EXISTENCE OF SUM AND PRODUCT INTEGRALS

BY

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This paper is dedicated to Professor H. S. Wall.

ABSTRACT. Functions are from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, where $\mathbb{R}$ represents the set of real numbers. If $c$ is a number and either (1) $\int_a^b G^2$ exists and $\int_a^b G$ exists, (2) $\int_a^b G$ exists and $\int_a^b (1 + G)$ exists and is not zero or (3) each of $\int_a^b (1 + G)$ and $\int_a^b (1 - G)$ exists and is not zero, then $\int_a^b cG$ exists,

$$\int_a^b (cG - eG) = 0,$$

and $\int_a^b (1 + G)$ exists for $a < x < y < b$ and $\int_a^b |1 + cG - 1(1 + cG)| = 0$. Furthermore, if $H$ is a function such that

$$\lim_{x \to p} H(x, p), \lim_{x \to p^+} H(p, x), \lim_{x, y \to p^+} H(x, y)$$

exist for each $p \in [a, b]$, $n \geq 2$ is an integer, and $G$ satisfies either (1), (2) or (3) of the above, then $\int_a^b HG^n$ exists, $\int_a^b |zVG^n - 1zVG^n| = 0$, and $\int_a^b |1 + zVG^n - 1(1 + zVG^n)| = 0$.

All integrals and definitions are of the subdivision-refinement type, and functions are from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, where $\mathbb{R}$ represents the set of real numbers. Furthermore, functions are assumed to be defined only for elements $\{x, y\}$ of $\mathbb{R} \times \mathbb{R}$ such that $x \leq y$, and $G(x, x) = 0$. The statements that $G$ is bounded, $G \in O^P$, $G \in O^Q$ and $G \in O^B$ on $[a, b]$ mean that there exist a subdivision $D$ of $[a, b]$ and positive numbers $B$ and $c$ such that if $J = \{x, y\}$ is a refinement of $D$, then

1. $|G(u)| < B$ for $u \in J(I)$,
2. $\left|\Pi_i^j (1 + G_q^i)\right| < B$ for $1 \leq i \leq j \leq n$,
3. $\left|\Pi_i^j (1 + G_q^i)\right| > c$ for $1 \leq i \leq j \leq n$, and
4. $\sum_{J(I)} |G| < B$,

respectively, where $G_q = G(x_{q-1}, x_q)$ and $J(I) = \{x_{q-1}, x_q\}$. Similarly, statements of the form $G > B$ should be interpreted in terms of subdivisions and refinements. The symbols $G(p, p^+)$, $G(p^-, p)$, $G(p^+, p^+)$ and $G(p^-, p^-)$ are used to denote $\lim_{x \to p^+} G(x, p)$, $\lim_{x \to p^-} G(x, p)$, $\lim_{x, y \to p^+} G(x, y)$, and $\lim_{x, y \to p^-} G(x, y)$, respectively. Further, $G \in O^L$ on $[a, b]$ only if $G(p, p^+)$, $G(p^-, p)$, $G(p^+, p^+)$ and $G(p^-, p^-)$ exist for each $p \in [a, b]$, and $G \in O^L_{14}$ on $[a, b]$ only if $G \in O^L$ on $[a, b]$ and $G(p^+, p^+)$ and $G(p^-, p^-)$ exist for each $p \in [a, b]$. For convenience in notation, when we consider a function $G$ defined only
on intervals \([x, y]\) such that \(a \leq x < y \leq b\), we adopt the convention that

\[ G(a^-, a^-) = G(a^-, a) = G(b^+, b^-) = G(b, b^-) = 0. \]

Also, \(G \in O^A\) on \([a, b]\) only if \(\int_a^b G \) exists and \(\int_a^b |G - \int G| = 0\), and \(G \in O^M\) on \([a, b]\) only if \(\int_x \Pi\ (1 + G)\) exists for \(a \leq x < y \leq b\) and \(\int_a^b |1 + G - \Pi(1 + G)| = 0\). The sources of these definitions are \([3, p. 299]\), \([4, p. 493]\), \([5]\) and \([7]\).

**Lemma 1.1.** If \(\int_a^b G^2\) exists and \(\int_a^b G\) exists, then \(G \in O L\) on \([a, b]\).

The proof of Lemma 1.1 is straightforward, and therefore we omit it.

**Lemma 1.2.** If \(\beta > 0\), \(|G| < 1 - \beta\) on \([a, b]\), \(\int_a^b G^2\) exists and \(\int_a^b G\) exists, then \(\int_a \Pi^b (1 + G)\) exists and is not zero \([6, \text{Theorem } 5]\).

**Lemma 1.3.** If \(G\) is bounded on \([a, b]\) and \(\int_a \Pi^b (1 + G)\) exists and is not zero, then \(G \in O^P\) and \(O^Q\) on \([a, b]\) \([7, \text{Theorem } 2]\).

**Lemma 1.4.** If \(G\) is bounded on \([a, b]\) and \(\int_a \Pi^b (1 + G)\) exists and is not zero, then \(G \in O^M\) on \([a, b]\).

**Indication of Proof.** This lemma follows from Lemma 1.3 and a result of B. W. Helton \([3, \text{Theorem } 4.2, p. 305}\).

**Lemma 1.5.** If \(\int_a^b G\) exists, then \(G \in O^A\) on \([a, b]\).

This result is due to A. Kolmogoroff \([8, p. 669]\). The reader is also referred to results by W. D. L. Appling \([1, \text{Theorems } 1, 2, p. 155]\) and B. W. Helton \([3, \text{Theorem } 4.1, p. 304]\).

**Lemma 1.6.** If \(E = \{p_i\}_{i=1}^N\) is a set of distinct points from \([a, b]\) and \(F, G\) and \(H\) are functions defined on \([a, b]\) such that

\begin{enumerate}
  \item \(G \in O^L\) on \([a, b]\) and \(\int_a^b G\) exists,
  \item if \(p \in E\), then \(H(p, p^+)\) and \(H(p^-, p)\) exist, and
  \item if \(a \leq x < y \leq b\), then \(F(x, y) = G(x, y)\) if \(x \notin E\) and \(y \notin E\) and \(F(x, y) = H(x, y)\) if \(x \in E\) or \(y \in E\),
\end{enumerate}

then \(\int_a^b F\) exists and is

\[ \int_a^b G + \sum_{p \in E} [H(p, p^+) + H(p^-, p) - G(p, p^+) - G(p^-, p)]. \]

The proof of Lemma 1.6 is straightforward, and therefore we omit it.
Lemma 1.7. If \( E = \{p_i\} \) is a set of distinct points from \([a, b]\) and \( F, G \) and \( H \) are functions on \([a, b]\) such that

1. \( G \in OL^0 \) on \([a, b]\) and \( a \prod_{i=1}^{b} (1 + G) \) exists and is not zero,
2. if \( p \in E \), then \( H(p, p^+) \) and \( H(p^-, p) \) exist, and
3. if \( a < x < y < b \), then \( F(x, y) = G(x, y) \) if \( x \not\in E \) and \( y \not\in E \) and \( F(x, y) = H(x, y) \) if \( x \in E \) or \( y \in E \),

then \( a \prod_{i=1}^{b} (1 + F) \) exists and is

\[
\left\{ \prod_{p \in E} \frac{(1 + G)(p, p^+)}{(1 + G^-)(p, p^-)} \right\}^{-1}.
\]

Furthermore, \( F \in OM^0 \) on \([a, b]\).

Proof. We first show that \( a \prod_{i=1}^{b} (1 + F) \) exists and is \( P \), where

\[
P = \left[ a \prod_{i=1}^{b} (1 + G) \right] \left( P_1 \right)^{-1},
\]

\[
P_1 = \prod_{p \in E} \left[ 1 + H(p, p^+) \right] \left( 1 + H(p^-, p) \right),
\]

and

\[
P_2 = \prod_{p \in E} \left[ 1 + G(p, p^+) \right] \left( 1 + G(p^-, p) \right).
\]

Since \( G \in OL^0 \) on \([a, b]\), it follows from the covering theorem that \( G \) is bounded on \([a, b]\). Hence, it follows from Lemma 1.3 that \( G \in OP^0 \) and \( OQ^0 \) on \([a, b]\).

Let \( \epsilon > 0 \). There exist a subdivision \( D_1 \) of \([a, b]\) and a number \( B > 1 \) such that if \( J = [x_q, y_q] \) is a refinement of \( D_1 \), then

1. \( \left| \prod_{i=1}^{n} (1 + G_q) \right| < B \) for \( 1 \leq i \leq n \),
2. \( |P_1| < B \) and \( |[P_2]| < B \), and
3. \( |a \prod_{i=1}^{b} (1 + G) - \prod_{i=1}^{n} (1 + G)| < \epsilon(3B^2)^{-1} \).

Let \( \delta \) be a positive number such that if \( p_i \)

\[
\left| p_i - x_i \right| < \delta \leq p_i < \left| p_i + \delta \right|
\]

for \( 1 \leq i \leq r \), then
Let $D$ denote the subdivision of $[a, b]$ consisting of

$$D_1 \cup \{p_i - \delta_i \cup \{p_i + \delta_i \cup \frac{1}{2} (p_i + p_{i+1})\}^{-1} \cup E$$

less any elements which are not in $[a, b]$. Suppose $J$ is a refinement of $D$. Let $K(l)$ be the subset of $J(l)$ such that $u \in K(l)$ only if neither end point of $u$ belongs to $E$. Note that no interval in $J(l)$ can have elements of $E$ at both end points. Let $L(l) = J(l) - K(l)$. Thus,

$$\left| \prod_{J(l)} (1 + F) - P \right| \leq \left| \prod_{J(l)} (1 + F) - \left[ \prod_{L(l)} (1 + G) \right] [P_1][P_2]^{-1} \right|$$

$$+ \left| \prod_{J(l)} (1 + G) \right| \left[ \prod_{L(l)} (1 + F) - \left[ \prod_{L(l)} (1 + G) \right] [P_1][P_2]^{-1} \right| + \epsilon (3B^2)^{-1} B^2$$

$$\leq B^{2r} \left| \prod_{L(l)} (1 + F) - \left[ \prod_{L(l)} (1 + G) \right] [P_1][P_2]^{-1} \right| + \frac{\epsilon}{3}$$

$$\leq B^{2r} \left| \prod_{L(l)} (1 + F) - P_1 \right| + B^{2r} |P_1| \left| 1 - \left[ \prod_{L(l)} (1 + G) \right] [P_2]^{-1} \right| + \frac{\epsilon}{3}$$

$$\leq \epsilon (3B^{2r})^{-1} B^{2r} + \epsilon (3B^{2r+1})^{-1} B^{2r+1} + \epsilon / 3 = \epsilon.$$

Therefore, $a \Pi^b (1 + F)$ exists and is $P$.

We now show that $F \in OM^o$ on $[a, b]$. Since it can be shown that $x \Pi^y (1 + F)$ exists for $a \leq x < y \leq b$ by an argument similar to the one used in the preceding paragraph, it is only necessary to show that

$$\int_a^b \left| 1 + F - \Pi (1 + F) \right| = 0.$$
Let $\epsilon > 0$. As noted in the previous paragraph, $G \in OP^0$ and $OQ^0$ on $[a, b]$. Hence, Lemma 1.4 implies that $G \in OM^0$ on $[a, b]$. There exist a subdivision $D_1$ of $[a, b]$ and a number $B > 1$ such that if $|x_q|^n_0$ is a refinement of $D_1$, then

1. $|\Pi_i^n (1 + G_{q_i})| < B$ for $1 \leq i \leq n$,
2. $|1 + F_{q_i}| < B$ and $|1 + G_{q_i}| > 1/B$ for $1 \leq q \leq n$, and
3. $\Sigma_{q=1}^n |1 + G_{q_i} - \Pi_{q(j)} (1 + G)| < \epsilon (5B^2)^{-1}$, where $J_q$ is a subdivision of $[x_{q-1}, x_q]$ for $1 \leq q \leq n$.

Let $\delta$ be a positive number such that if $1 \leq i \leq r$ and

\[ p_i^a \delta \leq x_i' < x_i < p_i < y_i < y_i < p_i + \delta, \]

then

1. $|F(x_i, p_i) - F(x_i', p_i)| < \epsilon (10r)^{-1}$,
2. $|F(p_i, y_i') - F(p_i, y_i)| < \epsilon (10r)^{-1}$,
3. $|G(x_i, p_i) - G(x_i', p_i)| < \epsilon (10r^3)^{-1}$, and
4. $|G(p_i, y_i) - G(p_i, y_i')| < \epsilon (10r^3)^{-1}$.

Let $D$ denote the subdivision of $[a, b]$ consisting of

\[ D_1 \cup \{|p_i - \delta| < x_i' < x_i < p_i < y_i < y_i < p_i + \delta, \}

less any elements which are not in $[a, b]$.

Suppose $J = \{x_q^n_0\}$ is a refinement of $D$. For $1 \leq q \leq n$, let $J_q = \{x_q^n_0(q)\}$ be a subdivision of $[x_{q-1}, x_q]$ such that

\[ \left| \sum_{q=1}^n \Pi_{q(j)} (1 + F) - \Pi_{J_q} (1 + F) \right| < \frac{\epsilon}{5n}. \]

Also, for $1 \leq q \leq n$, suppose

1. $q \in U$ only if $[x_{q-1}, x_q]$ does not have a point of $E$ as an end point,
2. $q \in V(1)$ only if $x_q \in E$, and
3. $q \in V(2)$ only if $x_{q-1} \in E$.

Note that $D$ is defined so that $q$ cannot belong to both $V(1)$ and $V(2)$. For $q \in V(1)$, let

1. $K_q = \{x_q, i_0^n(q)\}$,
2. $F_q = F(x_q, n(q)-1, x_q, n(q))$, and
3. $G_q = G(x_q, n(q)-1, x_q, n(q))$,

and for $q \in V(2)$, let

1. $K_q = \{x_q, i_1^n(q)\}$,
2. $F_q = F(x_q, 0, x_q, 1)$, and
3. $G_q = G(x_q, 0, x_q, 1)$.

If $q \in V(1)$ or $V(2)$, then $\Pi_{J_q} (1 + F) = (1 + F_q)\Pi_{K_q} (1 + G)$ and $\Pi_{J_q} (1 + G) = (1 + G_q)'\Pi_{K_q} (1 + G)$. Thus,
\[
\sum_{q=1}^{n} \left| 1 + F_q - \sum_{x_{q-1}} \prod_{x_q} (1 + F) \right| < \sum_{q=1}^{n} \left| 1 + F_q - \prod_{J_q(l)} (1 + F) \right| + \frac{\epsilon}{5}
\]

\[
= \sum_{q \in U} \left| 1 + G_q - \prod_{J_q(l)} (1 + G) \right| + \sum_{i=1}^{2} \sum_{q \in V(i)} \left| 1 + F_q - \prod_{K_q(l)} (1 + G) \right| + \frac{\epsilon}{5}
\]

\[
< \frac{\epsilon}{2} + \sum_{i=1}^{2} \sum_{q \in V(i)} \left| 1 + F_q \right| \left| 1 + \prod_{K_q(l)} (1 + G) \right| + \sum_{i=1}^{2} \sum_{q \in V(i)} \left| F_q - F'_q \right| + \frac{2\epsilon}{5}
\]

\[
< B \sum_{i=1}^{2} \sum_{q \in V(i)} \left| 1 - \prod_{K_q(l)} (1 + G) \right| + [2\pi][e(10\pi)^{-1}] + \frac{2\epsilon}{5}
\]

\[
= B \sum_{i=1}^{2} \sum_{q \in V(i)} \left| (1 + G_q)^{-1} \right| \left| 1 + G_q - \prod_{K_q(l)} (1 + G) \right| + \frac{3\epsilon}{5}
\]

\[
\leq B^2 \sum_{i=1}^{2} \sum_{q \in V(i)} \left| 1 + G_q - \prod_{J_q(l)} (1 + G) \right|
\]

\[
+ B^2 \sum_{i=1}^{2} \sum_{q \in V(i)} \left| G_q - G'_q \right| \left| \prod_{K_q(l)} (1 + G) \right| + \frac{3\epsilon}{5}
\]

\[
< B^2[e(5B^2)^{-1}] + [2\pi B^3][e(10\pi B^3)^{-1}] + \frac{3\epsilon}{5} = \epsilon.
\]

Therefore, \( F \in O^\circ \) on \([a, b]\).

**Theorem 1.** If \( \int_a^b G^2 \) exists, \( \int_a^b G \) exists and \( c \) is a number, then \( cG \in OM^\circ \) and \( OA^\circ \) on \([a, b]\).

**Proof.** Since it follows from Lemma 1.5 that \( cG \in OA^\circ \) on \([a, b]\), we need only show that \( cG \in OM^\circ \) on \([a, b]\). Since \( G \in OL^{14} \) on \([a, b]\) [Lemma 1.1], there exists a subdivision \( D = \{x_q\}_{0} \) of \([a, b]\) such that if \( 1 \leq q \leq n \) and \( x_{q-1} < x < y < x_q \), then \( |cG(x, y)| < \frac{1}{2} \). Let \( F \) be the function such that

(1) \( F(x, y) = cG(x, y) \) if \( x \notin D \) and \( y \notin D \), and

(2) \( F(x, y) = 0 \) if \( x \in D \) or \( y \in D \).

Thus, \( |F| < \frac{1}{2} \) and Lemma 1.6 implies that \( \int_a^b F^2 \) exists and \( \int_a^b F \) exists. Therefore, \( \int_a^b F \) exists and is not zero by Lemma 1.2, and hence, Lemma 1.7 implies that \( cG \in OM^\circ \) on \([a, b]\).
Lemma 2.1. If $G \in OM^0$ or $OA^0$ on $[a, b]$, $G \in OB^0$ on $[a, b]$ and $H \in OL^0$ on $[a, b]$, then $HG \in OM^0$ and $OA^0$ on $[a, b]$ [4, Theorem 2, p. 494], [3, Theorem 3.4, p. 301].

Theorem 2. If $\int_a^b G^2$ exists, $\int_a^b G$ exists, $H \in OL^0$ on $[a, b]$ and $n \geq 2$ is an integer, then $HG^n \in OM^0$ and $OA^0$ on $[a, b]$.

Proof. Since $G \in OL^{1+}$ on $[a, b]$ [Lemma 1.1], it follows that $HG^{n-2} \in OL^0$ on $[a, b]$. Therefore, since $G^2 \in OA^0$ on $[a, b]$ [Lemma 1.5] and $G \in OB^0$ on $[a, b]$, it follows from Lemma 2.1 that $HG^n \in OM^0$ and $OA^0$ on $[a, b]$.

Theorem 2 is not true for $n = 1$. The author [5, Theorem 10] has shown that for $\int_a^b HG$ to exist for every $H \in OL^0$ on $[a, b]$ it is necessary that $G \in OB^0$ on $[a, b]$. However, Davis and Chatfield [2, p. 747] define a function $G$ such that $\int_a^b G^2$ exists, $\int_a^b G$ exists and $G \not\in OB^0$ on $[a, b]$.

Lemma 3.1. If $\int_a^b (1 + G)$ exists and is not zero and $\int_a^b G$ exists, then $G \in OL^{1+}$ on $[a, b]$ [6, Theorem 9].

The conclusion of Theorem 9 [6] states that $G \in OL^0$ on $[a, b]$. However, the argument used to establish this also establishes that $G \in OL^{1+}$ on $[a, b]$.

Lemma 3.2. If $\int_a^b HG$ exists, $G \geq 0$ on $[a, b]$, $H \in OL^0$ on $[a, b]$ and $H$ is bounded away from zero on $[a, b]$, then $\int_a^b G$ exists.

Proof. There exists a subdivision $\{x_q\}_{q=0}^n$ of $[a, b]$ such that if $1 \leq q \leq n$, then $H$ does not change sign on $(x_{q-1}, x_q)$. Hence, $HG \in OB^0$ on $[x_{q-1}, x_q]$, and thus, $HG \in OB^0$ on $[a, b]$. Therefore, since $H^{-1} \in OL^0$ on $[a, b]$ and $HG \in OA^0$ on $[a, b]$ [Lemma 1.5], it follows from Lemma 2.1 that $\int_a^b G$ exists.

Lemma 3.3. If $\int_a^b (1 + G)$ exists and is not zero and $G > -1$ on $[a, b]$, then $\int_a^b \ln(1 + G)$ exists [6, Theorem 4].

Lemma 3.4. If $\int_a^b (1 + G)$ exists and is not zero and $\int_a^b G$ exists, then $\int_a^b G^2$ exists.

Proof. Since $G \in OL^{1+}$ on $[a, b]$ [Lemma 3.1], there exists a subdivision $D = \{x_q\}_{q=0}^n$ of $[a, b]$ such that if $1 \leq q \leq n$ and $x_{q-1} < x < y < x_q$, then $|G(x, y)| < \frac{1}{2}$. Let $F$ be the function such that

1. $F(x, y) = G(x, y)$ if $x \notin D$ and $y \notin D$, and
2. $F(x, y) = 0$ if $x \in D$ or $y \in D$.

Therefore, $\int_a^b (1 + F)$ exists and is not zero by Lemma 1.7, and $\int_a^b F$ exists by Lemma 1.6. Thus, from Lemma 3.3, $\int_a^b \ln(1 + F) = \int_a^b \sum_{n=1}^\infty (-1)^{n-1} F^n/n$ exists,
and hence,
\[
\int_a^b \sum_{n=2}^\infty (-1)^{n-1} \frac{F^n}{n} = \int_a^b F^2 \left[ -\frac{1}{2} + \sum_{n=3}^\infty (-1)^{n-1} \frac{F^{n-2}}{n} \right]
\]
exists. Thus, since \(-\frac{1}{2} + \sum_{n=3}^\infty (-1)^{n-1} \frac{F^{n-2}}{n}\) is in \(OL^\circ\) on \([a, b]\) and is bounded away from zero on \([a, b]\), Lemma 3.2 implies that \(\int_a^b F^2\) exists. Therefore, \(\int_a^b G^2\) exists by Lemma 1.6.

**Theorem 3.** If \(\int_a^b (1 + G)\) exists and is not zero, \(\int_a^b G\) exists and \(c\) is a number, then \(cG \in OM^\circ\) and \(OA^\circ\) on \([a, b]\).

**Proof.** It follows from Lemma 3.4 that \(\int_a^b G^2\) exists. Therefore, Theorem 3 follows from Theorem 1.

**Theorem 4.** If \(\int_a^b (1 + G)\) exists and is not zero, \(\int_a^b G\) exists, \(H \in OL^\circ\) on \([a, b]\) and \(n \geq 2\) is an integer, then \(HG^n \in OM^\circ\) and \(OA^\circ\) on \([a, b]\).

**Proof.** It follows from Lemma 3.4 that \(\int_a^b G^2\) exists. Therefore, Theorem 4 follows from Theorem 2.

**Lemma 5.1.** If each of \(\int_a^b (1 + G)\) and \(\int_a^b (1 - G)\) exists and is not zero, then \(G\) is bounded on \([a, b]\) [7, Lemma 6.1].

**Lemma 5.2.** If each of \(\int_a^b (1 + G)\) and \(\int_a^b (1 - G)\) exists and is not zero, then \(G \in OL^{14}\) on \([a, b]\).

**Proof.** Let \(p \in (a, b]\). It follows from Lemmas 5.1 and 1.3 that \(G \in OQ^\circ\) and \(-G \in OQ^\circ\) on \([a, b]\). By applying this result and the Cauchy criterion for product integrals, we have that

\[
(1) \quad 0 = \lim_{x,y \to p^-} \left[ (1 + G(x, p)) - [1 + G(x, y)](1 + G(y, p)) \right]
= \lim_{x,y \to p^-} \left( (G(x, p) - G(x, y)) - G(y, p) - G(x, y)G(y, p) \right),
\]

\[
(2) \quad 0 = \lim_{x,y \to p^-} \left[ (1 - G(x, p)) - [1 - G(x, y)](1 - G(y, p)) \right]
= \lim_{x,y \to p^-} \left( -G(x, p) + G(x, y) + G(y, p) - G(x, y)G(y, p) \right).
\]

Thus,

\[
(3) \quad 0 = \lim_{x,y \to p^-} [-G(x, p) - G(x, y) - G(y, p)],
\]

\[
(4) \quad 0 = \lim_{x,y \to p^-} G(x, y)G(y, p).
\]

Note that limits (1), (3) and (4) are the same as limits (2), (1) and (3), respectively, in Theorem 9 of a previous paper of the author [6]. It follows that \(G(p^-, p)\) exists by the argument used in Theorem 9 [6]. Thus, it follows from (3) and the existence of \(G(p^-, p)\) that \(G(p^-, p^-)\) exists and is zero. Therefore, since right limits can be treated similarly, \(G \in OL^{14}\) on \([a, b]\).
Lemma 5.3. If each of $\int_a^b (1 + G)$ and $\int_a^b (1 - G)$ exists and is not zero, then $\int_a^b G^2$ exists.

Proof. Since each of $\int_a^b (1 + G)$ and $\int_a^b (1 - G)$ exists and is not zero, $\int_a^b (1 - G^2)$ exists and is not zero. Thus, since $G$ is bounded on $[a, b]$ [Lemma 5.1], $-G^2 \in OQ^0$ on $[a, b]$ [Lemma 1.3]. Since $G \in O\L_{14}$ on $[a, b]$ [Lemma 5.2], there exists a subdivision $\{x_q\}_{0}^{n}$ of $[a, b]$ such that if $1 \leq q \leq n$ and $x_{q-1} < x < y < x_{q}$, then $|G(x, y)| < 1$. Thus, since $-G^2 \in OQ^0$ on $[a, b]$, it follows that $G^2 \in OB^0$ on $[x_{q-1}, x_{q}]$ for $1 \leq q \leq n$. This is true because if $F$ is a function such that $0 \leq F \leq 1$ on an interval $[r, s]$ and $F \notin OB^0$ on $[r, s]$, then $- F \notin OQ^0$ on $[r, s]$. Therefore, $G^2 \in OB^0$ on $[a, b]$. Lemma 1.4 implies that $-G^2 \in OM^0$ on $[a, b]$. Hence, it follows from Lemma 2.1 that $\int_a^b G^2$ exists.

Lemma 5.4. If $\beta > 0$, $|G| < 1 - \beta$ on $[a, b]$, $\int_a^b G^2$ exists and $\int_a^b (1 + G)$ exists and is not zero, then $\int_a^b G$ exists [6, Theorem 5].

Theorem 5. If each of $\int_a^b (1 + G)$ and $\int_a^b (1 - G)$ exists and is not zero and $c$ is a number, then $cG \in OM^0$ and $OA^0$ on $[a, b]$.

Proof. Since $G \in O\L_{14}$ on $[a, b]$ [Lemma 5.2], there exists a subdivision $D = \{x_q\}_{0}^{n}$ of $[a, b]$ such that if $1 \leq q \leq n$ and $x_{q-1} < x < y < x_{q}$, then $|G(x, y)| < \frac{1}{q}$. Let $F$ be the function such that

(1) $F(x, y) = G(x, y)$ if $x \notin D$ and $y \notin D$, and

(2) $F(x, y) = 0$ if $x \in D$ or $y \in D$.

Since $\int_a^b G^2$ exists [Lemma 5.3], $\int_a^b F^2$ also exists [Lemma 1.6]. Further, since $G \in O\L_{14}$ on $[a, b]$ and $\int_a^b (1 + G)$ exists and is not zero, $\int_a^b (1 + F)$ exists and is not zero [Lemma 1.7]. Therefore, $\int_a^b F$ exists [Lemma 5.4], and hence, $\int_a^b G$ exists [Lemma 1.6]. Thus, since $\int_a^b G^2$ exists and $\int_a^b G$ exists, it follows from Theorem 1 that $cG \in OM^0$ and $OA^0$ on $[a, b]$.

Theorem 6. If each of $\int_a^b (1 + G)$ and $\int_a^b (1 - G)$ exists and is not zero, $H \in O\L^0$ on $[a, b]$ and $n \geq 2$ is an integer, then $HG^n \in OM^0$ and $OA^0$ on $[a, b]$.

Proof. It follows as in Theorem 5 that $\int_a^b G^2$ exists and $\int_a^b G$ exists. Hence, Theorem 2 implies that $HG^n \in OM^0$ and $OA^0$ on $[a, b]$.

BIBLIOGRAPHY


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