Omega/\partial u| = 13/8 < 2$ and $|\partial \Omega/\partial v| = 1/4$. Putting the estimates together, $|f'(u)| \leq 4 x 7 + 7 x 7 + 4 x 2 + 5 + 2 = 92$.

We will show that $f(u) \geq d^2_0 + 1/100$ for $4/9 \leq u \leq 1$. Let us suppose that a computing machine has verified that $f(u) \geq d^2_0 + 1/50 + \epsilon$ for $4/9 = u_0 < u_1 < \ldots < u_n = 1$, where $|u_{i-1} - u_i| < (50 \times 92)^{-1}$. It then follows that $f(u) \geq d^2_0 + \epsilon$ for $4/9 \leq u \leq 1$. For $u \in [u_{i-1}, u_i]$ for some $i$, and

$$|f(u)| \geq |f(u_{i-1})| - \left| \int_{u_{i-1}}^u f'(t) \, dt \right|$$

$$\geq d^2_0 + 1/50 + \epsilon - 92|u_{i-1} - u_i| \geq d_0^2 + \epsilon.$$

A computing machine was programmed to find the minimum value of $f(u)$ for $1 \leq i \leq 8251$, where $u_i = 4/9 + (i-1)/14850$, and the answer was $3.51822$.

Had there been no roundoff error, we could say that $f(u) \geq d^2_0 + 1/6$. It is certainly safe to say that $f(u) \geq d^2_0 + 1/50 + 1/100$. Hence $\psi(a, a/2) = f(a^2) > d^2_0 + 1/100$ for $2/3 < a < 1$.

We shall use the same method for $\psi(a, b_m)$. Let us rename $f(a) = \psi(a, b_m) = a^2b_m^2 - (9/4)a^2 - b_m^2/4 - b_m(b_m - 3a)/2 - a^2b_m^2(b_m - 3a)/4$. It is unfortunate that $\psi(a, b_m)$ cannot be written simply as a function of $a^2$; all our functions are now functions of $a$. Let $g_1(a) = b_m^2 = g(a^2)$, and let $b(a) = b_m$. Put $\Phi(x, y, z) = 4 - (9/4)x^2 - y/4 - z(z - 3x)/2$ and $\Theta(x, y) = -x^2z^2(z - 3x)/4$. Then

$$f(a) = a^2g_1(a)\Phi(a, g_1(a), b(a)) - \Theta(a, b(a)),$$

and

$$f'(a) = 2ag_1(a)\Phi(a, g_1(a), b(a)) + a^2g_1'(a)\Phi(a, g_1(a), b(a)) + a^2g_1(a)[\partial \Phi/\partial x + (\partial \Phi/\partial y)g_1'(a) + (\partial \Phi/\partial z)b'(a)] - \partial \Theta/\partial x - (\partial \Theta/\partial z)b'(a).$$

Using some of the estimates from before and making some new ones, we see that $|g_1(a)| = |g(a^2)| < 4$, $|b(a)| = |\sqrt{4 - b_m^2} - a| \leq \frac{1}{2}$, $|\Phi| \leq 4 + 9/4 + 1 + b^2(a)/2 + 3|b(a)|/2 \leq 1/8 + 28/4 < 8$, $|g_1'(a)| < 14$, $\partial \Phi/\partial x = 9x/2 + 3z/2$, $|\partial \Phi/\partial x| \leq 9/2 + (3/2) \times \frac{1}{2} < 6$, $|\partial \Phi/\partial y| = \frac{1}{4}$, $|\partial \Phi/\partial z| = |z + 3x/2| \leq \frac{1}{2} + \frac{3}{2} = 2$, and $|b'(a)| < 13$. By calculation, $\partial \Theta/\partial x = xz^2/2 + 18x^2/2$. Taking the maximum of the positive and negative parts, $|\partial \Theta/\partial x| \leq b_m^2/2 \times \max|b^2_m + 18a^2|$, $|\partial \Theta/\partial x| \leq (b_m/2)\max|b^2_m + 9b_m| < 3$. Similarly, $\partial \Theta/\partial x = xz^2(z - 6xz + 9z^2 + z^2 - 3x)/2$, and $|\partial \Theta/\partial x| \leq (b_m/2)\max|b^2_m + 9, b_m| < 3$. Putting these estimates together, $|f(a)| \leq 2 \times 4 \times 8 + 14 \times 13 + 4(6 + 14/4 + 2 \times 13) + 3 + 3 \times 13 = 430$. To show that $f(a) \geq d^2_0 + \epsilon$, therefore, it is enough to show that $f(a_i) \geq d^2_0 + 1/50 + \epsilon$ for $2/3 = a_0 < \ldots < a_n = 1$ where $|a_{i-1} - a_i| < 1/25,000$.

A computing machine was programmed to find the minimum value of $f(a_i)$ for $1 \leq i \leq 100,001$ where $a_i = 2/3 + i/300,000$, and the answer was $3.40344 \ldots$. 

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Had there been no roundoff error we could say that \( f(\lambda) \geq d^2_0 + 1/15 \). It is certainly safe to say that \( f(\lambda) \geq d_0 + 1/50 + 1/100 \) so that \( \psi(\lambda, \beta_m) = f(\lambda) \geq d^2_0 + 1/100 \) for \( 2/3 \leq \lambda \leq 1 \). Therefore \( \psi(\lambda, \beta) \geq \min \{ \psi(\lambda, \alpha/2), \psi(\lambda, \beta_m) \} \geq d^2_0 + 1/100 \) and \( f_2^2 \geq \min \{ \psi(\lambda, \beta), \phi(\lambda, \beta) \} \geq \min \{ d^2_0 + 1/100, \rho(\lambda) \} \) as claimed. Thus (9) is proved.

We now prove (10). Let \( f'(t) = 2t^3 - 11t^2 + 12t - \lambda^2 \). Then \( f''(t) = 6t^2 - 22t + 12 \) and \( f'''(t) = 12t - 22 < 0 \) for \( 0 \leq t \leq 1 \). Hence \( f(t) \) is a concave function and has at most two zeroes in the range \([0, 1]\). In fact, \( f(4/9) = 0 \), and \( f(\alpha^2) = 0 \) where \( \alpha^2 = 0.90402... \) and \( f(t) > 0 \) for \( 4/9 < t < \alpha^2 \). Since \( f(\alpha^2) = p(\lambda) - d^2_0 \), we conclude that \( p(\lambda) > d^2_0 \) for \( 2/3 < \lambda < \alpha = 0.950802... \) and (10) is proved.

It now follows from (8), (9) and (10) that \( d(\lambda) \geq d_0 \) for \( 2/3 \leq \lambda \leq 0.9508 \) with equality only when \( \lambda = 2/3 \).

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