TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 183, September 1973

BICOHOMOLOGY THEORY

BY

DONOVAN H. VAN OSDOL[1]

ABSTRACT. Given a triple T and a cotriple G on a category $\mathfrak{D}$ such that T preserves group objects in $\mathfrak{D}$, let P and M be in $\mathfrak{D}$ with M an abelian group object. Applying the "hom functor" $\mathfrak{D}(-,-)$ to the (co)simplicial resolutions G^*P and T^*M yields a double complex $\mathfrak{D}(G^*P, T^*M)$. The nth homology group of this double complex is denoted $H^n(P, M)$, and this paper studies H^0 and H^1. When $\mathfrak{D}$ is the category of bialgebras arising from a triple, cotriple, and mixed distributive law, a complete description of H^0 and H^1 is given. The applications include a solution of the singular extension problem for sheaves of algebras.

I. Introduction. In his classic Tôhoku paper [17], Grothendieck solved the extension problem for sheaves of R-modules over a topological space. His technique consisted of deriving the hom functor using injective resolutions. In 1961 Gray initiated the study of the classification of extensions of sheaves in non-abelian categories [15]. His technique of solution consisted of taking an injective resolution of the second variable and a bar-like resolution of the first variable, and then taking homology of the double complex gotten by "homming" these resolutions together. Assuming the space to be paracompact Hausdorff and the first-variable sheaf to be coherent with projective stalks, he was able to locate the group of singular extensions somewhere in an exact sequence involving Ext's and Hom's.

Meanwhile, adjoint functors (and their logical equivalent, triples = standard constructions = monads = $\cdots$) were being actively studied ([19], [13]). Their applicability in the unification of homological algebra was realized by Barr and Beck, who in a series of papers ([1], [2], [3], [4], [5], [6], [9]) proved that nearly every known algebraic cohomology theory is realizable as a triple-theoretic cohomology theory. This paper may be viewed as an extension of their work, in order to cover the areas mentioned in the first paragraph.

When it was discovered (in [29]) that the stalk functor for sheaves on a space

Received by the editors February 14, 1972.

AMS (MOS) subject classifications (1970). Primary 14F99, 18C15, 55B99; Secondary 18F20, 18G25, 18H05.

Key words and phrases. Triple, principal homogeneous object, cotriplable, distributive law, double complex, homology, algebras, sheaves.

(1) This work was supported by NSF Grant No. GP-29067.

with values in an algebraic category is cotriplable, the dual of the results in Beck's thesis [9] made the Godement resolution [14] even more valuable than it had already been. One can also find in [29] a proof that algebra-valued sheaves are triplable over set-valued sheaves. (For generalizations of these two theorems, see [28] and [30].) It now becomes natural to replace Gray's bar-like and injective resolutions by cotriple and Godement resolutions. This was carried out in [29] for the following special case. Let X be a topological space, R a sheaf of commutative rings over X, P a sheaf of R-algebras, and M a sheaf of P-modules. Let G be the polynomial algebra cotriple lifted to the category of sheaves of R-algebras and T the Godement standard construction on the category of sheaves of P-modules. Then the first homology group of the double complex $\mathrm{Der}(G^{n+1}P, T^{m+1}M)$ (see §II for a description of the boundary operators) is in one-to-one correspondence with equivalence classes of singular extensions of P by M, where Der stands for the abelian group of global derivations. The techniques used in the proof of this theorem are completely analogous to those used in Beck's thesis [9].

It was subsequently proved [27] that the nth homology group of the double complex $\mathrm{Hom}(G^{i+1}P, T^{j+1}M)$ is isomorphic to $\mathrm{Ext}^n(P, M)$ for sheaves of abelian groups P and M over the space X. Here G is the free abelian group cotriple lifted to sheaves of abelian groups and T is the Godement standard construction. These last two theorems indicated that there is a very general classification theorem. There is, and it constitutes the content of this paper.

Although the proof is technical, long, and computational, the idea is relatively simple. A sheaf of algebras can (by virtue of (co)triplableness) be thought of as a sheaf of sets, each stalk of which is an algebra, such that the structure making it a sheaf of sets and the structure making its stalks algebras are compatible with each other. Now given sheaves P and M of algebras and a one-cocycle (a, b) in the double complex having boundary operators d and ∂, we can build a sheaf of algebras which acts like an extension of P by M (actually, like an M-principal homogeneous space over P [24]). We use the product sheaf of algebras $P \times M$ as a model, but "twist" its sheaf-of-sets and stalks-of-algebras structures using the cocycle. The fact that $\partial a = 0$ allows us to define a new algebra structure on the stalks $P_x \times M_x$, the fact that $db = 0$ allows us to define a new sheaf-of-sets structure on $P_x \times M_x$, and the fact that $da = \partial b$ allows us to show that these new structures are compatible with each other. Hence we have a new sheaf of algebras. On the other hand, any extension (or principal object) gives rise to a one-cocycle in the double complex in a natural way, and the resulting assignments are inverse to each other. The theorem is stated in a more general context, but the reader should keep the above discussion in mind as he reads the proof.

We call the group $H^n(P, M)$ which arises from our "standard" double complex the nth *bicohomology group* of P with coefficients in M. The proof that the first

bicohomology group $H^1(P, M)$ classifies M-principal objects over P consists of a generalization of the techniques used in [9].

§V of this paper is very technical and probably mysterious, so perhaps a few extra words about it are in order. There are at least two kinds of situations in which, for the purposes of cohomology, one does not wish to work in the category given. For example in the cohomology of groups one usually wants to take into account the effects of a given group operation on another group, and this leads to the concept of a module over a group. One defines the cohomology in the resulting category of modules rather than in the original category of groups. Another example is found in the cohomology of associative rings with identity; in this category the only abelian group object is the zero ring, and so again one passes to the category of modules over a given ring for his cohomology theory. §V is concerned with this process in general, and shows how the results in the earlier sections can be used in this *a priori* more general situation.

The discovery of the bicohomology groups leads to several questions, which will be dealt with in future papers. Since Hopf algebras (= bialgebras [26]) are bialgebraic over modules (in the sense of §IV), bicohomology or some suitable modification should apply to them. The immediate difficulty is that the appropriate underlying functors do not preserve abelian groups. Another problem is to classify the second bicohomology group, analogously to [2]. J. Duskin has some unpublished results which may also apply to this problem. An analysis of the bicohomology groups using spectral sequences should be carried out in order to facilitate computation. Also, one should try to find a set of axioms which characterize the theory as a functor of the first and/or second variable, as in [6].

Back to this paper, we will follow certain notational conventions. We will identify objects in a category with their identity morphisms, so that a symbol X will denote either itself or the identity morphism on X. Given morphisms $f: X \to Y$, $g: Y \to Z$ in a category, $g \cdot f$ will denote their composition. We will delete all parentheses which are not absolutely essential. Given objects X, Y in a category $\mathfrak{A}$, $\mathfrak{A}(X, Y)$ will denote the set of morphisms in $\mathfrak{A}$ with domain X and codomain Y. If $\mathfrak{A}$ is a small category and $\mathfrak{B}$ is any category, $\mathfrak{B}^{\mathfrak{A}}$ will denote the category of covariant functors $\mathfrak{A} \to \mathfrak{B}$, with natural transformations as morphisms. Given objects X, Y in a category $\mathfrak{A}$, their product in $\mathfrak{A}$ is $X \times Y$ with projections $p_1: X \times Y \to X$, $p_2: X \times Y \to Y$. Given an infinite set of objects $\{X_i\}$ in a category $\mathfrak{A}$, we denote their product in $\mathfrak{A}$ by ΠX_i with projections $p_j: \Pi X_i \to X_j$. Following a suggestion of Grillet [16], given $f: Z \to X$, $g: Z \to Y$ in the category $\mathfrak{A}$ the symbol $f\Pi g: Z \to X \times Y$ represents the morphism with projections f and g, whereas given $f: W \to X$, $g: Z \to Y$ the symbol $f \times g: W \times Z \to X \times Y$ represents the morphism with projections $f \cdot p_1$ and $g \cdot p_2$. All other notation is standard, such as in [22] or [23].

II. Basic definitions. Let $\mathfrak{D}$ be a category with finite products. Let $T' = (T', \eta', \mu')$ be a triple on $\mathfrak{D}$ which preserves finite products, and let $G' = (G', \epsilon', \delta')$ be a cotriple on $\mathfrak{D}$ [13]. Suppose we are given a (fixed) abelian group [11] $M = (M, m, u, z)$ in $\mathfrak{D}$:

$$\begin{tikzcd} M \times M \times M \arrow[r, "M \times m"] \arrow[d, "m \times M"'] & M \times M \arrow[d, "m"] \\ M \times M \arrow[r, "m"] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "z \Pi M"] \arrow[dr, "="'] & M \times M \arrow[d, "m"] \\ & M \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\Delta"'] & 1 \arrow[d, "z"] \\ M \times M \arrow[r, "u \times M"] & M \times M \arrow[r, "m"] & M \end{tikzcd} \qquad \begin{tikzcd} M \times M \arrow[r, "\text{twist}"] \arrow[dr, "m"'] & M \times M \arrow[d, "m"] \\ & M \end{tikzcd}$$

For convenience we will use the more intuitive notation $+, -, 0$; e.g. given a, b: $X \to M$ we write $a + b = X \xrightarrow{a \Pi b} M \times M \xrightarrow{m} M$. Since T' preserves products, $T'^n M = (T'^n M, T'^n m, T'^n u, T'^n z)$ is an abelian group in $\mathfrak{D}$ for each $n \geq 0$ [11]. We use the same $+, -, 0$ notation for these abelian groups.

Given any object P in $\mathfrak{D}$ we can now form a complex of complexes of abelian groups $C^{**}(P, M)$, as follows. Let $C^{m,n}(P, M) = \mathfrak{D}(G'^{m+1}P, T'^{n+1}M)$. Define $\eta'_i: T'^n M \to T'^{n+1}M$ by $\eta'_i = T'^i \eta' T'^{n-i} M$ and $\epsilon'_i: G'^{n+1}P \to G'^n P$ by $\epsilon'_i = G'^i \epsilon' G'^{n-i} P$. Then $\partial: \mathfrak{D}(G'^{m+1}P, T'^{n+1}M) \to \mathfrak{D}(G'^{m+1}P, T'^{n+2}M)$ and d: $\mathfrak{D}(G'^{m+1}P, T'^{n+1}M) \to \mathfrak{D}(G'^{m+2}P, T'^{n+1}M)$ are defined by $\partial(f) = \Sigma_{i=0}^{n+1}(-1)^i \eta'_i \cdot f$ and $d(f) = \Sigma_{i=0}^{m+1}(-1)^i f \cdot \epsilon'_i$. One shows rather easily that $d \cdot d = 0$, $\partial \cdot \partial = 0$, and $d \cdot \partial = \partial \cdot d$. We can then consider the associated double complex [20], the homology of whose associated total complex we will denote by $H^*(P, M)$ and call the *bicohomology groups* of P with coefficients in M.

In this paper we will be concerned only with $H^0(P, M)$ and $H^1(P, M)$, so we give a direct description of these. The group $H^0(P, M)$ is the intersection of $\mathrm{Ker}(\partial: \mathfrak{D}(G'P, T'M) \to \mathfrak{D}(G'P, T'^2M))$ and $\mathrm{Ker}(d: \mathfrak{D}(G'P, T'M) \to \mathfrak{D}(G'^2P, T'M))$. Explicitly, $H^0(P, M) = \{a: G'P \to T'M \mid a \cdot \epsilon' G'P - a \cdot G'\epsilon' P = 0$ and $T'\eta' M \cdot a - \eta' T'M \cdot a = 0\}$. To describe $H^1(P, M)$ we need some definitions. A *homogeneous one-cocycle* is an ordered pair (a, b) where $a: G'^2P \to T'M$, b: $G'P \to T'^2M$, and

(i) $a \cdot \epsilon' G'^2 P - a \cdot G'\epsilon' G' P + a \cdot G'^2 \epsilon' P = 0$,

(ii) $T'^2 \eta' M \cdot b - T'\eta' T' M \cdot b + \eta' T'^2 M \cdot b = 0$,

(iii) $b \cdot \epsilon' G' P - b \cdot G'\epsilon' P + \eta' T' M \cdot a - T'\eta' M \cdot a = 0$.

Let $Z^1(P, M)$ be the abelian group of all homogeneous one-cocycles. A *homogeneous one-coboundary* is any ordered pair $(c \cdot \epsilon' G' P - c \cdot G'\epsilon' P, T'\eta' M \cdot c - \eta' T' M \cdot c)$ where $c: G'P \to T'M$. Let $B^1(P, M)$ be the abelian group of all homogeneous one-coboundaries. Then $H^1(P, M) = Z^1(P, M)/B^1(P, M)$.

The interpretation of $H^1(P, M)$ will involve principal objects [24], and we now define these. An *M-object* in $\mathfrak{D}$ is a pair $(P', \circ: M \times P' \to P')$ in $\mathfrak{D}$ such that

$$\begin{tikzcd} M \times M \times P' \arrow[r, "M \times \circ"] \arrow[d, "m \times P'"'] & M \times P' \arrow[d, "\circ"] \\ M \times P' \arrow[r, "\circ"] & P' \end{tikzcd} \qquad \begin{tikzcd} P' \arrow[r, "z \Pi P'"] \arrow[dr, "="'] & M \times P' \arrow[d, "\circ"] \\ & P' \end{tikzcd}$$

both commute. Given maps $f: X \to M$, $g: X \to P'$ we write $f \circ g = X \xrightarrow{f \Pi g} M \times P' \xrightarrow{\circ} P'$. In more familiar notation, then, the above diagrams say $f \circ (g \circ h) = (f + g) \circ h$ and $0 \circ f = f$. Note that for $n \geq 0$, $(T'^n P', \circ = T'^n \circ)$ is a $T'^n M$-object. An *M-principal object over P* (relative to T' and G') consists of the following data:

(i) An M-object $(P', \circ)$.

(ii) A map $p: P' \to P$ in $\mathfrak{D}$.

(iii) For each $f: X \to M$, $g: X \to P'$ in $\mathfrak{D}$, $p \cdot (f \circ g) = p \cdot g$.

(iv) For each $n \geq 0$ and each $f, g: X \to T'^n P'$ in $\mathfrak{D}$ such that $T'^n p \cdot f = T'^n p \cdot g$ there exists a unique $\bar{f}: X \to T'^n M$ such that $g = \bar{f} \circ f$ (where $\circ = T'^n \circ$).

(v) There exists a fixed $s: G'P \to T'P'$ in $\mathfrak{D}$ such that $T'p \cdot s = \eta' P \cdot \epsilon' P$.

A *morphism* $h: (P', \circ, p, s) \to (P'', \circ, q, t)$ of M-principal objects over P is a morphism $h: P' \to P''$ in $\mathfrak{D}$ such that $q \cdot h = p$ and $h \cdot (f \circ g) = f \circ h \cdot g$ for maps $f: X \to M$, $g: X \to P'$. With the evident composition, we thus get a category $\mathcal{PO}(P, M)$ of M-principal objects over P and their morphisms.

The germ of the idea for such a definition of principal object comes from Serre [24]. Conditions (iii) and (iv) simply say that $(T'^n \circ, T'^n p_2)$ is the kernel pair of $T^n p$ for all $n \geq 0$. Condition (v) is a kind of local triviality condition; in the cases in which we are interested, it will mean that p is a split epimorphism in an underlying category. In those cases (see §IV) an M-principal object over P will be simply a map $p: P' \to P$ such that

$$T'^n M \times T'^n P' \rightrightarrows T'^n P' \longrightarrow T'^n P$$

is exact in the sense of Barr [2a] and $T'^n p$ is split in the underlying category. Alternatively, in those cases one should think of an M-principal object over P as a principal M-bundle whose image in the underlying category is a trivial bundle. For further interpretation, one should consult the examples in §VI.

III. The interpretation morphisms. Define $\theta: \mathfrak{D}(P, M) \to H^0(P, M)$ by setting $\theta(f) = \eta' M \cdot f \cdot \epsilon' P$. That the image of θ really lies in $H^0(P, M)$ is easily seen. For example,

$$T'\eta' M \cdot \theta(f) - \eta' T' M \cdot \theta(f) = T'\eta' M \cdot \eta' M \cdot f \cdot \epsilon' P - \eta' T' M \cdot \eta' M \cdot f \cdot \epsilon' P$$

$$= \eta' T' M \cdot \eta' M \cdot f \cdot \epsilon' P - \eta' T' M \cdot \eta' M \cdot f \cdot \epsilon' P = 0.$$

It is equally obvious that θ is a homomorphism of abelian groups. We thus have the following result.

Proposition 1. *There exists a homomorphism* $\theta: \mathfrak{D}(P, M) \to H^0(P, M)$ *of abelian groups, defined by* $\theta(f) = \eta' M \cdot f \cdot \epsilon' P$.

We now turn to the more complicated task of interpreting $H^1(P, M)$, and we do this in terms of M-principal objects over P. Let $(P', \circ, p, s)$ be an M-principal object over P, and consider the two morphisms $s \cdot \epsilon' G' P, s \cdot G'\epsilon' P: G'^2 P \to T' P'$. We have

$$T' p \cdot s \cdot \epsilon' G' P = \eta' P \cdot \epsilon' P \cdot \epsilon' G' P = \eta' P \cdot \epsilon' P \cdot G'\epsilon' P = T' p \cdot s \cdot G'\epsilon' P,$$

so by property (iv) of principal objects there exists a unique $a: G'^2 P \to T' M$ such that $a \circ s \cdot \epsilon' G' P = s \cdot G'\epsilon' P$. On the other hand,

$$T'^2 p \cdot \eta' T' P' \cdot s = \eta' T' P \cdot T' p \cdot s = \eta' T' P \cdot \eta' P \cdot \epsilon' P$$

$$= T'\eta' P \cdot \eta' P \cdot \epsilon' P = T'\eta' P \cdot T' p \cdot s = T'^2 p \cdot T'\eta' P' \cdot s$$

so that again by property (iv), there exists a unique $b: G' P \to T'^2 M$ such that $b \circ T'\eta' P' \cdot s = \eta' T' P' \cdot s$.

Proposition 2. *There exists a function* $\Lambda: |\mathcal{PO}(P, M)| \to Z^1(P, M)$ *given by* $\Lambda(P', \circ, p, s) = (a, b)$.

Proof. We must verify that (a, b) satisfies the three conditions which make it a homogeneous one-cocycle. To prove that $a \cdot \epsilon' G'^2 P + a \cdot G'^2 \epsilon' P = a \cdot G'\epsilon' G' P$, we let both maps operate on $s \cdot \epsilon' G' P \cdot \epsilon' G'^2 P$:

$$(a \cdot G'^2 \epsilon' P + a \cdot \epsilon' G'^2 P) \circ s \cdot \epsilon' G' P \cdot \epsilon' G'^2 P$$

$$= a \cdot G'^2 \epsilon' P \circ (a \cdot \epsilon' G'^2 P \circ s \cdot \epsilon' G' P \cdot \epsilon' G'^2 P)$$

$$= a \cdot G'^2 \epsilon' P \circ [(a \circ s \cdot \epsilon' G' P) \cdot \epsilon' G'^2 P] = a \cdot G'^2 \epsilon' P \circ s \cdot G'\epsilon' P \cdot \epsilon' G'^2 P$$

$$= a \cdot G'^2 \epsilon' P \circ s \cdot \epsilon' G' P \cdot G'^2 \epsilon' P = (a \circ s \cdot \epsilon' G' P) \cdot G'^2 \epsilon' P = s \cdot G'\epsilon' P \cdot G'^2 \epsilon' P,$$

whereas

$$a \cdot G'\epsilon'G'P \circ s \cdot \epsilon'G'P \cdot \epsilon'G'^2P = a \cdot G'\epsilon'G'P \circ s \cdot \epsilon'G'P \cdot G'\epsilon'G'P$$

$$= (a \circ s \cdot \epsilon'G'P) \cdot G'\epsilon'G'P = s \cdot G'\epsilon'P \cdot G'\epsilon'G'P = s \cdot G'\epsilon'P \cdot G'^2\epsilon'P.$$

Thus by property (iv) of principal objects, $a \cdot G'^2\epsilon'P + a \cdot \epsilon'G'^2P = a \cdot G'\epsilon'G'P$. A similar computation shows that $\eta'T'^2M \cdot b + T'^2\eta'M \cdot b = T'\eta'T'M \cdot b$. Finally, $b \cdot G'\epsilon'P + T'\eta'M \cdot a = \eta'T'M \cdot a + b \cdot \epsilon'G'P$ because $(b \cdot G'\epsilon'P + T'\eta'M \cdot a) \circ T'\eta'P' \cdot s \cdot \epsilon'G'P = (\eta'T'M \cdot a + b \cdot \epsilon'G'P) \circ T'\eta'P' \cdot s \cdot \epsilon'G'P$. This completes the proof.

Proposition 3. *If there exists* $h: (P', \circ, p, s) \to (P'', \circ, q, t)$ *in* $\mathcal{PO}(P, M)$ *then* $\Lambda(P', \circ, p, s) - \Lambda(P'', \circ, q, t)$ *is in* $B^1(P, M)$.

Proof. Consider the two maps $T'h \cdot s, t: G'P \to T'P''$. We have $T'q \cdot T'h \cdot s = T'p \cdot s = \eta'P \cdot \epsilon'P = T'q \cdot t$, so by property (iv) of principal objects there exists a unique $c: G'P \to T'M$ such that $c \circ T'h \cdot s = t$. Let $\Lambda(P', \circ, p, s) = (a, b)$ and $\Lambda(P'', \circ, q, t) = (a', b')$. Then $c \cdot \epsilon'G'P' - c \cdot G'\epsilon'P = a - a'$ and $T'\eta'M \cdot c - \eta'T'M \cdot c = b - b'$ by computations similar to those given in Proposition 2.

Letting $\pi_0(\mathcal{PO}(P, M))$ be the set of connected components of $\mathcal{PO}(P, M)$, we obtain the following corollary to Propositions 2 and 3.

Theorem 4. *There is a function* $\omega: \pi_0(\mathcal{PO}(P, M)) \to H^1(P, M)$ *given by* $\omega(\text{class}(P', \circ, p, s)) = \text{class}(\Lambda(P', \circ, p, s))$.

IV. When the interpretation is an isomorphism. We are of course most interested in θ and ω when they are isomorphisms. It is easy to give sufficient conditions for θ to be an isomorphism.

Proposition 5. *If* $\eta'M$ *is the equalizer of* $(\eta'T'M, T'\eta'M)$ *and* $\epsilon'P$ *is the coequalizer of* $(\epsilon'G'P, G'\epsilon'P)$, *then* $\theta: \mathfrak{D}(P, M) \to H^0(P, M)$ *is an isomorphism.*

Proof. If $\eta'M$ is a monomorphism and $\epsilon'P$ is an epimorphism then θ is one-to-one. For suppose $\theta(f) = \theta(g)$. Then $\eta'M \cdot f \cdot \epsilon'P = \eta'M \cdot g \cdot \epsilon'P$, so that $f \cdot \epsilon'P = g \cdot \epsilon'P$ and $f = g$. Now assume the full hypotheses of the proposition and let $h: G'P \to T'M$ be in $H^0(P, M)$. Then $h \cdot \epsilon'G'P = h \cdot G'\epsilon'P$, so there exists a unique $g: P \to T'M$ such that $g \cdot \epsilon'P = h$. Since $\epsilon'P$ is an epimorphism, $\eta'T'M \cdot g \cdot \epsilon'P = \eta'T'M \cdot h = T'\eta'M \cdot h = T'\eta'M \cdot g \cdot \epsilon'P$ implies $\eta'T'M \cdot g = T'\eta'M \cdot g$. Hence there exists a unique $f: P \to M$ such that $\eta'M \cdot f = g$, and then $\theta(f) = \eta'M \cdot f \cdot \epsilon'P = g \cdot \epsilon'P = h$. Thus θ is onto, and the proof is complete.

To make ω an isomorphism is a more complicated task. We assume that we are given a base category $\mathfrak{A}$ equipped with a triple $T = (T, \eta, \mu)$ and a cotriple $G = (G, \epsilon, \delta)$ [13] which are linked via a *mixed distributive law* $\lambda: TG \to GT$ [8]. This means that λ is a natural transformation satisfying

(i) $G\lambda \cdot \lambda G \cdot T\delta = \delta T \cdot \lambda$,

(ii) $G\mu \cdot \lambda T \cdot T\lambda = \lambda \cdot \mu G$,

(iii) $\lambda \cdot \eta G = G\eta$,

(iv) $\epsilon T \cdot \lambda = T\epsilon$.

We can then form the category $\mathfrak{A}_G^T$ of *bialgebras* whose objects are all (A, σ_1, σ_2) with A in $|\mathfrak{A}|$ and $\sigma_1 \colon TA \to A$, $\sigma_2 \colon A \to GA$ in $\mathfrak{A}$ satisfying

(i) $\sigma_1 \cdot T\sigma_1 = \sigma_1 \cdot \mu A$,

(ii) $\sigma_1 \cdot \eta A = A$,

(iii) $G\sigma_2 \cdot \sigma_2 = \delta A \cdot \sigma_2$,

(iv) $\epsilon A \cdot \sigma_2 = A$,

(v) $G\sigma_1 \cdot \lambda A \cdot T\sigma_2 = \sigma_2 \cdot \sigma_1$,

and whose morphisms $f \colon (A, \sigma_1, \sigma_2) \to (A', \tau_1, \tau_2)$ are all $f \colon A \to A'$ in $\mathfrak{A}$ such that $\tau_1 \cdot Tf = f \cdot \sigma_1$, $Gf \cdot \sigma_2 = \tau_2 \cdot f$. We essentially want $\mathfrak{D}$ to "be" $\mathfrak{A}_G^T$, but we must clarify what this means. The idea originates in [9].

Assume that we have a diagram

$$\begin{tikzcd}
\mathfrak{D} \arrow[r, shift left, "S'"] & \mathfrak{B} \arrow[l, shift left, "Q'"] \\
\mathcal{C} \arrow[u, shift left, "U'"] \arrow[r, shift left, "S"] & \mathfrak{A} \arrow[u, shift left, "U"] \arrow[l, shift left, "Q"]
\end{tikzcd}$$

with T arising from the adjoint pair $F \dashv U$, G arising from the adjoint pair $S \dashv Q$, T' arising from the adjoint pair $S' \dashv Q'$, and G' arising from the adjoint pair $F' \dashv U'$. Suppose that U, U' are triplable and S, S' are cotriplable, and that there exists a natural isomorphism $\rho \colon SU' \to US'$. Then $\mathfrak{D}$ has a chance of "being" $\mathfrak{A}_G^T$.

To make it so, we first recall that we have (adjoint) equivalences $\Psi \colon \mathcal{C} \to \mathfrak{A}_G$, $\Phi \colon \mathfrak{B} \to \mathfrak{A}^T$, $\Psi' \colon \mathfrak{D} \to \mathfrak{B}_{G''}$, and $\Phi' \colon \mathfrak{D} \to \mathcal{C}^{T''}$ where G'' arises from $S' \dashv Q'$ and T'' arises from $F' \dashv U'$. We want to lift Ψ to Θ and Φ to Γ, i.e. we want commutative squares:

$$\begin{tikzcd}
\mathfrak{A}_G^T \arrow[d, "U_G^T"'] & \mathcal{C}^{T''} \arrow[l, "\Theta"'] \arrow[d, "U^{T''}"] \\
\mathfrak{A}_G & \mathcal{C} \arrow[l, "\Psi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{B}_{G''} \arrow[r, "\Gamma"] \arrow[d, "S_{G''}"'] & \mathfrak{A}_G^T \arrow[d, "S_G^T"] \\
\mathfrak{B} \arrow[r, "\Phi"] & \mathfrak{A}^T.
\end{tikzcd}$$

In addition, we want Θ and Γ to be (adjoint) equivalences. For this it suffices to assume the existence of natural equivalences $\alpha\colon U_G^T F_G^T \Psi \to \Psi U' F'$, $\beta\colon \Phi S' Q' \to S_G^T Q_G^T \Phi$ such that α is a relative triple map and β is a relative cotriple map. We interpret what this means in the case of α, allowing the reader to write down the (dual) conditions for β. For each object C of $|\mathcal{C}|$, $\alpha C\colon TSC \to ST'C$ should be an isomorphism such that $\alpha C \cdot \eta SC = S\eta' C$, $S\mu' C \cdot \alpha T' C \cdot T\alpha C = \alpha C \cdot \mu SC$, and $G\alpha C \cdot \lambda SC \cdot TS\eta C = S\eta T' C \cdot \alpha C$. In the presence of such α and β, define $\Theta(C, \gamma) = (SC, S\gamma \cdot \alpha C, S\eta C)$ and $\Gamma(B, \tau) = (UB, U\epsilon B, \beta B \cdot U\tau)$. The verification that Θ and Γ are adjoint equivalences can safely be left to the reader. Finally, we require that

$$\begin{tikzcd}
\mathfrak{D} \arrow[r, "\Phi'"] \arrow[d, "\Psi'"'] & \mathcal{C}^{T''} \arrow[d, "\Theta"] \\
\mathfrak{B}_{G''} \arrow[r, "\Gamma"] & \mathfrak{A}_G^T
\end{tikzcd}$$

commute up to (unique) isomorphism. This imposes two conditions on α, β, ρ, namely,

$$\begin{tikzcd}
UFSU' \arrow[r, "\alpha U'"] \arrow[d, "UF\rho"'] & SU'F'U' \arrow[r, "SU'\epsilon'"] & SU' \arrow[rr, "S\eta U'"] \arrow[d, "\rho"] & & SQSU' \arrow[d, "SQ\rho"] \\
UFUS' \arrow[rr, "U\epsilon S'"] & & US' \arrow[r, "US'\eta'"] & US'Q'S' \arrow[r, "\beta S'"] & SQUS'
\end{tikzcd}$$

should both commute.

The effect of the foregoing discussion is to replace $\mathfrak{A}$, $\mathfrak{B}$, $\mathcal{C}$, $\mathfrak{D}$, etc. by the following:

$$\begin{tikzcd}
\mathfrak{A}_G^T \arrow[r, shift left, "S_G^T"] \arrow[d, shift left, "U_G^T"] & \mathfrak{A}^T \arrow[l, shift left, "Q_G^T"] \arrow[d, shift left, "U^T"] \\
\mathfrak{A}_G \arrow[r, shift left, "S_G"] \arrow[u, shift left, "F_G^T"] & \mathfrak{A} \arrow[l, shift left, "Q_G"] \arrow[u, shift left, "F^T"]
\end{tikzcd}$$

where, for example, $Q_G^T(A, \sigma) = (GA, G\sigma \cdot \lambda A, \delta A)$. For purposes of cohomology, we still need to know a little more. It is necessary that $\Gamma\Psi' = \Theta\Phi'$ preserves abelian groups and that $C^{m,n}(P, M) \to \mathfrak{A}_G^T((F_G^T U_G^T)^{m+1}\Gamma\Psi' P, (Q_G^T S_G^T)^{n+1}\Gamma\Psi' M)$ induced by $\Gamma\Psi' = \Theta\Phi'$ is an isomorphism of complexes of complexes of abelian groups. A necessary and sufficient condition for this is that S commute with finite products. One can either verify this directly, or combine results in [9] and [28] together with the observation: S' preserves finite products because SU' does, and U creates (inverse) limits and is faithful [21].

For the convenience of the reader, we now list our assumptions and notational conventions.

(1) There is a diagram

$$\begin{tikzcd}
\mathfrak{D} \arrow[r, shift right, "Q'"'] \arrow[d, shift right, "U'"] & \mathfrak{B} \arrow[l, shift right, "S'"'] \arrow[d, shift right, "U"] \\
\mathcal{C} \arrow[r, shift right, "Q"'] \arrow[u, shift right, "F'"] & \mathfrak{A} \arrow[l, shift right, "S"'] \arrow[u, shift right, "F"]
\end{tikzcd}$$

in which $\mathfrak{A}$ and $\mathcal{C}$ have finite products, $S \dashv Q$, $F \dashv U$, $F' \dashv U'$, $S' \dashv Q'$, and S commutes with finite products (up to unique isomorphism).

(2) $T = UF$, $G = SQ$, $T' = Q'S'$, $G' = F'U'$, $G'' = S'Q'$, $T'' = U'F'$, S and S' are cotriplable, U and U' are triplable.

(3) $\lambda: TG \to GT$ is a mixed distributive law.

(4) $\rho: SU' \to US'$, $\alpha: TS \to ST'$, $\beta: US'Q' \to SQU$ are natural equivalences, with α a triple map relative to $\Psi: \mathcal{C} \to \mathfrak{A}_G$ and β a cotriple map relative to $\Phi: \mathfrak{B} \to \mathfrak{A}^T$.

(5) α and ρ are compatible with the counits ϵ, ϵ'; and β, ρ are compatible with the units η, η'.

The data (1) through (5) allow the replacement of the diagram in (1) by the diagram:

$$\begin{tikzcd}
\mathfrak{A}_G^T \arrow[r, shift left, "S_G^T"] \arrow[d, shift left, "U_G^T"] & \mathfrak{A}^T \arrow[l, shift left, "Q_G^T"] \arrow[d, shift left, "U^T"] \\
\mathfrak{A}_G \arrow[r, shift left, "S_G"] \arrow[u, shift left, "F_G^T"] & \mathfrak{A} \arrow[l, shift left, "Q_G"] \arrow[u, shift left, "F^T"]
\end{tikzcd}$$

Under these conditions, we say that $\mathfrak{D}$ is *bialgebraic* over $\mathfrak{A}$.

We now need a "dictionary" to allow us to translate from the homogeneous situation of §II to the nonhomogeneous situation in which we now find ourselves. The translator is $\Gamma\Psi' = \Theta\Phi' : \mathfrak{D} \to \mathfrak{A}_G^T$, and for P in $|\mathfrak{D}|$ we will write $\Gamma\Psi'(P) = (P, \pi_1, \pi_2)$ where $\pi_1 : TP \to P$ and $\pi_2 : P \to GP$ in $\mathfrak{A}$, etc. This is not strictly true, since $\Gamma\Psi'(P) = (US'P, -, -)$, but we feel that notation has proliferated to a sufficient extent already in this paper.

An abelian group $M = (M, \gamma_1, \gamma_2)$ in $\mathfrak{D} = \mathfrak{A}_G^T$ is a bialgebra (M, γ_1, γ_2) together with bialgebra morphisms $+ : (M, \gamma_1, \gamma_2) \times (M, \gamma_1, \gamma_2) \to (M, \gamma_1, \gamma_2)$, $0 : 1 \to (M, \gamma_1, \gamma_2)$, and $- : (M, \gamma_1, \gamma_2) \to (M, \gamma_1, \gamma_2)$ subject to the usual conditions (see §II). Among other things, this means that the diagrams

$$\begin{tikzcd}
T(M \times M) \arrow[r, "T+"] \arrow[d, "Tp_1 \Pi Tp_2"'] & TM \arrow[dd, "\gamma_1"] \\
TM \times TM \arrow[d, "\gamma_1 \cdot p_1 \times \gamma_1 \cdot p_2"'] & \\
M \times M \arrow[r, "+"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times M \arrow[r, "+"] \arrow[d, "\gamma_2 \cdot p_1 \times \gamma_2 \cdot p_2"'] & M \arrow[dd, "\gamma_2"] \\
GM \times GM \arrow[d] & \\
G(M \times M) \arrow[r, "G+"] & GM
\end{tikzcd}$$

both commute, where $GM \times GM \to G(M \times M)$ is the inverse to the map $Gp_1 \Pi Gp_2$ (which is an isomorphism because of our assumptions). Again $T'^n M = (Q_G^T S_G^T)^n (M, \gamma_1, \gamma_2)$ is an abelian group for each $n \geq 0$ because $Q_G^T S_G^T$ preserves finite products.

Given an object $P = (P, \pi_1, \pi_2)$ in $\mathfrak{D} = \mathfrak{A}_G^T$ the complex of complexes $C^{m,n}(P, M)$ translates into the complex of complexes $D^{m,n}(P, M) = \mathfrak{A}(T^m P, G^n M)$ with boundaries $\partial : \mathfrak{A}(T^m P, G^n M) \to \mathfrak{A}(T^m P, G^{n+1} M)$ and $d : \mathfrak{A}(T^m P, G^n M) \to \mathfrak{A}(T^{m+1} P, G^n M)$ defined as follows. Let $\lambda_i : G^i T G^{n-i} M \to G^{i+1} T G^{n-i-1} M$ denote $G^i \lambda G^{n-i-1}$, and $\lambda^j : T^{j+1} G T^{m-j-1} P \to T^j G T^{m-j} P$ denote $T^j \lambda T^{m-j-1} P$. Define $\partial^i : D^{m,n}(P, M) \to D^{m,n+1}(P, M)$ by

$$\partial^i(f) = \begin{cases} Gf \cdot \lambda^0 \cdot \lambda^1 \ \cdots \ \lambda^{m-1} \cdot T^m \pi_2 & \text{if } i = 0, \\ G^{i-1} \delta G^{n-i} M \cdot f & \text{if } 1 \leq i \leq n, \\ G^n \gamma_2 \cdot f & \text{if } i = n+1, \end{cases}$$

and then $\partial = \Sigma(-1)^i \partial^i$. Define $d^j : D^{m,n}(P, M) \to D^{m+1,n}(P, M)$ by

$$d^j(f) = \begin{cases} G^n \gamma_1 \cdot \lambda_{n-1} \cdot \lambda_{n-2} \cdots \lambda_0 \cdot Tf & \text{if } j = 0, \\ f \cdot T^{j-1} \mu T^{m-j} P & \text{if } 1 \leq j \leq m, \\ f \cdot T^m \pi_1 & \text{if } j = m+1, \end{cases}$$

and then $d = \Sigma(-1)^j d^j$. As an illustration of the proof that this is the correct translation, we consider ∂^{n+1}. Following f down and across the bottom in

$$\begin{tikzcd}
\mathfrak{A}(T^m P, G^n M) \arrow[r, "\sim"] \arrow[d, "\partial^{n+1}"'] & \mathfrak{A}_G^T((F_G^T U_G^T)^{m+1}(P, \pi_1, \pi_2), (Q_G^T S_G^T)^{n+1}(M, \gamma_1, \gamma_2)) \arrow[d, "\eta^*_{n+1}"] \\
\mathfrak{A}(T^m P, G^{n+1} M) \arrow[r, "\sim"] & \mathfrak{A}_G^T((F_G^T U_G^T)^{m+1}(P, \pi_1, \pi_2), (Q_G^T S_G^T)^{n+2}(M, \gamma_1, \gamma_2))
\end{tikzcd}$$

yields

$$f \mapsto G^n\gamma_2 \cdot f \mapsto G(G^{n+1}\gamma_1 \cdot \lambda_n \cdots \lambda_0) \cdot \lambda G^{n+1} M \cdot TG(G^n\gamma_2 \cdot f) \cdot T(\lambda^0 \cdots \lambda^{m-1} \cdot T^m \pi_2)$$

which is $G^{n+2}\gamma_1 \cdot \lambda_{n+1} \cdots \lambda_0 \cdot TG^{n+1}\gamma_2 \cdot TGf \cdot \lambda^1 \cdots \lambda^m \cdot T^{m+1}\pi_2$, whereas following f across the top and down yields $f \mapsto G(G^n\gamma_1 \cdot \lambda_{n-1} \cdots \lambda_0) \cdot \lambda G^n M \cdot TGf \cdot T(\lambda^0 \cdots \lambda^{m-1} \cdot T^m \pi_2) \mapsto G^{n+1}\gamma_2 \cdot G^{n+1}\gamma_1 \cdot \lambda_n \cdots \lambda_0 \cdot TGf \cdot \lambda^1 \cdots \lambda^m \cdot T^{m+1}\pi_2$. The end result is the same because $G^{n+1}\gamma_2 \cdot G^{n+1}\pi_1 \cdot \lambda_n \cdots \lambda_0 = G^{n+2}\gamma_1 \cdot G^{n+1}\lambda M \cdot G^{n+1}T\gamma_2 \cdot \lambda_n \cdots \lambda_0 = G^{n+2}\gamma_1 \cdot G^{n+1}\lambda M \cdot \lambda_n \cdots \lambda_0 \cdot T^{n+1}G\gamma_2$. The formulas for the boundary maps in $D^{*,*}(P, M)$ are the reasons for calling this the "nonhomogeneous" situation [9].

Since our hypotheses on $\mathfrak{D} = \mathfrak{A}_G^T$ imply those of Proposition 5, we turn immediately to $H^1(P, M)$. A *nonhomogeneous one-cocycle* is an ordered pair (a, b) where $a: GP \to M$, $b: P \to TM$ in $\mathfrak{A}$ such that

(i) $a \cdot T\pi_1 - a \cdot \mu P + \gamma_1 \cdot Ta = 0$,

(ii) $G\gamma_2 \cdot b - \delta H \cdot b + Gb \cdot \pi_2 = 0$,

(iii) $b \cdot \pi_1 - G\gamma_1 \cdot \lambda M \cdot Tb - Ga \cdot \lambda P \cdot T\pi_2 + \gamma_2 \cdot a = 0$.

Let $NZ^1(P, M)$ be the abelian group of all nonhomogeneous one-cocycles. A *nonhomogeneous one-coboundary* is an ordered pair $(\gamma_1 \cdot Tc - c \cdot \pi_1, \gamma_2 \cdot c - Gc \cdot \pi_2)$ where $c: P \to H$ in $\mathfrak{A}$. Let $NB^1(P, M)$ be the abelian group of all nonhomogeneous one-coboundaries. Then $NH^1(P, M) = NZ^1(P, M)/NB^1(P, M)$, and $H^1 \cong NH^1$.

An *M-object* in $\mathfrak{D} = \mathfrak{A}_G^T$ is a pair $((P', \pi_1', \pi_2'), \circ)$ where $\circ: (M, \gamma_1, \gamma_2) \times (P', \pi_1', \pi_2') \to (P', \pi_1', \pi_2')$ is a bialgebra morphism subject to the two conditions of §II. Again, we will write $f \circ g$ for $X \xrightarrow{f \Pi g} M \times P' \xrightarrow{\circ} P'$ whenever it is convenient to do so. Given $f: X \to T(M \times P)$, $g: X \to M$, $h: X \to P$ we have the equations $\pi_1 \cdot T\circ \cdot f = \gamma_1 \cdot Tp_1 \cdot f \circ \pi_1 \cdot Tp_2 \cdot f$ and $\pi_2 \cdot (g \circ h) = \gamma_2 \cdot g \circ \pi_2 \cdot h$. Note that for $n \geq 0$, $((Q_G^T S_G^T)^n P', \circ = (Q_G^T S_G^T)^n \circ)$ is a $(Q_G^T S_G^T)^n$ M-object.

An *M-principal object over P* is an M-object $(P', \circ)$ and a bialgebra map $p: P' \to P$ as in §II. We will partially translate conditions (iv) and (v), leaving the rest to the reader.

(iv) For each $n \geq 0$ and each $f, g\colon (X, \xi_1, \xi_2) \to (G^n P', G^n \pi_1' \cdot \lambda_{n-1} \cdots \lambda_0, \delta G^{n-1} P')$ in $\mathfrak{A}_G^T$ such that $G^n p \cdot f = G^n p \cdot g$ there exists a unique $\bar{f}\colon (X, \xi_1, \xi_2) \to (G^n M, G^n \gamma_1 \cdot \lambda_{n-1} \cdots \lambda_0, \delta G^{n-1} M)$ in $\mathfrak{A}_G^T$ such that $g = \bar{f} \circ f$.

(v) There exists a fixed $s\colon P \to P'$ in $\mathfrak{A}$ such that $p \cdot s = P$.

A *morphism* $g\colon (P', \circ, p, s) \to (P'', \circ, q, s')$ of M-principal objects over P is a map $g\colon P' \to P''$ in $\mathfrak{A}$ such that $q \cdot g = p$, $g \cdot (f \circ h) = f \circ g \cdot h$, $\pi_1'' \cdot Tg = g \cdot \pi_1'$, and $\pi_2'' \cdot g = Gg \cdot \pi_2'$. As in §II we get a category $\mathfrak{N}\mathcal{P}\mathcal{O}(P, M)$ of M-principal objects over P and their morphisms, and $\Gamma\Psi'\colon \mathcal{P}\mathcal{O}(P, M) \to \mathfrak{N}\mathcal{P}\mathcal{O}(P, M)$ is an equivalence of categories.

We complete the translation by giving $\Lambda\colon |\mathfrak{N}\mathcal{P}\mathcal{O}(P, M)| \to NZ^1(P, M)$ (see Proposition 2). Given $(P', \circ, p, s)$ in $|\mathfrak{N}\mathcal{P}\mathcal{O}(P, M)|$, $\Lambda(P', \circ, p, s) = (a, b)$ where $a \circ s \cdot \pi_1 = \pi_1' \cdot Ts$ and $b \circ Gs \cdot \pi_2 = \pi_2' \cdot s$.

Theorem 6. *There exists a function* $\Lambda^{-1}\colon \mathfrak{N}Z^1(P, M) \to |\mathfrak{N}\mathcal{P}\mathcal{O}(P, M)|$ *such that* $\Lambda \cdot \Lambda^{-1} = NZ^1(P, M)$. *Moreover, both* $\Lambda^{-1} \cdot \Lambda(P', \circ, p, s)$ *and* $(P', \circ, p, s)$ *are in the same connected component of* $\mathfrak{N}\mathcal{P}\mathcal{O}(P, M)$ *for each* $(P', \circ, p, s)$ *in* $|\mathfrak{N}\mathcal{P}\mathcal{O}(P, M)|$.

Proof. Given a nonhomogeneous one-cocycle (a, b) we define $\Lambda^{-1}(a, b) = (B, \beta_1, \beta_2)$ by setting $B = P \times M$ in $\mathfrak{A}$, $\beta_1 = [\pi_1 \cdot p_1 \times (a \cdot p_1 + \gamma_1 \cdot p_2)] \cdot Tp_1 \Pi Tp_2$, and $\beta_2 = c \cdot [\pi_2 \cdot p_1 \times (b \cdot p_1 + \gamma_2 \cdot p_2)]$ where $c = (Gp_1 \Pi Gp_2)^{-1}$ and p_i is the ith projection. There are a number of things to verify. First, β_1 is associative because

$$\begin{aligned}
\beta_1 \cdot T\beta_1 &= [\pi_1 \cdot p_1 \times (a \cdot p_1 + \gamma_1 \cdot p_2)] \cdot Tp_2 \prod Tp_2 \\
&\quad \cdot T[\pi_1 \cdot p_1 \times (a \cdot p_1 + \gamma_1 \cdot p_2)] \cdot T(Tp_1 \prod Tp_2) \\
&= [\pi_1 \cdot p_1 \times (a \cdot p_1 + \gamma_1 \cdot p_2)] \cdot T\pi_1 \cdot T^2 p_1 \prod T(a \cdot Tp_1 + \gamma_1 \cdot Tp_2) \\
&= \pi_1 \cdot T\pi_1 \cdot T^2 p_1 \prod [a \cdot T\pi_1 \cdot T^2 p_1 + \gamma_1 \cdot T(a \cdot Tp_1 \cdot + \gamma_1 \cdot Tp_2)] \\
&= \pi_1 \cdot T\pi_1 \cdot T^2 p_1 \prod [a \cdot T\pi_1 \cdot T^2 p_1 + (\gamma_1 \cdot Ta \cdot T^2 p_1 + \gamma_1 \cdot T\gamma_1 \cdot T^2 p_2)] \\
&= \pi_1 \cdot T\pi_1 \cdot T^2 p_1 \prod [(a \cdot T\pi_1 + \gamma_1 \cdot Ta) \cdot T^2 p_1 + \gamma_1 \cdot T\gamma_1 \cdot T^2 p_2] \\
&= \pi_1 \cdot T\pi_1 \cdot T^2 p_1 \prod [a \cdot \mu P \cdot T^2 p_1 + \gamma_1 \cdot T\gamma_1 \cdot T^2 p_2] \\
&= \pi_1 \cdot \mu P \cdot T^2 p_1 \prod [a \cdot \mu P \cdot T^2 p_1 + \gamma_1 \cdot \mu M \cdot T^2 p_2] \\
&= [\pi_1 \cdot \mu P \times (a \cdot \mu P + \gamma_1 \cdot \mu M)] \cdot T^2 p_1 \prod T^2 p_2 \\
&= [\pi_1 \cdot p_1 \times (a \cdot p_1 + \gamma_1 \cdot p_2)] \cdot [\mu P \times \mu M] \cdot T^2 p_1 \prod T^2 p_2 \\
&= [\pi_1 \cdot p_1 \times (a \cdot p_1 + \gamma_1 \cdot p_2)] \cdot Tp_1 \prod Tp_2 \cdot \mu(P \times M) = \beta_1 \cdot \mu B.
\end{aligned}$$

The fourth equal sign follows because + is a T-morphism, the fifth because + is associative, and the sixth because a is a cocycle. The coassociativity of β_2 is proved dually. To show that β_2 is counitary, we first demonstrate that $\epsilon M \cdot b \cdot p_1 \colon P \times M \longrightarrow M$ is 0.

$$\begin{aligned}
\epsilon M \cdot b \cdot p_1 &= \epsilon M \cdot \epsilon GM \cdot \delta M \cdot b \cdot p_1 = \epsilon M \cdot \epsilon GM \cdot (Gb \cdot \pi_2 + G\gamma_2 \cdot b) \cdot p_1 \\
&= \epsilon M \cdot (\epsilon GM \cdot Gb \cdot \pi_2 + \epsilon GM \cdot G\gamma_2 \cdot b) \cdot p_1 \\
&= (\epsilon M \cdot \epsilon GM \cdot Gb \cdot \pi_2 + \epsilon M \cdot \epsilon GM \cdot G\gamma_2 \cdot b) \cdot p_1 \\
&= (\epsilon M \cdot b \cdot \epsilon P \cdot \pi_2 + \epsilon M \cdot G\epsilon M \cdot G\gamma_2 \cdot b) \cdot p_1 = (\epsilon M \cdot b + \epsilon M \cdot b) \cdot p_1 \\
&= \epsilon M \cdot b \cdot p_1 + \epsilon M \cdot b \cdot p_1
\end{aligned}$$

and M is an abelian group, so $\epsilon M \cdot b \cdot p_1 = 0$. Hence

$$\begin{aligned}
\epsilon B \cdot \beta_2 &= \epsilon(P \times M) \cdot c \cdot [\pi_2 \cdot p_1 \times (b \cdot p_1 + \gamma_2 \cdot p_2)] \\
&= \epsilon P \cdot \pi_2 \cdot p_1 \times \epsilon M \cdot (b \cdot p_1 + \gamma_2 \cdot p_2) \\
&= p_1 \times (\epsilon M \cdot b \cdot p_1 + \epsilon M \cdot \gamma_2 \cdot p_2) = p_1 \times (\epsilon M \cdot b \cdot p_1 + p_2) = p_1 \times p_2 = {}^{P \times M}.
\end{aligned}$$

The (dual) proof that β_1 is unitary is left to the reader. To see that (B, β_1, β_2) is in $|\mathfrak{A}_G^T|$ we must still verify that β_1 and β_2 are "coherent". Since the computation is similar to that given above, we delete it. Hence $G\beta_1 \cdot \lambda B \cdot T\beta_2 = \beta_2 \cdot \beta_1$, and $\Lambda^{-1} \colon NZ^1(P, M) \longrightarrow |\mathfrak{A}_G^T|$. We define an action of M on B by $\circ \colon M \times B \xrightarrow{\tau \times M} P \times M \times M \xrightarrow{P \times +} B$ where τ is the twist map, $p_1 \cdot \tau = p_2$ and $p_2 \cdot \tau = p_1$. It is easy to see that $\circ \cdot (M \times \circ) = \circ \cdot (+ \times B)$ and $\circ \cdot (\circ \times B) = B$, so we verify that $\circ$ is a morphism in $\mathfrak{A}_G^T$. It is a coalgebra morphism because

$$\begin{aligned}
&Gp_1 \prod Gp_2 \cdot G(P \times +) \cdot G(\tau \times M) \cdot c \cdot (\gamma_2 \cdot p_1 \times \beta_2 \cdot p_2) \\
&\quad = (GP \times +) \cdot (\tau \times GM) \cdot \left(\gamma_2 \cdot p_1 \prod [\pi_2 \cdot p_1 \times (b \cdot p_1 + \gamma_2 \cdot p_2)]\right) \\
&\quad = (GP \times +) \cdot ([\pi_2 \cdot p_1 \times (b \cdot p_1 + \gamma_2 \cdot p_2)] \cdot p_1 \times \gamma_2 \cdot p_2) \cdot (\tau \times M) \\
&\quad = (\pi_2 \cdot p_1 \times [b \cdot p_1 + \gamma_2 \cdot p_2 + \gamma_2 \cdot p_3]) \cdot (\tau \times M) \\
&\quad = [\pi_2 \cdot p_1 \times (b \cdot p_1 + \gamma_2 \cdot p_2)] \cdot (P \times +) \cdot (\tau \times M) = Gp_1 \prod Gp_2 \cdot \beta_2 \cdot \circ
\end{aligned}$$

and $Gp_1 \Pi Gp_2$ is an isomorphism. A similar proof yields $\beta_1 \cdot T(\circ) = \circ \cdot (\gamma_1 \cdot p_1 \times \beta_1 \cdot p_2) \cdot Tp_1 \Pi Tp_2$. Hence $(B, \circ)$ is an M-object. The first projection

$p_1: B \to P$ makes it an object over P (the verifications that p_1 is a morphism in $\mathfrak{A}_G^T$ are trivial), and $s = P\Pi 0: P \to B$ satisfies $p_1 \cdot s = P$. Only condition (iv) for principal objects remains to be verified in order to show that $\Lambda^{-1}(a, b) = ((B, \beta_1, \beta_2), \circ, p_1, s)$ is in $|\mathfrak{N}\mathfrak{P}\mathfrak{O}(P, M)|$. So suppose $f, g: (X, \xi_1, \xi_2) \to (G^n B, G^n\beta_1 \cdot \lambda_{n-1} \cdots \lambda_0, \delta G^{n-1} B)$ in $\mathfrak{A}_G^T$ (where $\delta G^{-1} B = \beta_2$) such that $G^n p_1 \cdot f = G^n p_1 \cdot g$. Define $\bar{f}: X \to G^n M$ in $\mathfrak{A}$ by $\bar{f} = G^n p_2 \cdot g - G^n p_2 \cdot f$. Then $\bar{f} \circ f = g$ and $\bar{f}$ is unique with respect to the given property. This essentially says that B is an M-principal object over P in $\mathfrak{A}$, and we need only show that $\bar{f}$ is in $\mathfrak{A}_G^T$. We do not give the computations, but remark that as an aid in demonstrating that $\bar{f}$ is a T-morphism, one needs the following "multiplicative" property of β_1: given $h: X \to G^n M$ and $k: X \to G^n B$ in $\mathfrak{A}$, $G^n\gamma_1 \cdot \lambda_{n-1} \cdots \lambda_0 \cdot Th \circ G^n\beta_1 \cdot \lambda_{n-1} \cdots \lambda_0 \cdot Tk = G^n\beta_1 \cdot \lambda_{n-1} \cdots \lambda_0 \cdot T(h \circ k)$. This can be verified by straightforward calculation. Hence $\Lambda^{-1}: NZ^1(P, M) \to |\mathfrak{N}\mathfrak{P}\mathfrak{O}(P, M)|$. Next, $\Lambda \cdot \Lambda^{-1}(a, b) = \Lambda(B, \beta_1, \beta_2) = (a', b')$ where $a' \circ (P\Pi 0) \cdot \pi_1 = \beta_1 \cdot T(P\Pi 0)$ and $b' \circ G(P\Pi 0) \cdot \pi_2 = \beta_2 \cdot (P\Pi 0)$. But $\beta_1 \cdot T(P\Pi 0) = \pi_1 \Pi(a + \gamma_1 \cdot 0) = \pi_1 \Pi a = \pi_1 \Pi(a + 0 \cdot \pi_1) = a \circ (P\Pi 0) \cdot \pi_1$ so that $a = a'$. Similarly, $b = b'$, and $\Lambda \cdot \Lambda^{-1} = NZ^1(P, M)$. Finally, let $(P', \circ, p, s)$ be in $|\mathfrak{N}\mathfrak{P}\mathfrak{O}(P, M)|$. We define $g: P' \to B = P \times M$ in $\mathfrak{A}$ by letting $g = p\Pi z$ where $z \circ s \cdot p = P'$. We claim that $g: (P', \circ, p, s) \to \Lambda^{-1} \cdot \Lambda(P', \circ, p, s)$ is in $\mathfrak{N}\mathfrak{P}\mathfrak{O}(P, M)$. Since $\beta_2 \cdot (p\Pi z) = \pi_2 \cdot p\Pi(b \cdot p + \gamma_2 \cdot z)$ and $G(p\Pi z) \cdot \pi_2' = Gp \cdot \pi_2' \Pi Gz \cdot \pi_2' = \pi_2 \cdot p\Pi Gz \cdot \pi_2'$ (up to unique isomorphism), g will be a G-morphism if and only if $b \cdot p + \gamma_2 \cdot z = Gz \cdot \pi_2'$. But

$$(b \cdot p + \gamma_2 \cdot z) \circ G(s \cdot p) \cdot \pi_2' = \gamma_2 \cdot z \circ (b \cdot p \circ Gs \cdot \pi_2 \cdot p) = \gamma_2 \cdot z \circ \pi_2' \cdot s \cdot p$$
$$= \pi_2' \cdot (z \circ s \cdot p) = \pi_2' = Gz \cdot \pi_2' \circ G(s \cdot p) \cdot \pi_2'$$

which implies the result. One can show that $z \cdot \pi_1' = a \cdot Tp + \gamma_1 \cdot Tz$ by letting them both operate on $s \cdot p \cdot \pi_1'$, and then $\beta_1 \cdot T(p\Pi z) = \pi_1 \cdot p\Pi(a \cdot Tp + \gamma_1 \cdot Tz) = \pi_1 \cdot p\Pi z \cdot \pi_1' = p \cdot \pi_1' \Pi z \cdot \pi_1' = (p\Pi z) \cdot \pi_1'$. Hence g is a T-morphism. That $p_1 \cdot g = p$ is easy, and $g \cdot \circ = \circ \cdot (p_1 \times g)$ follows because $(z \cdot \circ) \circ (s \cdot p \cdot \circ) = (p_1 + z \cdot p_2) \circ (s \cdot p \cdot \circ)$. Hence $\Lambda^{-1} \cdot \Lambda(P', \circ, p, s)$ and $(P', \circ, p, s)$ are in the same connected component of $\mathfrak{N}\mathfrak{P}\mathfrak{O}(P, M)$. This completes the proof of the theorem.

Theorem 7. *If $\mathfrak{D}$ is bialgebraic over $\mathfrak{A}$ then $\omega: \pi_0(\mathfrak{P}\mathfrak{O}(P, M)) \to H^1(P, M)$ is an isomorphism.*

Proof. This is a corollary of Theorems 4 and 6.

V. What to do if there are not enough group objects. We restrict our attention to the case when $\mathfrak{D}$ is bialgebraic over $\mathfrak{A}$. Often $\mathfrak{D} = \mathfrak{A}_G^T$ will not have "enough"

abelian group objects, in the sense that there is only one and it is trivial. We then have to pass to a category of modules for our coefficients, and we now say what this means in the generality of $\mathfrak{A}_G^T$.

Fix an object (P, π_1, π_2) in $\mathfrak{A}_G^T$. Let $\mathfrak{A}_G^T/E$ (E = "exponential") be the category having morphisms $(P', \pi_1', \pi_2') \to T'^n(P, \pi_1, \pi_2)$ in $\mathfrak{A}_G^T$, $n \geq 0$, as its objects where $T' = Q_G^T S_G^T$. Morphisms in $\mathfrak{A}_G^T/E$ are described as follows:

$$\mathfrak{A}_G^T/E((P', \pi_1', \pi_2') \to T'^n(P, \pi_1, \pi_2), (P'', \pi_1'', \pi_2'') \to T'^m(P, \pi_1, \pi_2))$$

$$= \begin{cases} \text{Commutative squares} \quad \begin{tikzcd} P' \arrow[r] \arrow[d] & P'' \arrow[d] \\ T'^n P \arrow[r, "\eta'"] & T'^m P \end{tikzcd} & \text{in } \mathfrak{A}_G^T \text{ if } m < n, \\[2ex] \text{Commutative triangles} \quad \begin{tikzcd} P' \arrow[rr] \arrow[dr] & & P'' \arrow[dl] \\ & T'^n P & \end{tikzcd} & \text{in } \mathfrak{A}_G^T \text{ if } m = n, \\[2ex] \text{Commutative squares} \quad \begin{tikzcd} P' \arrow[r] \arrow[d] & P'' \arrow[d] \\ T'^n P \arrow[r, "\eta'"] & T'^m P \end{tikzcd} & \text{in } \mathfrak{A}_G^T \text{ if } m > n, \end{cases}$$

where

$$\eta' = \begin{cases} \eta' T'^{m-1} P \cdot \eta' T'^{m-2} P \cdots \eta' T'^{m-n} P & \text{if } m > n, \\ \mu' T'^{m-1} P \cdots \mu' T'^{n-2} P & \text{if } 0 < m < n. \end{cases}$$

Composition of morphisms in $\mathfrak{A}_G^T/E$ is via juxtaposition of triangles and/or squares.

Recall that $\eta' P\colon (P, \pi_1, \pi_2) \to T'(P) = (GP, G\pi_1 \cdot \lambda P, \delta P)$ is just π_2. Thus $\eta'\colon T'^n P \to T'^m P$ for $n < m$ is $\delta G^{m-n-2} P \cdots \delta G^{n-1} P$ where $\delta G^{-1} P = \pi_2$. Let $\mathfrak{A}^T/E$ be the category of objects over $S_G^T T'^n P = (G^n P, G^n \pi_1 \cdot \lambda_{n-1} \cdots \lambda_0)$, $n \geq 0$, in $\mathfrak{A}^T$ with morphisms analogous to those in $\mathfrak{A}_G^T/E$. Let $\mathfrak{A}_G/E$ be the category of objects over $U_G^T T'^n P = (G^n P, \delta G^{n-1} P)$, $n \geq 0$, in $\mathfrak{A}_G$ with morphisms analogous to those in $\mathfrak{A}_G^T/E$. Let $\mathfrak{A}/E$ be the category of objects over

$$S_G U_G^T T'^n P = U^T S_G^T T'^n P = G^n P, \qquad n \geq 0,$$

in $\mathfrak{A}$ with morphisms as before. This means

$$\mathfrak{A}/E(P' \to G^nP,\ P'' \to G^mP)$$

$$= \begin{cases} \text{Commutative squares} \quad \begin{tikzcd} P' \arrow[r] \arrow[d] & P'' \arrow[d] \\ G^nP \arrow[r, "\epsilon"] & G^mP \end{tikzcd} & \text{in } \mathfrak{A} \text{ if } m < n, \\[2ex] \text{Commutative triangles} \quad \begin{tikzcd} P' \arrow[rr] \arrow[dr] & & P'' \arrow[dl] \\ & G^nP & \end{tikzcd} & \text{in } \mathfrak{A} \text{ if } m = n, \\[2ex] \text{Commutative squares} \quad \begin{tikzcd} P' \arrow[r] \arrow[d] & P'' \arrow[d] \\ G^nP \arrow[r, "\delta"] & G^mP \end{tikzcd} & \text{in } \mathfrak{A} \text{ if } m > n, \end{cases}$$

where $\delta = \delta G^{m-n-2}P \cdots \delta G^{n-1}P$, $\delta G^{-1}P = \pi_2$, and $\epsilon = \epsilon G^mP \cdots \epsilon G^{n-1}P$. We now have the following picture:

$$\begin{tikzcd} \mathfrak{A}_G^T/E \arrow[r, shift left, "S_G^T/E"] \arrow[d, shift left, "U_G^T/E"] & \mathfrak{A}^T/E \arrow[l, shift left, "Q_G^T/E"] \arrow[d, shift left, "U^T/E"] \\ \mathfrak{A}_G/E \arrow[r, shift left, "S_G/E"] \arrow[u, shift left, "F_G^T/E"] & \mathfrak{A}/E \arrow[l, shift left, "Q_G/E"] \arrow[u, shift left, "F^T/E"] \end{tikzcd}$$

where the functors pointing to the right or downward forget structure morphisms, and

$$Q_G/E(P' \to G^nP) = (GP', \delta P') \to (G^{n+1}P, \delta G^nP),$$

$$F^T/E(P' \to G^nP) = (TP', \mu P') \to (TG^nP, \mu G^nP)$$

$$\xrightarrow[G^n\pi_1 \cdot \lambda_{n-1} \cdots \lambda_0]{} (G^nP, G^n\pi_1 \cdot \lambda_{n-1} \cdots \lambda_0),$$

$$Q_G^T/E((P', \pi_1') \to (G^nP, G^n\pi_1 \cdot \lambda_{n-1} \cdots \lambda_0))$$

$$= (GP', G\pi_1' \cdot \lambda P', \delta P') \to (G^{n+1}P, G^{n+1}\pi_1 \cdot \lambda_n \cdots \lambda_0, \delta G^nP),$$

$$F_G^T/E((P', \pi_2') \to (G^nP, \delta G^{n-1}))$$

$$= (TP', \mu P', \lambda P' \cdot T\pi_2') \to (TG^nP, \mu G^nP, \lambda G^nP \cdot T\delta G^{n-1}P)$$

$$\xrightarrow[G^n\pi_1 \cdot \lambda_{n-1} \cdots \lambda_0]{} (G^nP, G^n\pi_1 \cdot \lambda_{n-1} \cdots \lambda_0, \delta G^{n-1}P).$$

Now $F_G^T/E \dashv U_G^T/E$, $F^T/E \dashv U^T/E$, $S_G^T/E \dashv Q_G^T/E$, and $S_G/E \dashv Q_G/E$. For example, $\eta: \mathfrak{A}_G/E \to Q_G/E\, S_G/E$ and $\epsilon: S_G/E\, Q_G/E \to \mathfrak{A}/E$ are given by

$$\eta[(P', \pi_2') \to (G^nP, \delta G^{n-1}P)]: [(P', \pi_2') \to (G^nP, \delta G^{n-1}P)]$$

$$\to [(GP', \delta P') \to (G^{n+1}P, \delta G^nP)]$$

where

$$\begin{tikzcd} P' \arrow[r, "\pi_2'"] \arrow[d, "p"'] & GP' \arrow[d, "Gp"] \\ G^nP \arrow[r, "\delta G^{n-1}P"] & G^{n+1}P \end{tikzcd}$$

and

$$\epsilon[P' \to G^nP]: [GP' \to G^{n+1}P] \to [P' \to G^nP]$$

where

$$\begin{tikzcd} GP' \arrow[r, "\epsilon P'"] \arrow[d, "Gp"'] & P' \arrow[d, "p"] \\ G^{n+1}P \arrow[r, "\epsilon G^nP"] & G^nP. \end{tikzcd}$$

Similarly, for $F^T/E \dashv U^T/E$ we have $\epsilon[(P', \pi_1') \to (P, \pi_1)] = \pi_1'$ and $\eta[P' \to P] = \eta P'$. Note that a $U^T/E\, F^T/E$-algebra structure on $P' \xrightarrow{p} G^nP$ in $\mathfrak{A}/E$ is a commutative diagram

$$\begin{tikzcd} TP' \arrow[r, "\pi_1'"] \arrow[d, "Tp"'] & P' \arrow[dd, "p"] \\ TG^nP \arrow[d, "G^n\pi_1 \cdot \lambda_{n-1} \cdots \lambda_0"'] & \\ G^nP \arrow[r, equal] & G^nP \end{tikzcd}$$

such that $\pi_1' \cdot \eta P' = P'$ and $\pi_1' \cdot \mu P' = \pi_1' \cdot T\pi_1'$. But these are precisely the conditions that $(P', \pi_1') \xrightarrow{p} (G^n P, G^n \pi_1 \cdot \lambda_{n-1} \cdots \lambda_0)$ be an object in $\mathfrak{A}^T/E$. The conditions for morphisms are also the same, and hence U^T/E is triplable. Moreover, S_G/E is cotriplable because a $S_G/E\, Q_G/E$-coalgebra structure on $P' \xrightarrow{p} G^n P$ in $\mathfrak{A}/E$ is a commutative diagram

$$\begin{tikzcd} P' \arrow[r, "\pi_2'"] \arrow[d, "p"'] & GP' \arrow[d, "Gp"] \\ G^n P \arrow[r, "\delta G^{n-1} P"] & G^{n+1} P \end{tikzcd}$$

such that $\epsilon P' \cdot \pi_2' = P'$ and $\delta P' \cdot \pi_2' = G\pi_2' \cdot \pi_2'$. But these are the conditions for $(P', \pi_2') \xrightarrow{p} (G^n P, \delta G^{n-1} P)$ to be an object in $\mathfrak{A}_G/E$, and similarly for morphisms.

Does $\lambda: TG \to GT$ "lift" to λ/E in this situation? Well, $U^T/E\, F^T/E\, S_G/E\, Q_G/E(P' \xrightarrow{p} G^n P) = TGP' \to TG^{n+1}P \to G^{n+1}P$ and $S_Q/E\, Q_G/E\, U^T/E\, F^T/E(P' \xrightarrow{p} G^n P) = GTP' \to GTG^n P \to G^{n+1}P$ so we consider

$$\begin{tikzcd} TGP' \arrow[r, "\lambda P'"] \arrow[d, "TGp"'] & GTP' \arrow[d, "GTp"] \\ TG^{n+1}P \arrow[r, "\lambda G^n P"] \arrow[d, "G^{n+1}\pi_1 \cdot \lambda_n \cdots \lambda_0"'] & GTG^n P \arrow[d, "G^{n+1}\pi_1 \cdot G\lambda_{n-1} \cdots G\lambda_0"] \\ G^{n+1}P \arrow[r, equal] & G^{n+1}P \end{tikzcd}$$

This diagram obviously commutes and defines λ/E. That it is a mixed distributive law follows easily. In this situation α, β, and ρ of §IV are each identity natural transformations. Finally, if we assume that $\mathfrak{A}$ and $\mathfrak{A}_G$ have pullbacks and S_G preserves them, then $\mathfrak{A}/E$ and $\mathfrak{A}_G/E$ will have (nonempty) finite products and S_G/E will preserve them. This is immediate as soon as one realizes that the product of $p: P' \to G^n P$ and $q: P'' \to G^m P$ in $\mathfrak{A}/E$ is the pullback of p and $\epsilon G^n P \cdot \epsilon G^{n+1} P \cdots \epsilon G^{m-1} P \cdot q$ in $\mathfrak{A}$ for $n < m$, and the pullback of p and q in $\mathfrak{A}$ for $n = m$. As a result of all we have done so far in this section, we know that $\mathfrak{A}_G^T/E$ is bialgebraic over $\mathfrak{A}/E$.

However, there is no terminal object in $\mathfrak{A}_G^T/E$ and this makes it difficult to talk about abelian group objects there. This problem is bypassed by noting that the only reason we required an abelian group object in §§II through IV was to get a complex of complexes of abelian groups. For this something slightly weaker

will do. Let $(P', \pi_1', \pi_2') \xrightarrow{p} (P, \pi_1, \pi_2)$ be a *P-module* [9]. This means that p is an abelian group object in the comma category $(\mathfrak{A}_G^T, (P, \pi_1, \pi_2))$, or equivalently that there exist commutative diagrams

$$\begin{tikzcd} P' \times_P P' \arrow[rr, "m"] \arrow[dr] & & P' \arrow[dl, "p"] \\ & P & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[rr, "z"] \arrow[dr, "="'] & & P' \arrow[dl, "p"] \\ & P & \end{tikzcd} \qquad \begin{tikzcd} P' \arrow[rr, "u"] \arrow[dr, "p"'] & & P' \arrow[dl, "p"] \\ & P & \end{tikzcd}$$

in $\mathfrak{A}_G^T$ with m, z, u subject to the usual abelian group axioms. Then for each $(P'', \pi_1'', \pi_2'') \xrightarrow{q} (P, \pi_1, \pi_2)$ in $\mathfrak{A}_G^T/E$, $\mathfrak{A}_G^T/E((F_G^T/E\ U_G^T/E)^{n+1}q, p)$ is a complex of abelian groups. Moreover, $Q_G^T/E\ S_G^T/E\ p$ is a $Q_G^T/E\ S_G^T/E(P, \pi_1, \pi_2)$-module because $Q_G^T/E\ S_G^T/E$ preserves finite products, and $(P', \pi_1', \pi_2') \xrightarrow{p}$ $(P, \pi_1, \pi_2) \xrightarrow{\eta P} T'(P, \pi_1, \pi_2) = (GP, G\pi_1 \cdot \lambda P, \delta P)$ is also a $Q_G^T/E\ S_G^T/E(P, \pi_1, \pi_2)$-module (essentially because ηP is a monomorphism). In addition $\eta p\colon p \to Q_G^T/E\ S_G^T/Ep$ is a homomorphism of abelian groups because $Q_G^T/E\ S_G^T/E$ preserves finite products. From this (and induction) it follows that $\mathfrak{A}_G^T/E((F_G^T/E\ U_G^T/E)^{n+1}q, (Q_G^T/E\ S_G^T/E)^{m+1}p)$ is a complex of complexes of abelian groups. Let $H^n(P'', P')_P$ be the nth homology group of the associated double complex.

The proofs of §§II through IV are valid for coefficients equal to a (P, π_1, π_2)-module, and we now interpret explicitly the classification theorems for the special case of the identity map in the first variable. By Proposition 5, $H^0(P, P')_P \cong \mathfrak{A}_G^T/E(P \xrightarrow{p} P, P' \xrightarrow{p} P) = (\mathfrak{A}_G^T, P)(P, p)$ which is just all commutative triangles

$$\begin{tikzcd} (P, \pi_1, \pi_2) \arrow[rr] \arrow[dr, "="'] & & (P', \pi_1', \pi_2') \arrow[dl, "p"] \\ & (P, \pi_1, \pi_2) & \end{tikzcd} \quad \text{in } \mathfrak{A}_G^T.$$

The group $H^1(P, P')_P$ will be isomorphic to equivalence classes of principal objects, which in this situation are called *extensions* of P by the P-module $P' \xrightarrow{p} P$. An extension of P by p consists of

(1) An object $(P'', \pi_1'', \pi_2'') \xrightarrow{q} (P, \pi_1, \pi_2)$ in $\mathfrak{A}_G^T/E$ and an operation

$$\begin{tikzcd} P' \times_P P'' \arrow[rr, "\circ"] \arrow[dr] & & P'' \arrow[dl, "q"] \\ & P & \end{tikzcd}$$

which is compatible with the addition and zero of $P' \xrightarrow{p} P$.

(2) For each

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & P' \arrow[d, "p"] \\ & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "g"] \arrow[dr] & P'' \arrow[d, "q"] \\ & P \end{tikzcd}$$

in $\mathfrak{A}_G^T/E$, $q \cdot (f \circ g) = q \cdot g$.

(3) For each $n \geq 0$ and each

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & T'^n P'' \arrow[d, "T'^n q"] \\ T'^m P \arrow[r] & T'^n P \end{tikzcd}$$

in $\mathfrak{A}_G^T/E$ there exists a unique

$$\begin{tikzcd} X \arrow[r, "\bar{f}"] \arrow[d] & T'^n P' \arrow[d, "T'^n p"] \\ T'^m P \arrow[r] & T'^n P \end{tikzcd}$$

in $\mathfrak{A}_G^T/E$ such that $g = \bar{f} \circ f$ (where $\circ = T'^n \circ$).

(4) There exists a fixed $s\colon P \to P''$ in $\mathfrak{A}$ such that $q \cdot s = P$ in $\mathfrak{A}$. The reader can provide the definition of a morphism of extensions (see §II).

VI. **Examples.** In this section we offer some examples in which bicohomology is applicable, and see what H^0 and H^1 are in these cases.

Example 1. For the simplest case, we take $\mathfrak{A} = \mathfrak{C}$, $\mathfrak{B} = \mathfrak{D}$, $U = U'$, $F = F'$, and S, Q, S', Q' the appropriate identity functors. This puts us in the situation of Beck's thesis [9], and we get a plethora of examples.

(a) With $\mathfrak{A} = \mathbf{Sets}$, $\mathfrak{B} = \mathbf{Groups}$, and F = free group functor, one has $H^1(P, M) \cong EM^2(P, M)$ where EM is the Eilenberg-Mac Lane theory [12] with P operating trivially on M. More generally Barr and Beck [5] have proved that $H^n(P, M) \cong EM^{n+1}(P, M)$ for $n \geq 1$.

(b) With $\mathfrak{A} = \mathbf{Sets}$, $\mathfrak{B} = K$-modules, and F = free K-module functor, one has $H^n(P, M) \cong \mathrm{Ext}^n(P, M)$ for $n \geq 0$ [9].

(c) Let X be a topological space, $\mathfrak{A}$ = the category of sheaves of sets over

X, $\mathfrak{B}$ = the category of sheaves of abelian groups over X, U the obvious forgetful functor, and F the left adjoint of U (see Example 3). Then for any two sheaves P, M in $\mathfrak{D}$, $H^0(P, M) \cong \mathfrak{D}(P, M)$ and $H^1(P, M) \cong$ equivalence classes of short exact sequences $0 \to M \to P' \to P \to 0$ in $\mathfrak{D}$ such that there is a map $UP \to UP'$ of sheaves of sets with $UP \to UP' \to UP$ the identity.

(d) Let K be a field and take $\mathfrak{A}$ = graded, connected, commutative K-algebras, $\mathfrak{B}$ = graded, connected bicommutative Hopf algebras, and F = graded tensor algebra functor. Then $H^1(P, M)$ classifies sequences of Hopf algebras which look like $M \to M \otimes P \to P$ in $\mathfrak{A}$ [9].

(e) Let P be a fixed group, $\mathfrak{A}$ = Sets, $\mathfrak{B}$ = Groups, and F = free group functor. Then a P-module as defined in §V can be naturally identified with a right P-module in the classical sense ($P' \to P$ gets identified with its kernel M). Barr and Beck have shown [5]

$$H^n(P, P')_P \cong \begin{cases} \mathrm{Der}(P, M), & n = 0, \\ EM^{n+1}(P, M), & n > 0, \end{cases}$$

where EM is the Eilenberg-Mac Lane theory [12].

(f) Let K be a commutative ring, $\mathfrak{A}$ = Sets, $\mathfrak{B}$ = K-algebras with 1, F = (noncommutative) polynomial K-algebra functor, and P a fixed K-algebra. Then a P-module (§V) "is" a two-sided P-module in the classical sense, again via the kernel functor. If $p: P' \to P$ is a P-module then Barr has shown [4]

$$H^n(P, P')_P \cong \begin{cases} \mathrm{Der}_K(P, \mathrm{Ker}\, p), & n = 0, \\ \mathrm{Sh}^{n+1}(P, \mathrm{Ker}\, p), & n > 0, \end{cases}$$

where Sh is the Shukla theory [25].

(g) This is the same as (f), except we let $\mathfrak{A}$ = K-modules, and F = tensor algebra functor. Then (see [3])

$$H^n(P, P')_P \cong \begin{cases} \mathrm{Der}_K(P, \mathrm{Ker}\, p), & n = 0, \\ \mathrm{Hoch}^{n+1}(P, \mathrm{Ker}\, p), & n > 0, \end{cases}$$

where Hoch is the Hochschild theory [18].

Example 2. Somewhat dually to Example 1 we can take $\mathfrak{A} = \mathfrak{B}$, $\mathfrak{C} = \mathfrak{D}$, $Q = Q'$, $S = S'$, and U, F, U', F' the appropriate identity functors.

(a) Let X be a topological space, $\mathfrak{A} = \mathrm{Sets}^{|X|}$, $\mathfrak{C} = \mathfrak{F}(X, \mathrm{Sets})$ = the category of sheaves of sets on X, $S: \mathfrak{C} \to \mathfrak{A}$ the stalk functor (which takes a sheaf of sets P to the set of stalks $\{P_x \mid x \in X\}$, and $Q: \mathfrak{A} \to \mathfrak{C}$ the functor whose value

at $\{A_x \mid x \in X\}$ is the sheaf which takes an open subset V of X to ΠA_x (product over all $x \in V$). The cotriplableness of S is proved in [28] and [29], see also [16]. Note that QS is the Godement standard construction [14], the first triple. An abelian group M in $\mathcal{C}$ is precisely a sheaf of abelian groups, but what is an M-principal object over a sheaf of sets P? It is a sheaf of sets P' lying over P, being freely operated on (on the left) by M, and having stalks $P'_x \cong M_x \times P_x$. (The "triviality" of the stalks is part of the proof of Theorem 6.) One can verify, then, that P is the sheaf coequalizer of $M \times P' \underset{p_2}{\overset{\circ}{\rightrightarrows}} P'$. A map of principal objects will be a sheaf morphism $f\colon P' \to P''$ such that the diagram

$$\begin{tikzcd}
M \times P' \arrow[r, shift left, "\circ"] \arrow[r, shift right, "p_2"'] \arrow[d, "M \times f"'] & P' \arrow[r, "p"] \arrow[d, "f"] & P \arrow[d, equal] \\
M \times P'' \arrow[r, shift left, "\circ"] \arrow[r, shift right, "p_2"'] & P'' \arrow[r, "q"] & P
\end{tikzcd}$$

commutes. Intuitively, $H^1(P, M)$ tells how many (equivalence classes of) sheaves there are having stalks $M_x \times P_x$ and P as quotient of the group action $M \times P' \to P'$. Of course $H^0(P, M) \cong \mathcal{C}(P, M)$.

(b) Let X be a topological space, $\mathfrak{Ab}$ = the category of abelian groups, $\mathfrak{A} = \mathfrak{Ab}^{|X|}$, $\mathcal{C} = \mathcal{F}(X, \mathfrak{Ab})$ and S, Q as in Example 2(a). Again S is cotriplable, and an abelian group M in $\mathcal{C}$ is just any object of $\mathcal{C}$. One checks that an M-principal object over P is simply a short exact sequence $0 \to M \to P' \to P \to 0$ of sheaves of abelian groups such that for each x in X the sequence $0 \to M_x \to P'_x \to P_x \to 0$ is a split exact sequence of abelian groups. Two principal objects are equivalent if there is a commutative diagram

$$\begin{tikzcd}
 & P' \arrow[dr] \arrow[dd] & \\
0 \arrow[r] & M \arrow[ur] \arrow[dr] & P \arrow[r] & 0. \\
 & P'' \arrow[ur] &
\end{tikzcd}$$

Hence $H^1(P, M)$ is a relative Ext, that is $H^1(P, M) \subseteq \mathrm{Ext}^1(P, M)$ is the subgroup consisting of extensions of P by M which have split stalks. If P has projective stalks then $H^1(P, M) = \mathrm{Ext}^1(P, M)$. Using standard techniques of homological algebra (as in [20, Theorem 8.2]), one can prove that $H^n(P, M) \cong R\mathrm{Ext}^n(P, M)$ where $R\mathrm{Ext}^n(P, M)$ is the subgroup of $\mathrm{Ext}^n(P, M)$ consisting of n-fold extensions in which the corresponding short exact sequences all have split stalks.

(c) One can carry out the analysis of the first part of Example 2(b) for sheaves of (not necessarily abelian) groups, and conclude that $H^1(P, M)$ classifies

extensions of P by M having stalks split as groups.

(d) To illustrate §V in this simple case let X be a topological space, K a sheaf of commutative rings over X, K_x-alg = the category of associative K_x-algebras with identity, $\mathfrak{A} = \Pi\, K_x$-alg (product of categories indexed by all $x \in X$), $\mathcal{C}$ = the category of sheaves of commutative K-algebras over X, S the stalk functor, and Q as in Example 2(a)–(c). The cotriplableness of S is proved in [29]. Since the only abelian group in $\mathcal{C}$ is the zero sheaf, the techniques of §V are necessary. Let P be a fixed sheaf of K-algebras and let $P' \xrightarrow{p} P$ be a P-module in the sense of §V. Then the zero map $P \xrightarrow{z} P'$ splits p, so that as sheaves of K-modules $P' \cong P \oplus M$ where M = the kernel of p. The fact that the addition map $P' \times_P P' \xrightarrow{m} P'$ is a map of sheaves of K-algebras implies that multiplication in M is trivial. The action of P on M is given by $ab = (a, 0)(0, b)$ in P', $ba = (0, b)(a, 0)$. Hence M is a sheaf of two-sided P-modules, and P' is isomorphic to the split extension of P by the sheaf of P-modules M. The assignment of the abelian group $P' \to P$ to its kernel M is in fact an equivalence between P-modules (in the sense of §V) and sheaves of two-sided P-modules in the usual sense [9]. Now let $P'' \xrightarrow{q} P$ be in $\mathcal{C}/E$ and consider f in $\mathcal{C}/E(q, p)$. This means $f\colon P'' \to P'$ is a map of sheaves of K-algebras and $p \cdot f = q$. Probing more deeply, given an open set V in X, $fV\colon P''V \to PV \oplus MV$ being a map of KV-algebras means $fV(xy) = fV(x)fV(y) = (q(x), \hat{f}(x))(q(y), \hat{f}(y)) = (q(x)q(y), q(x)\hat{f}(y) + \hat{f}(x)q(y))$. Hence f corresponds to a morphism $\hat{f}\colon P'' \to M$ of sheaves of K-modules satisfying $\hat{f}V(xy) = qV(x)\hat{f}V(y) + \hat{f}V(x)qV(y)$. On the other hand, given such an $\hat{f}\colon P'' \to M$ we can define g in $\mathcal{C}/E(q, p)$ by $g = q\Pi\hat{f}$, and these passages are mutually inverse. Now a morphism of sheaves of K-modules $\hat{f}\colon P'' \to M$ satisfying the above condition is called a *q-derivation* of P'' into the sheaf of P-modules M, and we let $\mathrm{Der}_q(P'', M)$ be the set of all such [7]. Then $\mathcal{C}/E(q, p) \cong \mathrm{Der}_q(P'', M)$ as abelian groups, where the addition in $\mathrm{Der}_q(P'', M)$ is pointwise. We now examine the structure of $\mathcal{C}/E(q, QSp)$. An element f of this group will be a morphism $f\colon P'' \to QSP'$ of sheaves of K-algebras such that $QSp \cdot f = \eta P \cdot q$ (see §V). Using the good exactness properties of QS one sees that the QSP-module $QSP' \to QSP$ can also be interpreted as $QSP \oplus QSM \to QSP$, and then f corresponds to an $\eta P \cdot q$-derivation of P'' into the sheaf of QSP-modules QSM. Hence $\mathcal{C}/E(q, QSp) \cong \mathrm{Der}_{\eta P \cdot q}(P'', QSM)$. Note also that QSM is a sheaf of P-modules via $\eta P\colon P \to QSP$ ("change of ring" [10]) so that we can consider $\mathcal{C}/E(q, QSp) \cong \mathrm{Der}_q(P'', QSM)$. Specializing to the identity map in the first variable, we are interested in the 0th and 1st homology groups of the complex $\mathrm{Der}_P(P, QSM) \to \mathrm{Der}_P(P, (QS)^2M) \to \cdots$. According to the theory of §V, $H^0(P, P')_P \cong \mathrm{Der}_P(P, M)$ and $H^1(P, P')_P$ will classify extensions of P by p. As soon as one notices that M_x is a two-sided P_x-module for each x in X, it is not difficult to see that a typical element

of $H^1(P, P')_p$ is represented by a short exact sequence $0 \to M \to P'' \to P \to 0$ of sheaves of K-algebras such that $0 \to M_x \to P''_x \to P_x \to 0$ is split exact as K_x-algebras for all x in X.

(e) The techniques of Example 2(d) can be mimicked in order to derive analogous results about sheaves of Lie algebras, commutative algebras, etc.

Example 3. We now use the full generality of §§II–V.

(a) Let X be a topological space, R a sheaf of commutative rings over X, $\mathfrak{D}$ = the category of sheaves of R-modules over X, $\mathcal{C}$ = the category of sheaves of sets over X, $\mathfrak{B} = \Pi\, R_x$-**modules**, $\mathfrak{A} = \Pi$ **Sets** (where the products are taken over all x in X). Let S and S' be the stalk functors, Q and Q' as in Example 2(a)–(e), U and U' the obvious forgetful functors, $F = \Pi\, F_x$ = the product of the free R_x-module functors, and F' = the "free" functor which associates to a sheaf P of sets the sheaf of R-modules associated to the presheaf $F'P$ where $F'P(V)$ = the free $R(V)$-module on the set $P(V)$. Once one knows the distributive law, the rest of the requirements of §IV are easy to check. The distributive law λ: $UFSQ \to SQUF$ is defined in [28], but we review that construction here for the convenience of the reader. Given $\{A_x \mid x \in X\}$ in $\mathfrak{A}$ and V open in X, define ϕV: $U'F'Q\{A_x\} \to QUF\{A_x\}$ by requiring that

$$\begin{tikzcd}
UF \prod_{x \in V} A_x \arrow[r, "\phi V"] \arrow[dr, "UF(\mathrm{proj}_x)"'] & \prod_{x \in V} UFA_x \arrow[d, "\mathrm{proj}_x"] \\
& UFA_x
\end{tikzcd}$$

commute. Then $\lambda\{A_x\} = UFSQ\{A_x\} \xrightarrow{\cong} SU'F'Q\{A_x\} \xrightarrow{S\phi} SQUF\{A_x\}$. Given sheaves P, M in $\mathfrak{D}$ the general term of the double complex which we are to look at is $\mathfrak{D}((F'U')^{n+1}P, (Q'S')^{m+1}M)$ with induced boundary operators. This means that we are "homming" the "free" resolution of P into the Godement resolution of M [14]. By the general theory of §IV, $H^0(P, M) \cong \mathfrak{D}(P, M)$ = the $R(X)$-module of sheaf homomorphisms of P into M, and $H^1(P, M) \cong \mathrm{Ext}^1(P, M)$ = equivalence classes of short exact sequences $0 \to M \to P' \to P \to 0$ in $\mathfrak{D}$. The requirement of a "section" s: $US'P \to US'P'$ in $\mathfrak{A}$ is trivially met because $P'_x \to P_x$ is onto for each $x \in X$ (the reader should compare this with Example 2(b)). One conjectures that $H^n(P, M) \cong \mathrm{Ext}^n(P, M)$, and this has been proved in [27].

(b) Let $\mathfrak{A}$, $\mathcal{C}$, S, and Q be as in Example 3(a). Let $\mathfrak{D}$ = the category of sheaves of groups on X, $\mathcal{C} = \Pi$ **Groups**, S' the stalk functor, Q' its right adjoint, U and U' forgetful functors, F and F' "free" functors and λ analogous to Example 3(a).

An abelian group M in $\mathfrak{D}$ is just a sheaf of abelian groups, and the double complex for P in $\mathfrak{D}$ looks like $\mathfrak{D}((F'U')^{*+1}P, (Q'S')^{*+1}M)$. Here $H^0(P, M) \cong \mathfrak{D}(P, M)$ and $H^1(P, M) \cong \mathrm{Ext}^1(P, M) =$ equivalence classes of exact sequences $0 \to M \to P' \to P \to 1$ in $\mathfrak{D}$.

(c) If one desires a group action, he should proceed as follows. Let everything be as in Example 3(b) and fix a sheaf of groups P. Then a P-module $P' \xrightarrow{p} P$ in the sense of §V corresponds, via its kernel M, to a sheaf of P-modules in the usual sense. Under this correspondence, the double complex $\mathfrak{D}/E((F'/E\ U'/E)^{*+1}P, (Q'/E\ S'/E)^{*+1}p)$ becomes $\mathrm{Der}((F'U')^{*+1}P, (Q'S')^{*+1}M)$ where $Q'S'$ is now the Godement triple in the category of sheaves of P-modules. The group $H^0(P, P')_P$ "is" $\mathrm{Der}(P, M)$ and $H^1(P, P')_P$ classifies short exact sequences $0 \to M \to P' \to P \to 1$ in $\mathfrak{D}$.

(d) Let $\mathfrak{A}$, $\mathcal{C}$, S, and Q be as in Example 3(a). Let R be a sheaf of rings over X, $\mathfrak{D}$ = the category of sheaves of associative R-algebras (with 1) over X, $\mathfrak{B} = \Pi\, R_x$-algebras, S' and Q' as in Example 2(d), and λ analogous to Example 3(a). If we fix a sheaf P in $\mathfrak{D}$ and an abelian group $P' \xrightarrow{p} P$ in $(\mathfrak{D}, P)$ then as in Example 2(d) we have $\mathfrak{D}/E((F'/E\ U'/E)^{*+1}P, (Q'/E\ S'/E)^{*+1}p) \cong \mathrm{Der}_P((F'U')^{*+1}P, (Q'S')^{*+1}M)$ where M = kernel of p. We conclude that $H^0(P, P')_P \cong \mathrm{Der}_P(P, M)$ and $H^1(P, P')_P$ classifies singular extensions of P by the P-module M. This example was worked out *ad hoc* in [29].

(e) There is no inherent reason in Example 3(d) to descend all the way to the set level. For example let $\mathfrak{D}$, $\mathfrak{B}$, S', and Q be as in Example 3(d) but let $\mathcal{C}$ = the category of sheaves of R-modules over X and $\mathfrak{A} = \Pi\, R_x$-modules (with by now obvious U, F, U', F', S, Q, and λ). The double complex $\mathfrak{D}/E((F'/E\ U'/E)^{*+1}P, (Q'/E\ S'/E)^{*+1}p) \cong \mathrm{Der}_P((F'U')^{*+1}P, (Q'S')^{*+1}M)$ has the same symbols as Example 2(d), but now $F'U'$ is "the tensor algebra on the underlying module" lifted to sheaves instead of "the polynominal algebra on the underlying set" lifted to sheaves. This time $H^0(P, P')_P \cong \mathrm{Der}_P(P, M)$ and $H^1(P, P')_P$ is in one-to-one correspondence with equivalence classes of singular extensions of P by M such that $0 \to M_x \to P'_x \to P_x \to 0$ is split as R_x-modules for each x in X.

(f) One can use similar techniques to interpret the absolute and relative bicohomology theories for sheaves of Lie-algebras, commutative algebras, etc.

(g) Apparently bicohomology applies to Hopf algebras (bialgebras in the sense of [26]) as well as to sheaves. This will be studied in a future paper.

REFERENCES

1. M. Barr, *Cohomology in tensored categories*, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 344–354. MR 35 #235.

2. ———, *Cohomology and obstructions: Commutative algebras*, Sem. Triples and

Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, p. 357. MR 42 #6075.

2a. M. Barr, *Exact categories*, Lecture Notes in Math., vol. 236, Springer-Verlag, Berlin and New York, 1971.

3. ———, *Harrison cohomology, Hochschild cohomology, and triples*, J. Algebra 8 (1968), 314–323. MR 36 #3851.

4. ———, *Shukla cohomology and triples*, J. Algebra 5 (1967), 222–231. MR 36 #1510.

5. M. Barr and J. Beck, *Acyclic models and triples*, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 336–343. MR 39 #6955.

6. ———, *Homology and standard constructions*, Sem. Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 245–335. MR 41 #3562.

7. M. Barr and G. Rinehart, *Cohomology as the derived functor of derivations*, Trans. Amer. Math. Soc. 122 (1966), 416–426. MR 33 #159.

8. J. Beck, *Distributive laws*, Sem. Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 119–140. MR 39 #2842.

9. ———, *Triples, algebras, and cohomology*, Dissertation, Columbia University, New York, 1967 (submitted).

10. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956. MR 17, 1040.

11. B. Eckmann and P. Hilton, *Group-like structures in general categories. I. Multiplications and comultiplications*, Math. Ann. 145 (1961/62), 227–255. MR 25 #108.

12. S. Eilenberg and S. Mac Lane, *Cohomology theory in abstract groups. I*, Ann. of Math. (2) 48 (1947), 51–78. MR 8, 367.

13. S. Eilenberg and J. Moore, *Adjoint functors and triples*, Illinois J. Math. 9 (1965), 381–398. MR 32 #2455.

14. R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Sci. Indust., no. 1252, Hermann, Paris, 1958. MR 21 #1583.

15. J. Gray, *Extensions of sheaves of algebras*, Illinois J. Math. 5 (1961), 159–174. MR 23 #A186.

16. P. Grillet, *Regular categories*, Lecture Notes in Math., vol. 236, Springer-Verlag, Berlin and New York, 1971.

17. A Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) 9 (1957), 119–221. MR 21 #1328.

18. G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math. (2) 46 (1945), 58–67. MR 6, 114.

19. D. Kan, *Adjoint functors*, Trans. Amer. Math. Soc. 87 (1958), 294–329. MR 24 #A1301.

20. S. Mac Lane, *Homology*, Die Grundlehren der math. Wissenschaften, Band 114, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR 28 #122.

21. E. Manes, *A triple theoretic construction of compact algebras*, Sem. Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 91–118. MR 39 #5657.

22. B. Mitchell, *Theory of categories*, Pure and Appl. Math., vol. 17, Academic Press, New York, 1965. MR 34 #2647.

23. B. Pareigis, *Kategorien und Funktoren*, Mathematische Leitfäden, Teubner, Stuttgart, 1969; English transl., Pure and Appl. Math., vol. 39, Academic Press, New York, 1970. MR 42 #337a, b.

24. J.-P. Serre, *Cohomologie galoisienne*, 3rd ed., Lecture Notes in Math., no. 5, Springer-Verlag, Berlin and New York, 1965. MR 34 #1328.

25. U. Shukla, *Cohomologie des algèbres associatives*, Ann. Sci. École Norm. Sup. (3) 78 (1961), 163–209. MR 24 #A2605.

26. M. Sweedler, *Hopf algebras*, Math. Lecture Note Series, Benjamin, New York, 1969. MR 40 #5705.

27. D. Van Osdol, *Remarks concerning* $\mathrm{Ext}^*(M, -)$, Bull. Amer. Math. Soc. 76 (1970), 612–617. MR 42 #6081.

28. ———, *Sheaves in regular categories*, Lecture Notes in Math., vol. 236, Springer-Verlag, Berlin and New York, 1971.

29. ———, *Some applications of triples to sheaf theory*, Dissertation, University of Illinois, Chicago, Ill., 1969 (submitted).

30. H. Wolff, *V-localizations and V-triples*, Dissertation, University of Illinois, Chicago, Ill., 1970 (submitted).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NEW HAMPSHIRE 03824

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, OSLO, NORWAY