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ALMOST MAXIMAL INTEGRAL DOMAINS AND FINITELY GENERATED MODULES

BY

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ABSTRACT. We present a class of integral domains with all finitely generated modules isomorphic to direct sums of cyclic modules. This class contains all previously known examples (i.e., the principal ideal domains and the almost maximal valuation rings) and, by an example, at least one more domain. The class consists of the integral domains satisfying (1) every finitely generated ideal is principal (obviously a necessary condition) and (2) every proper homomorphic image of the domain is linearly compact. We call an integral domain almost maximal if it satisfies (2). This is one of eleven conditions which, for valuation rings, is equivalent of E. Matlis' "almost maximal." An arbitrary integral domain R is almost maximal if and only if it is h-local and R_M is almost maximal for every maximal ideal M of R. Finally, equivalent conditions for a Prüfer domain to be almost maximal are studied, and in the process some conjectures of E. Matlis are answered.

1. **Preliminaries.** All rings will be commutative with an identity, and all modules will be unitary modules. R will always denote a ring and Ω will denote the set of maximal ideals of R. If A is a module, then A^* will denote the set of nonzero elements of A. If R happens to be an integral domain, then Q will denote the field of fractions of R, and K will denote the R-module Q/R. If R is an integral domain and A is an R-module, $t(A)$ will denote the set of torsion elements of A. If A is an R-module, then the R-topology on A is the topology with the submodules rA, $r \in R^*$, being a subbase for the open neighborhoods of 0 in A. If R is an integral domain, H will denote the completion of R in the R-topology. H is a ring and

$$H \cong \varprojlim_{r \in R^*} R/Rr \cong \mathrm{Hom}_R(K, K).$$

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See [11, §§5 and 6] for a discussion of H and the R-topology. We say that a family of sets has the finite intersection property, denoted f.i.p., if the intersection of every finite subfamily is nonempty.

R is said to be a generalized valuation ring if for every pair of elements of R, one divides the other. Thus a generalized valuation ring is a valuation ring if and only if it is an integral domain. An integral domain is a Prüfer domain if every finitely generated ideal is projective. Equivalently, R is a Prüfer domain if and only if R_M is a valuation ring for all $M \in \Omega$.

Definition. An R-module A is *linearly compact* if every family of cosets of submodules of A that has the f.i.p. has a nonempty intersection.

It follows trivially from the definition that homomorphic images and submodules of linearly compact modules are linearly compact. If $0 \to A' \to A \to A'' \to 0$ is an exact sequence of modules and A' and A'' are linearly compact, then A is linearly compact [20, Proposition 9]. Thus a finite direct sum of linearly compact modules is linearly compact. A module with the descending chain condition is linearly compact [20, Proposition 5]. A linearly compact module cannot contain an infinite direct sum of nonzero submodules [20, Proposition 6].

Linear compactness with respect to topologies have been studied by D. Zelinsky [20], H. Leptin [8] and [9], and S. Warner [18]. Our definition amounts to assuming that the topology on the module is always the discrete topology.

Definition. An R-module A is *algebraically compact* if whenever $\{\Sigma_{i\in I} r_{i\alpha} x_i = a_\alpha\}_{\alpha\in\Gamma}$ is finitely solvable, the family of equations is solvable, where $r_{i\alpha} \in R$, $a_\alpha \in A$, and for each $\alpha \in \Gamma$ $\{i \in I: r_{i\alpha} \neq 0\}$ is finite.

R. B. Warfield, Jr. has studied algebraically compact modules in [16]. In particular, every linearly compact module is algebraically compact [16, Proposition 9], and every injective module is algebraically compact [16, Theorem 2].

1.1 **Proposition.** *Let R be either a Noetherian ring or a generalized valuation ring, and let I be an ideal of R. Then R/I is linearly compact if and only if R/I is algebraically compact.*

Proof. It follows directly from the definition that R/I is a linearly compact R-module if and only if R/I is a linearly compact R/I-module; and similarly for algebraic compactness. Thus we may assume $I = \{0\}$. If R is Noetherian, the result follows from [16, Proposition 9]. [16, Proposition 9] also gives a proof for valuation rings which is valid for generalized valuation rings. Q.E.D.

1.2 **Lemma.** *Let R be a generalized valuation ring and I an ideal of R. If R/I is linearly compact, then R/I^2 is linearly compact.*

Proof. Let $\{r_\alpha + I_\alpha\}_{\alpha\in\Gamma}$ be a family of cosets of submodules of R with the f.i.p. such that $I_\alpha \supseteq I^2$ for all $\alpha \in \Gamma$. We must show that this family has a nonempty

intersection. If $I_\alpha \supseteq I$ for all $\alpha \in \Gamma$ the desired result follows from the hypothesis. Thus assume $\exists \alpha_0 \in \Gamma$ such that $I \supsetneqq I_{\alpha_0}$. Let $y \in I - I_{\alpha_0}$. Define $f: R/I \longrightarrow R/I^2$ by $f(r + I) = ry + I^2$. f is an R-homomorphism and $f(R/I) = Ry/I^2$. Thus Ry/I^2 is a linearly compact R-module.

Let $\Delta = \{\alpha \in \Gamma: I_\alpha \subseteq I_{\alpha_0}\}$ and let a prime denote modulo I^2. $\{r'_\alpha - r'_{\alpha_0} + I'_\alpha\}_{\alpha \in \Delta}$ is a family of cosets of submodules of $Ry/I^2 = Ry'$ with the f.i.p. Ry' is linearly compact implies $\exists x' \in \bigcap_{\alpha \in \Delta} r'_\alpha - r'_{\alpha_0} + I'_\alpha$. It follows that $x + r_{\alpha_0} \in \bigcap_{\alpha \in \Delta} r_\alpha + I_\alpha = \bigcap_{\alpha \in \Gamma} r_\alpha + I_\alpha$. Q.E.D.

1.3 **Proposition.** *Let R be a generalized valuation ring and $\mathfrak{F} = \{I: I$ is an ideal of R such that R/I is not linearly compact$\}$. Then either*

1. $\mathfrak{F} = \varnothing$ *or*
2. *there exists a prime ideal P of R such that $\mathfrak{F} = \{I: I \subseteq P\}$ or*
3. *there exists a prime ideal P of R such that $\mathfrak{F} = \{I: I \subsetneqq P\}$.*

Proof. Let $\mathfrak{G} = \{I: I$ is a prime ideal of R such that $I \supseteq \bigcup \mathfrak{F}\}$. If M is the maximal ideal of R, then $M \in \mathfrak{G}$, so $\mathfrak{G} \neq \varnothing$. Let $P = \bigcap \mathfrak{G}$. P is a prime ideal of R, and clearly if $I \supsetneqq P$, then $I \notin \mathfrak{F}$.

Suppose there exists an ideal I of R such that $P \supsetneqq I$ and $I \notin \mathfrak{F}$. Let $x \in P - I$. Then $Rx \supsetneqq I$, so $Rx \notin \mathfrak{F}$. Let $P' = \bigcap_{n=1}^{\infty} Rx^n$. Then either P' is a prime ideal of R or x is nilpotent. We claim that P' is not a prime ideal. For suppose P' is a prime ideal. If $P' \notin \mathfrak{G}$, there exists a $J \in \mathfrak{F}$ such that $J \supsetneqq P'$. Then for some positive integer n, $Rx^n \subseteq J$. But $Rx^n \notin \mathfrak{F}$ by the lemma and an easy induction. Thus $J \notin \mathfrak{F}$. This is contrary to $J \in \mathfrak{F}$, so it must be the case that $P' \in \mathfrak{G}$. But $P \supsetneqq P'$ which contradicts the definition of P. We have shown that P' is not a prime ideal, so x is nilpotent. Then $x^n = 0$ for some positive integer n. $Rx \notin \mathfrak{F}$ implies $\{0\} = Rx^n \notin \mathfrak{F}$ by the lemma and an easy induction. Thus $\mathfrak{F} = \varnothing$. We have shown that if there exists an ideal I of R such that $P \supsetneqq I$ and $I \notin \mathfrak{F}$, then $\mathfrak{F} = \varnothing$. The desired conclusion follows. Q.E.D.

We now wish to give three examples of valuation rings to indicate that each of the three possibilities of the last proposition do occur. The $\mathfrak{F}$ in these examples refer to the $\mathfrak{F}$ of the proposition.

Example 1.1. Let R be a complete discrete rank one valuation ring. If I is a nonzero ideal of R, then R/I satisfies the descending chain condition, so is linearly compact. Thus $I \notin \mathfrak{F}$. By [10, Theorem 9], $\mathfrak{F} = \varnothing$.

Example 1.2. Let R be a discrete rank one valuation ring which is not complete. As above, if I is a nonzero ideal of R, then $I \notin \mathfrak{F}$. By [10, Theorem 9], $\{0\} \in \mathfrak{F}$. Thus $\mathfrak{F} = \{I: I \subseteq \{0\}\}$.

Example 1.3. The following valuation ring is discussed in [19, p. 101]. Let k be a field, t an indeterminant over k, and R the set of all elements of the form

$\sum_{n=1}^{\infty} a_n t^{r_n}$ where $a_n \in k$, r_n is a real number, $r_n \geq 0$, $r_{n+1} \geq r_n$, and $\lim_{n\to\infty} r_n = \infty$. With formal addition and multiplication, R becomes a valuation ring of Krull dimension one. If r is a positive real number, let $I_r = \{\sum_{n=1}^{\infty} a_n t^{r_n} \in R\colon a_n = 0$ for all n, or if n is the smallest integer such that $a_n \neq 0$, then $r_n \geq r\}$. I_r is an ideal of R. For every positive integer n, choose a real number s_n such that $3 - 1/n < s_n < 3 - 1/(n+1)$. It is not difficult to see that $\{\sum_{i=1}^{n-1} t^{s_i} + I_{3-1/n}\}_{n=2}^{\infty}$ is a family of cosets of submodules of R with the f.i.p. and with empty intersection. $\bigcap_{n=2}^{\infty} I_{3-1/n} = I_3$, so we have $I_3 \in \mathfrak{F}$. If M is a maximal ideal of R, then $M \notin \mathfrak{F}$. By the proposition it must be the case that $\mathfrak{F} = \{I\colon I \subsetneqq M\}$.

1.4 **Proposition.** *Let R be a Noetherian ring and $\mathfrak{F} = \{I\colon I$ is an ideal of R such that R/I is not linearly compact$\}$. Then either $\mathfrak{F} = \emptyset$ or the maximal elements under set inclusion of $\mathfrak{F}$ are prime ideals of R.*

Proof. Suppose $\mathfrak{F} \neq \emptyset$ and I is a maximal element of $\mathfrak{F}$. We first claim that I is irreducible. If not, there exists ideals J and L of R such that $I = J \cap L$, $J \neq I$, and $L \neq I$. $J, L \notin \mathfrak{F}$, so R/J and R/L are linearly compact. $R/I \longrightarrow R/J \oplus R/L$ by $r + I \longrightarrow (r + J, r + L)$ is an embedding, and so R/I is linearly compact contrary to $I \in \mathfrak{F}$. Thus I is irreducible.

I is irreducible implies I is P-primary for some prime ideal P of R. We wish to show $P = I$. Suppose not. Then $P \supsetneqq I$, and so R/P is linearly compact. We first claim that P/P^n is linearly compact for all $n \geq 1$. This is trivial if $n = 1$. Suppose it is true for n. P^n/P^{n+1} is a finitely generated R/P-module, and hence linearly compact. From the exact sequence $0 \longrightarrow P^n/P^{n+1} \longrightarrow P/P^{n+1} \longrightarrow P/P^n \longrightarrow 0$ and P/P^n and P^n/P^{n+1} are linearly compact, it follows that P/P^{n+1} is linearly compact. We have shown that P/P^n is linearly compact for all $n \geq 1$.

From $0 \longrightarrow P/P^n \longrightarrow R/P^n \longrightarrow R/P \longrightarrow 0$ exact and R/P and P/P^n are linearly compact, we deduce R/P^n is linearly compact for all $n \geq 1$. R is Noetherian implies there exists a positive integer n such that $P^n \subseteq I$. R/P^n is linearly compact implies R/I is linearly compact. Contradiction. Thus $I = P$ and I is a prime ideal of R. Q.E.D.

2. Definition and basic properties.

Definitions. R is a *maximal ring* if every homomorphic image of R is linearly compact. R is an *almost maximal ring* if every proper homomorphic image of R is linearly compact.

For valuation rings, the above definitions agree with the commonly accepted ones, as for example in E. Matlis' [10]. Homomorphic images of linearly compact modules are linearly compact, so R is a maximal ring if and only if R is linearly

compact. D. Zelinsky proved [20, Theorem 2] that a maximal ring is a finite direct sum of local maximal rings.

Let R be a local Noetherian ring with maximal ideal M. If R is complete in the M-adic topology, then R is a maximal ring [20, Theorem 3]. If R is a maximal ring, then R is linearly compact, hence algebraically compact, and by [16, Corollary 7] R is complete in the R-topology. If R is also an integral domain of Krull dimension one, then the M-adic topology and the R-topology are the same, and so R is a maximal ring if and only if R is complete.

A similar result for valuation rings is due to E. Matlis. Suppose R is an almost maximal valuation ring. Then R is a maximal ring if and only if R is complete in the R-topology [10, Theorem 9].

Example 2.1. If R is a Noetherian integral domain of Krull dimension one, then R is an almost maximal ring. For if I is a nonzero ideal of R, R/I satisfies the descending chain condition and hence is linearly compact.

Example 2.2. If R is a polynomial ring in two variables over a field, then R is not an almost maximal ring. This follows from the later result 2.6 and the fact that R is not b-local.

Example 2.3. Let k be a field, X and Y indeterminants over k, $S = k[[X, Y]]$, and Q the field of fractions of S. Let G be the free abelian group on two generators ordered lexicographically, i.e., $(x, y) \geq (x', y')$ if and only if $x > x'$ or $(x = x'$ and $y \geq y')$. There exists a unique valuation $v\colon Q^* \longrightarrow G$ such that $v(X) = (1, 0)$, $v(Y) = (0, 1)$, and $v \mid k^* = (0, 0)$. Let R be the valuation ring of v. Then R is an almost maximal ring, but R is not a maximal ring.

Proof. We will only give a brief summary, leaving most of the details to the reader. To show R is an almost maximal ring suppose $\{r_\alpha + I_\alpha\}_{\alpha\in\Gamma}$ is a family of cosets of submodules of R with the f.i.p. such that $\bigcap_{\alpha\in\Gamma} I_\alpha \neq \{0\}$. If $\bigcap_{\alpha\in\Gamma} I_\alpha$ is a cyclic ideal, the result follows easily. Let P be the nontrivial nonmaximal prime ideal of R. The only noncyclic ideals of R are P^n for $n \geq 1$. Thus assume $\bigcap_{\alpha\in\Gamma} I_\alpha = P^e$ for e a positive integer. $\exists\{\alpha_n\}_{n=1}^\infty \subseteq \Gamma$ such that $I_{\alpha_n} \supseteq I_{\alpha_{n+1}}$, $\bigcap_{\alpha\in\Gamma} r_\alpha + I_\alpha = \bigcap_{n=1}^\infty r_{\alpha_n} + I_{\alpha_n}$, and $X^{e-1}Y^n \notin I_{\alpha_n}$. For each $n \geq 1$, there exists an integer b_n such that

$$r_{\alpha_n} = \sum_{i=-b_n}^{\infty} c_{n,0,i}X^0Y^i + \sum_{i=-2b_n}^{\infty} c_{n,1,i}X^1Y^i + \cdots$$

$$+ \sum_{i=-eb_n}^{\infty} c_{n,e-1,i}X^{e-1}Y^i + r_{n,e} \quad \text{where } c_{n,j,i} \in k \text{ and } r_{n,e} \in P^e.$$

By the f.i.p. we can assume $b_n = b_1$ for all $n \geq 1$. Then

$$\sum_{i=-b_1}^{\infty} c_{1,0,i} X^0 Y^i + \sum_{i=-2b_1}^{\infty} c_{1,1,i} X^1 Y^i + \cdots + \sum_{i=-eb_1}^{0} c_{1,e-1,i} X^{e-1} Y^i$$
$$+ \sum_{i=1}^{\infty} c_{i,e-1,i} X^{e-1} Y^i \in \bigcap_{n=1}^{\infty} r_{\alpha_n} + I_{\alpha_n}.$$

This shows that R is an almost maximal ring. $\{\Sigma_{i=0}^{n-1} X^i Y^{-i^2} + P^n\}_{n=1}^{\infty}$ is a family of cosets of submodules of R with the f.i.p. and with empty intersection showing R is not a maximal ring. Q.E.D.

Example 2.4. Let k be a field, X, Y and Z indeterminants over k, $S = k[[X, Y, Z]]$, and Q the quotient field of S. Let G be the free abelian group on two generators ordered lexicographically. Let $v: Q^* \longrightarrow G$ be the unique valuation determined by $v(X) = (1, 0)$, $v(Y) = (0, 1)$, $v(Z) = (0, 0)$, and $v \mid k^* = (0, 0)$. Let R be the valuation ring of v. Then R is not an almost maximal ring. For if M is the maximal ideal of R, then $X \in \bigcap_{n=1}^{\infty} M^n$, and it can be shown that $\{\Sigma_{i=0}^{n} Y^i Z^{-i^2} + M^n\}_{n=1}^{\infty}$ is a family of cosets of submodules of R with the f.i.p. and with empty intersection.

The following result is due to D. T. Gill [3, Proposition 1]. It is an easy corollary of 1.3, and is included for the convenience of the reader.

2.1 **Proposition.** *Let R be a generalized valuation ring which is not an integral domain. Then R is a maximal ring if and only if R is an almost maximal ring.*

Proof. The one implication is obvious. Suppose R is an almost maximal ring and not a maximal ring. If $\mathfrak{F}$ is as in 1.3, then $\mathfrak{F} = \{\{0\}\}$. By the hypothesis and 1.3, there is a prime ideal P and R such that $\{\{0\}\} = \{I: I \subsetneqq P\}$. P is a simple module, so linearly compact. From the exact sequence $0 \longrightarrow P \longrightarrow R \longrightarrow R/P \longrightarrow 0$ we deduce that R is linearly compact. Contradiction. Q.E.D.

2.2 **Proposition.** *Let R be a Noetherian ring which is not an integral domain. Then R is a maximal ring if and only if R is an almost maximal ring.*

Proof. This follows directly from 1.4. Q.E.D.

The following theorem summarizes several of the known conditions for a valuation ring to be an almost maximal valuation ring.

2.3. **Theorem.** *Let R be a valuation ring. The following statements are equivalent:*

1. *Every R-homomorphic image of Q is injective.*
2. *K is injective.*
3. *Every proper R-homomorphic image of Q is linearly compact.*
4. *K is linearly compact.*

5. *R is an almost maximal ring.*

6. *Every R-homomorphic image of Q is algebraically compact.*

7. *K is algebraically compact.*

8. *Every proper homomorphic image of R is algebraically compact.*

9. *Every finitely generated R-module is a direct sum of cyclic R-modules.*

10. *$H \cong \mathrm{Hom}_R(C, C)$ for some injective R-module C.*

11. *H is a maximal ring.*

Proof. $1 \Rightarrow 2$: Trivial.

$2 \Leftrightarrow 3 \Leftrightarrow 5$: [10].

$4 \Rightarrow 3$: Let A be a nonzero R-submodule of Q. We wish to show that Q/A is linearly compact. Let $x \in A^*$. Then $Q/Rx \cong Q/R = K$. Q/A is a homomorphic image of the linearly compact module K, and hence is linearly compact.

$3 \Rightarrow 4$: Trivial.

$3 \Rightarrow 6$ and $5 \Rightarrow 8$: Linearly compact modules and injective modules are algebraically compact.

$6 \Rightarrow 7$: Trivial.

$8 \Rightarrow 5$: 1.1.

$7 \Rightarrow 2$ and $6 \Rightarrow 1$: An R-module A is said to be semicompact if every family of cosets of submodules of A, $\{a_\alpha + A_\alpha\}_{\alpha \in \Gamma}$, has a nonempty intersection if it has the f.i.p. and if each A_α is the annihilator of some ideal of R. By [16, Proposition 8] every algebraically compact module is semicompact. By [10, Proposition 3] a divisible semicompact module over a Prüfer domain is injective.

$2 \Leftrightarrow 9$: [12, Lemma 5.5].

$2 \Leftrightarrow 10$: [12, Proposition 4.8].

$2 \Leftrightarrow 11$: [12, Proposition 4.7]. Q.E.D.

2.4 **Proposition.** *If R is an almost maximal Prüfer domain, A is an R-module, and $t(A)$ is finitely generated, then $A \cong t(A) \oplus A/t(A)$.*

Proof. By an induction on the number of generators and the fact that linear compactness is closed under extensions, every finitely generated torsion R-module is linearly compact. Thus $t(A)$ is algebraically compact. Since R is a Prüfer domain, it follows from [16, Corollary 5 and Theorem 2] that the exact sequence $0 \to t(A) \to A \to A/t(A) \to 0$ splits. Q.E.D.

2.5 **Lemma.** *If R is a maximal ring, then R/P is a local ring for all prime ideals P of R. In particular, a maximal integral domain is local.*

Proof. A maximal ring is a finite direct sum of local rings [20, Theorem 2], say $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$. A prime ideal P of R is of the form $P = R_1 \oplus \cdots \oplus R_{i-1} \oplus P' \oplus R_{i+1} \oplus \cdots \oplus R_n$ for some prime ideal P' of R_i. $R/P \cong R_i/P'$, and so R/P is local since R_i is local. Q.E.D.

Definition. R is *h-local* if for all $r \in R^*$, R/Rr is a semilocal ring, and for all nonzero prime ideals P of R, R/P is a local ring.

This definition is an obvious generalization of the definition in [11, p. 45] to rings which are not necessarily integral domains. E. Matlis has proved that, for integral domains R, R is h-local if and only if $T \cong \bigoplus \Sigma_{M \in \Omega} T_M$ for every torsion R-module T [12, Theorem 3.1].

2.6 **Proposition.** *If R is an almost maximal ring, then R is h-local.*

Proof. Suppose R is an almost maximal ring. If $r \in R^*$, then R/Rr is a maximal ring. R/Rr is then a finite direct sum of local rings, and hence is semilocal. If P is a nonzero prime ideal of R, then R/P is a maximal integral domain, hence local by 2.5. Q.E.D.

2.7 **Lemma.** *Let R be an h-local integral domain, $M \in \Omega$, and T a torsion R_M-module. Then every R-submodule of T is an R_M-submodule of T, i.e., the set of R-submodules of T is equal to the set of R_M-submodules of T.*

Proof. Let U be an R-submodule of T, $N \in \Omega$, and $N \neq M$. Then we get an exact sequence of R-modules $0 \longrightarrow U \otimes_R R_N \longrightarrow T \otimes_R R_N$. From T is a torsion R_M-module, R is h-local and [11, Corollary 8.2]

$$T \otimes_R R_N \cong (T \otimes_R R_M) \otimes_R R_N \cong T \otimes_R (R_M \otimes_R R_N) \cong T \otimes_R Q \cong \{0\}.$$

Thus $U \otimes_R R_N \cong \{0\}$, i.e., $U_N \cong \{0\}$. $U \cong U_M \oplus (\bigoplus \Sigma_{N \in \Omega,\, N \neq M} U_N) \cong U_M$, and so U is an R_M-module. Q.E.D.

2.8 **Corollary.** *Let R be an h-local integral domain and $M \in \Omega$. If T is a cyclic torsion R_M-module, then T is a cyclic R-module.*

Proof. Suppose $T = R_M/J$ for J a nonzero ideal of R_M. By the lemma, $R(1+J)$ is an R_M-submodule of R_M/J, and hence $R(1+J) = R_M/J$. Q.E.D.

Definition. R is a *locally almost maximal ring* if R_M is an almost maximal ring for all $M \in \Omega$.

2.9 **Theorem.** *Let R be an integral domain. Then R is an almost maximal ring if and only if R is an h-local locally almost maximal ring.*

Proof. Suppose R is an almost maximal ring. R is h-local by 2.6. Let $M \in \Omega$ and let J be a nonzero ideal of R_M. Let $I = J \cap R$. $R/I \cong (R/I)_M \oplus (\bigoplus \Sigma_{N \in \Omega,\, N \neq M} (R/I)_N)$. $(R/I)_M \cong R_M/I_M \cong R_M/R_M I = R_M/R_M(J \cap R) = R_M/J$. R_M/J is an R-direct summand of the linearly compact R-module R/I, so R_M/J is R-linearly compact, and hence trivially R_M-linearly compact. Thus R_M is an almost maximal ring.

Conversely, suppose R is a locally almost maximal ring and R is h-local. Let I be a nonzero ideal of R. We need to show that R/I is a linearly compact R-module. $R/I \cong \bigoplus \Sigma_{M \in \Omega} R_M/I_M$ since R is h-local. By hypothesis R_M/I_M is a linearly compact R_M-module, and hence a linearly compact R-module by 2.7. R/I is a finite direct sum of linearly compact R-modules, hence linearly compact. Q.E.D.

Let R be the valuation ring of Example 2.4. R is not an almost maximal ring. Let P be the nontrivial nonmaximal prime ideal of R. R/P and R_P are discrete rank one valuation rings, hence almost maximal rings by Example 2.1. Thus if R is a valuation ring, P a prime ideal of R, and R/P and R_P are both almost maximal rings, then this does not imply that R is an almost maximal ring.

We have seen a number of results regarding linear compactness for valuation rings that are also true for Noetherian integral domains. Let R be a generalized valuation ring and P a prime ideal of R. If R is a maximal (respectively almost maximal) ring, then R_P is a maximal (respectively almost maximal) ring [3, Lemma 2]. This result does not carry over to Noetherian integral domains. For let $R = k[[X, Y, Z]]$, where k is a field and X, Y and Z are indeterminants over k. If M is the maximal ideal of R, then R is complete in the M-adic topology, so R is a maximal ring. If we let $P = (Y, Z)$ and $I = (Z)$, then $R_P/R_P I \cong (R/I)_{P/I} \cong k[[X, Y]]_{(Y)}$. It can be shown that the last ring is not linearly compact as a module over itself, hence it is not linearly compact as an R_P-module. Thus R_P is not an almost maximal ring. We have shown that R is a maximal Noetherian integral domain with prime ideal P such that R_P is not an almost maximal ring.

If X and Y are indeterminants over the field k, then $k[[X, Y]]$ is a local Noetherian integral domain complete in the maximal ideal topology, and hence is a maximal ring. $k[[X, Y]]$ is a unique factorization domain, and so integrally closed. Clearly $k[[X, Y]]$ is not a Prüfer domain. Thus a maximal integrally closed integral domain need not be a Prüfer domain.

3. Finitely generated modules.

Definitions. R is a *domain with gcd's* if R is an integral domain and for all $a, b \in R$ $\exists g \in R$ such that $Rg = Ra + Rb$. R is a *domain with lcm's* if R is an integral domain and for all $a, b \in R$ $\exists l \in R$ such that $Rl = Ra \cap Rb$.

Thus R is a domain with gcd's if and only if R is an integral domain in which every finitely generated ideal is principal. If $g, a, b \in R$ and $Rg = Ra + Rb$, then g is the greatest common divisor of a and b. Similarly, if $Rl = Ra \cap Rb$, then l is the least common multiple of a and b. Domains with gcd's are also known as Bèzout domains in the literature.

If R is a domain with gcd's, then R is a Prüfer domain. For finitely generated ideals of R, being principal, are projective. If R is a semilocal Prüfer domain, then R is a domain with gcd's [4, Corollary 5].

The following is comparable to the result that if R is an integral domain, then R is a Prüfer domain if and only if every finitely generated torsion-free R-module is projective [2, Proposition 4.1, p. 133].

3.1 **Proposition.** *Let R be an integral domain. Then R is a domain with gcd's if and only if every finitely generated torsion-free R-module is free.*

Proof. If R is a domain with gcd's, then a finitely generated torsion-free R-module is free by induction on the rank of the module. The converse is trivial. Q.E.D.

3.2 **Proposition.** *If R is a domain with gcd's, then R is a domain with lcm's.*

Proof. Suppose R is a domain with gcd's and $a, b \in R$. We get an exact sequence of R-modules $0 \longrightarrow Ra \cap Rb \longrightarrow Ra \oplus Rb \longrightarrow Ra + Rb \longrightarrow 0$. $Ra + Rb$ is principal, by hypothesis, hence projective, so the exact sequence splits. Thus $Ra \cap Rb$ is a finitely generated ideal of R, so by the hypothesis it is principal. Q.E.D.

The converse of 3.2 is false. It is not difficult to see that a unique factorization domain is a domain with lcm's, yet there are unique factorization domains which are not domains with gcd's, for example, a polynomial ring in two variables over a field.

3.3 **Proposition.** *Let R be an integral domain.*

1. *If R is an h-local domain with gcd's, then every finitely presented R-module is a direct sum of cyclic R-modules.*

2. *If every finitely presented R-module is a direct sum of cyclic R-modules, then R is a domain with gcd's.*

Proof. 1. Suppose R is an h-local domain with gcd's, and A is a finitely presented R-module. $A/t(A)$ is a finitely generated torsion-free R-module, hence free by 3.1. Thus $A \cong t(A) \oplus A/t(A)$ with $A/t(A)$ free. $t(A)$ being a direct summand of the finitely presented module A, is finitely presented. Thus we can assume A is a torsion finitely presented R-module. Since R is h-local, $A \cong \bigoplus \Sigma_{M \in \Omega} A_M$. R is a Prüfer domain, so R_M is a valuation ring. Clearly A_M is a finitely presented R_M-module. By [17, Theorem 1], A_M is a direct sum of cyclic R_M-modules. By 2.8, A_M is a direct sum of cyclic R-modules, and hence A is a direct sum of cyclic R-modules.

2. Suppose every finitely presented R-module is a direct sum of cyclic R-modules. Let $I = Ra + Rb$ with $a, b \in R$. We want to show I is principal. We can find an exact sequence of R-modules $0 \longrightarrow L \longrightarrow R^2 \longrightarrow I \longrightarrow 0$. By [16, Proposition 5] R is a Prüfer domain, so I is a projective R-module. The sequence splits, L is finitely generated, and so I is finitely presented. By hypothesis, I is a direct sum of cyclic R-modules, and so I is principal. Q.E.D.

This proposition could be restated in the following form. If R is an h-local integral domain, then R is a domain with gcd's if and only if every finitely presented R-module is a direct sum of cyclic R-modules.

In [5] I. Kaplansky defines and studies elementary divisor rings. For integral domains, these are exactly the integral domains R for which every finitely presented R-module is a direct sum of cyclic R-modules.

The converse of the first implication of 3.3 is not true. Namely there exist integral domains which are not h-local, but every finitely presented module is a direct sum of cyclics. We will give such an example. Translating [5, Theorem 5.2] to our terminology we have that if R is an integral domain, then every finitely presented R-module is a direct sum of cyclic R-modules if and only if (1) R is a domain with gcd's and (2) for all $a, b, c \in R$ such that $Ra + Rb + Rc = R$, $\exists p, q \in R$ such that $Rpa + R(pb + qc) = R$. Suppose R is a Prüfer doamin with exactly two maximal ideals. Then R satisfies condition (1) since R is a semilocal Prüfer domain, and by considering the cases where a, b and c are contained or not contained in each of the maximal ideals of R, it follows that R satisfies condition (2). Thus for an example of a non-h-local integral domain with the property that every finitely presented module is a direct sum of cyclics, it suffices to take a Prüfer domain with exactly two maximal ideals which is not h-local. This amounts to finding a field with two valuations that are "dependent" but neither is a "specialization" of the other. Such an example is given in [1, Exercise 1, p. 187].

3.4 **Theorem.** *If R is an almost maximal domain with gcd's, then every finitely generated R-module is a direct sum of cyclic R-modules.*

Proof. Let A be a finitely generated R-module. $A/t(A)$ is a free R-module by 3.1, and so $A \cong t(A) \oplus A/t(A)$. We may thus assume that A is a finitely generated torsion R-module. Since R is h-local and R is a locally almost maximal ring by 2.9, the desired result follows from [12, Theorem 5.7]. Q.E.D.

In [13] E. Matlis defines a ring of type I to be an h-local Prüfer domain R with exactly two maximal ideals such that R_M is a maximal valuation ring for all $M \in \Omega$. Such a ring is a domain with gcd's since it is a semilocal Prüfer domain. Thus by 2.9 and 3.4, if R is a ring of type I, then every finitely generated R-module is a direct sum of cyclic R-modules.

In [6, Notes, p. 80] I. Kaplansky conjectured that if R is an integral domain, then every finitely generated R-module is a direct sum of cyclic R-modules if and only if R is a principal ideal domain or R is an almost maximal valuation ring. This conjecture is false, because in [13] an example due to B. Osofsky is given of a ring of type I which is neither a principal ideal domain or an almost maximal valuation ring.

Every principal ideal domain and every almost maximal valuation ring is an almost maximal domain with gcd's. Thus the class of almost maximal domain with gcd's contains all the known integral domains R for which every finitely generated R-module is a direct sum of cyclic R-modules. Are there any more? In other words, is the converse of 3.4 true?

3.5 **Lemma.** *Let R be an integral domain and S a multiplicatively closed subset of R. If every finitely generated R-module is a direct sum of cyclic R-modules, then every finitely generated R_S-module is a direct sum of cyclic R_S-modules.*

Proof. Let A be a finitely generated R_S-module. Suppose $a_1, a_2, \cdots, a_n$ generate A as an R_S-module. Let $B = Ra_1 + Ra_2 + \cdots + Ra_n$. B is a direct sum of cyclic R-modules. $A \cong B \otimes_R R_S$, and so A is a direct sum of cyclic R_S-modules. Q.E.D.

3.6 **Proposition.** *Let R be an integral domain. If every finitely generated R-module is a direct sum of cyclic R-modules, then R is a locally almost maximal domain with gcd's.*

Proof. Suppose every finitely generated R-module is a direct sum of cyclic R-modules. Every finitely generated ideal of R is a direct sum of cyclics, and hence must be principal. Thus R is a domain with gcd's. R is then a Prüfer domain, and so R_M is a valuation ring if $M \in \Omega$. By the last lemma, every finitely generated R_M-module is a direct sum of cyclic R_M-modules. By 2.3 (5) $\Leftrightarrow$ (9) R_M is an almost maximal ring. Thus R is a locally almost maximal ring. Q.E.D.

Let R be an integral domain. Suppose it is true that if every finitely generated R-module is a direct sum of cyclic R-modules, then R is h-local. By 2.9 and the last proposition, R is an almost maximal domain with gcd's, i.e., the converse of 3.4 would be true. Thus the converse of 3.4 is true if R is h-local whenever every finitely generated R-module is a direct sum of cyclic R-modules. It is an open question whether this is the case.

Let R and P be as in the paragraph directly after the proof of 2.9. Then R/P and R_P are both almost maximal valuation rings, and R is a valuation ring which is not an almost maximal ring. By 2.3, (5) $\Leftrightarrow$ (9) it follows that both the rings R/P and R_P have the property that every finitely generated module is a direct sum of cyclics, yet it is not the case that every finitely generated R-module is a direct sum of cyclic R-modules.

4. Equivalent conditions for being an almost maximal ring.

4.1 **Proposition.** *Let R be a Prüfer domain. The following statements are equivalent:*

1. *R is an almost maximal ring and Ω is finite.*

2. *Every proper R-homomorphic image of Q is linearly compact.*

3. *K is linearly compact.*

Proof. $2 \Rightarrow 1$. If I is a nonzero ideal of R, then R/I is a submodule of the linearly compact module Q/I, hence linearly compact. Thus R is an almost maximal ring. By 2.6 R is h-local. Thus $K \cong \bigoplus \Sigma_{M \in \Omega} K_M$. A linearly compact module cannot contain an infinite direct sum of nonzero submodules, so Ω is finite.

$1 \Rightarrow 3$. R is h-local and a locally almost maximal ring by 2.9. Thus $K \cong \bigoplus \Sigma_{M \in \Omega} K_M$ and R_M is an almost maximal valuation ring. K_M is an R_M-linearly compact module by 2.3 (4) $\Leftrightarrow$ (5), and hence K_M is an R-linearly compact module by 2.7. K is a finite direct sum of linearly compact R-modules, so K is linearly compact.

$3 \Rightarrow 2$. Repeat the argument given in the proof of 2.3 (4) $\Rightarrow$ (3). Q.E.D.

4.2 **Proposition.** *If R is an almost maximal Prüfer domain, then every R-homomorphic image of Q is injective.*

Proof. Let A be a nonzero R-submodule of Q. R is h-local by 2.6, so $Q/A \cong \bigoplus \Sigma_{M \in \Omega} (Q/A)_M \cong \bigoplus \Sigma_{M \in \Omega} Q/A_M$. R_M is an almost maximal valuation ring by 2.9, so Q/A_M is an injective R_M-module by 2.3 (5) $\Rightarrow$ (1). By [12, Theorem 3.3] $\operatorname{inj\,dim}_R Q/A = \sup\{\operatorname{inj\,dim}_{R_M} (Q/A)_M : M \in \Omega\} = 0$. Thus Q/A is an injective R-module. Q.E.D.

A ring is said to be pre-self-injective if every proper homomorphic image of R is a self-injective ring. If we restrict ourselves to integral domains, then [7, Theorem 3.5] says that if R is an integral domain and not a field, then R is an almost maximal Prüfer domain of Krull dimension one if and only if R is a pre-self-injective ring. [7, Theorem 4.5] says that if R is an integral domain and not a field, then R is a pre-self-injective ring if and only if every R-homomorphic image of Q is injective and the Krull dimension of R is one. Combining these two results we get that if R is an integral domain of Krull dimension one, then R is an almost maximal Prüfer domain if and only if every R-homomorphic image of Q is injective. Thus the converse of 4.2 is true if R is of Krull dimension one.

4.3 **Proposition.** *Let R be a semilocal integral domain. The following statements are equivalent:*

1. *R is an almost maximal Prüfer domain.*

2. *Every R-homomorphic image of Q is injective.*

3. *Every R-homomorphic image of Q is algebraically compact and R is a Prüfer domain.*

4. *Every proper homomorphic image of R is algebraically compact and R is a Prüfer domain.*

Proof. $1 \Rightarrow 2$. This is a special case of 4.2.

$2 \Rightarrow 3$. That R is a Prüfer domain follows from [10, Theorem 5]. Every injective module is algebraically compact.

$3 \Rightarrow 1$. We will first show that every proper R-homomorphic image of Q is linearly compact. Suppose $\{r_\alpha + A_\alpha\}_{\alpha\in\Gamma}$ is a family of cosets of submodules of Q with the f.i.p. and if $A = \bigcap_{\alpha\in\Gamma} A_\alpha$, then $A \neq \{0\}$. We need to show that this family has a nonempty intersection. Without loss of generality we may assume that the family is closed under finite intersections. R is a semilocal Prüfer domain, so there exist valuations $v_1, v_2, \cdots, v_n$ of Q with associated valuation rings R_{v_i} such that $R = \bigcap_{i=1}^n R_{v_i}$, and no v_i is a specialization of any other v_j, i.e., $R_{v_i} \not\supset R_{v_j}$ if $i \neq j$. Suppose that for a given j $\exists \alpha \in \Gamma$ such that $v_j(A_\alpha^*) = v_j(A^*)$. Then for this j we may assume $v_j(A_\alpha^*) = v_j(A^*)$ for all $\alpha \in \Gamma$. Thus with a possible renumbering, we may assume that there exists an integer k, $0 \leq k \leq n$, such that for $1 \leq i \leq k$, $v_i(A_\alpha^*) \neq v_i(A^*)$ for all $\alpha \in \Gamma$, and for $k+1 \leq i \leq n$, $v_i(A_\alpha^*) = v_i(A^*)$ for all $\alpha \in \Gamma$. We may assume $k \geq 1$ for otherwise the desired result is trivial. We will use a prime to denote modulo A.

For $1 \leq i \leq k$, and $\alpha \in \Gamma$, $v_i(A_\alpha^*) \supsetneqq v_i(A^*)$ so $\exists s_{i\alpha} \in A_\alpha$ such that $v_i(s_{i\alpha}) \notin v_i(A^*)$. R is a semilocal Prüfer domain, so R is a domain with gcd's, and so $\exists s_\alpha \in Q$ such that $Rs_\alpha = Rs_{1\alpha} + Rs_{2\alpha} + \cdots + Rs_{k\alpha}$. We have $s_\alpha \in A_\alpha$ and $v_i(s_\alpha) \notin v_i(A^*)$ for all $1 \leq i \leq k$. For each i, $1 \leq i \leq k$, $\exists \beta_i \in \Gamma$ such that $v_i(s_\alpha) \notin v_i(A_{\beta_i}^*)$. Let $A'_\beta = A'_{\beta_1} \cap A'_{\beta_2} \cap \cdots \cap A'_{\beta_k}$. Since $\{r'_\alpha + A'_\alpha\}_{\alpha\in\Gamma}$ is closed under finite intersections, A'_β is an R-submodule of Q' with an associated coset appearing in this family. With a possible change of the coset representative s'_α, for all $\alpha \in \Gamma$ $\exists s_\alpha \in A_\alpha$ $\exists \beta \in \Gamma$ such that $r'_\beta + A'_\beta \subseteq r'_\alpha + Rs'_\alpha \subseteq r'_\alpha + A'_\alpha$. Thus from $\{r'_\alpha + A'_\alpha\}_{\alpha\in\Gamma}$ we have formed the family $\{r'_\alpha + Rs'_\alpha\}_{\alpha\in\Gamma}$ with the f.i.p. such that $\bigcap_{\alpha\in\Gamma} r'_\alpha + A'_\alpha = \bigcap_{\alpha\in\Gamma} r'_\alpha + Rs'_\alpha$. But that $\{r'_\alpha + Rs'_\alpha\}_{\alpha\in\Gamma}$ has a nonempty intersection follows from [16, Proposition 8] since Rs'_α is a finitely generated R-module. We have shown that every proper R-homomorphic image of Q is linearly compact. By 4.1 R is an almost maximal ring.

$1 \Rightarrow 4$. Every linearly compact module is algebraically compact.

$4 \Rightarrow 1$. We need to show that a family of cosets of submodules of R, $\{r_\alpha + I_\alpha\}_{\alpha\in\Gamma}$ with the f.i.p. has a nonempty intersection provided that $\bigcap_{\alpha\in\Gamma} I_\alpha \neq \{0\}$. Let a prime denote modulo $\bigcap_{\alpha\in\Gamma} I_\alpha$. As in the proof of $(3) \Rightarrow (1)$ above, we can assume each I'_α is a cyclic ideal of R'. Then $\{r'_\alpha + I'_\alpha\}_{\alpha\in\Gamma}$ has a nonempty intersection by [16, Proposition 8] since I'_α is a finitely generated R-module. Thus $\{r_\alpha + I_\alpha\}_{\alpha\in\Gamma}$ has a nonempty intersection. Q.E.D.

4.4 **Corollary.** *If R is a semilocal integral domain such that every R-homomorphic image of Q is injective, then every finitely generated R-module is a direct sum of cyclic R-modules.*

Proof. By the proposition, R is an almost maximal semilocal Prüfer domain. A semilocal Prüfer domain is a domain with gcd's. The result follows from 3.4. Q.E.D.

4.5 **Corollary.** *Let R be a semilocal integral domain such that every R-homomorphic image of Q is injective. Then R is a Prüfer domain and so can be described by valuations $v_1, v_2, \cdots, v_n$ with corresponding valuation rings R_{v_i}, $R = \bigcap_{i=1}^{n} R_{v_i}$, and $R_{v_i} \not\supseteq R_{v_j}$ if $i \neq j$. Moreover, the approximation theorem remains valid, i.e., if $u_i \in Q$ and $\alpha_i \in v_i(Q^*)$ are given, then $\exists u \in Q$ such that $v_i(u - u_i) = \alpha_i$ for all $i = 1, 2, \cdots, n$.*

Proof. That R is a Prüfer domain follows from [10, Theorem 5], and clearly R can be described by the valuations. By 4.3, R is an almost maximal ring and hence h-local by 2.6. Thus every nonzero prime ideal of R is contained in only one maximal ideal. This is exactly the condition that makes $v_1, v_2, \cdots, v_n$ independent valuations [19, Theorem 18′, p. 47, and Remark, p. 48]. Q.E.D.

In [10, Remark 2] three unsolved problems are posed. We are now able to answer all three of them negatively. The first asks if R is a locally almost maximal Prüfer domain, is it true that every R-homomorphic image of Q is injective? In [14, Example 2], an example is given of a locally almost maximal Prüfer domain R with exactly two maximal ideals and a nonzero prime ideal contained in both maximal ideals. Thus R is not h-local. If every R-homomorphic image of Q were injective, then by 4.3 and 2.6 R would be h-local, which it is not. Thus this is an example of a locally almost maximal Prüfer domain R where every R-homomorphic image of Q is not injective as desired. Another counterexample to this conjecture is described in [7, p. 418].

2.9 says that if R is an integral domain, then R is an almost maximal ring if and only if R is a locally almost maximal ring and R is h-local. By the example used in the last paragraph, it follows that the conditions R is h-local and R is a locally almost maximal ring are independent conditions. Put another way, R is a locally almost maximal ring does not imply that R is an almost maximal ring.

An R-module A is said to be semicompact if every family of cosets of submodules of A, $\{a_\alpha + A_\alpha\}_{\alpha \in \Gamma}$, has a nonempty intersection if it has the f.i.p. and if each A_α is the annihilator of some ideal of R. The second unsolved problem asks: For a Prüfer domain, is every proper R-homomorphic image of Q linearly compact equivalent to every R-homomorphic image of Q semicompact? For the counterexample, let R be the ring of integers. Then $K \cong \bigoplus \Sigma_{M \in \Omega} K_M$, and so K is not linearly compact since it contains an infinite direct sum of nonzero submodules. Every R-homomorphic image of Q is injective and hence semicompact [10, Proposition 2]. Thus every R-homomorphic image of Q is semicompact, but

not every R-homomorphic image of Q is linearly compact, as desired. The converse which is true, follows trivially from [10, Proposition 3].

The third unsolved problem asks whether a Prüfer domain with every proper R-homomorphic image of Q linearly compact is either an almost maximal valuation ring or a Dedekind domain. For the counterexample, let R be the ring of type I appearing in [13]. R is neither a valuation ring nor a Dedekind domain. Ω has two elements and R is an h-local locally almost maximal Prüfer domain, hence R is an almost maximal ring by 2.9. By 4.1 every proper R-homomorphic image of Q is linearly compact. Thus every proper R-homomorphic image of Q is linearly compact, yet R is neither an almost maximal valuation ring nor a Dedekind domain, as desired.

4.6 **Lemma.** *Let R be a Prüfer domain. The following statements are equivalent*:

1. *H is h-local.*
2. *R is h-local and Ω is finite.*

Proof. $1 \Rightarrow 2$. Suppose H is h-local and I is a nonzero ideal of R. By [11, Proposition 5.10], $f\colon R/I \to H/HI$ by $f(r+I) = r + HI$ is an isomorphism, and clearly this is a ring isomorphism. R/I is then a semilocal ring since H/HI is a semilocal ring. If P is a nonzero prime ideal of R, then HP is a nonzero prime ideal of H. Thus $R/P \cong H/HP$ is a local integral domain, so R is h-local. By [11, Corollary 8.5], $H \cong \Pi_{M\in\Omega} H(M)$. Clearly H is h-local implies Ω is finite.

$2 \Rightarrow 1$. Suppose R is h-local and Ω is finite. Then $H \cong \Pi_{M\in\Omega} H(M)$ and $H(M)$ is the completion of R_M, a valuation ring, so $H(M)$ is a valuation ring. Thus H is a finite direct sum of local rings, hence h-local. Q.E.D.

The following generalizes [12, Proposition 4.7].

4.7. **Proposition.** *Let R be a Prüfer domain. The following statements are equivalent*:

1. *H is a maximal ring.*
2. *R is an almost maximal ring and Ω is finite.*

Proof. $1 \Rightarrow 2$. Suppose H is a maximal ring. Then H is h-local by 2.6. By 4.6, R is h-local and Ω is finite. $H \cong \Pi_{M\in\Omega} H(M)$ by [11, Corollary 8.5], and H is a maximal ring implies $H(M)$ is a maximal ring. $H(M)$ is a maximal valuation ring, so by the local case [12, Proposition 4.7], R_M is an almost maximal ring. By 2.9 R is an almost maximal ring.

$2 \Rightarrow 1$. Suppose R is an almost maximal ring and Ω is finite. By 2.9, R is h-local and R_M is an almost maximal valuation ring for all $M \in \Omega$. By the local case [12, Proposition 4.7], $H(M)$ is a maximal valuation ring. $H \cong \Pi_{M\in\Omega} H(M)$ is a finite direct sum of maximal rings, hence H is a maximal ring. Q.E.D.

The following three theorems are an attempt to organize the results which generalize 2.3, the eleven equivalent conditions for a valuation ring to be almost maximal. We will number the conditions the same way they were numbered in 2.3, and add a twelfth condition.

4.8 **Theorem.** *Let R be a semilocal Prüfer domain. The following seven statements are equivalent*:

1. *Every R-homomorphic image of Q is injective.*
3. *Every proper R-homomorphic image of Q is linearly compact.*
4. *K is linearly compact.*
5. *R is an almost maximal ring.*
6. *Every R-homomorphic image of Q is algebraically compact.*
8. *Every proper homomorphic image of R is algebraically compact.*
11. *H is a maximal ring.*

Moreover, these seven statements imply the remaining five statements:

2. *K is injective.*
7. *K is algebraically compact.*
9. *Every finitely generated R-module is a direct sum of cyclic R-modules.*
10. *$H \cong \mathrm{Hom}_R(C, C)$ for some injective R-module C.*
12. *R is a locally almost maximal ring.*

Proof. $5 \Leftrightarrow 3 \Leftrightarrow 4$. 4.1.

$5 \Leftrightarrow 1 \Leftrightarrow 6 \Leftrightarrow 8$. 4.3.

$5 \Leftrightarrow 11$. 4.7.

So the first seven conditions are equivalent.

$1 \Rightarrow 2$ and $6 \Rightarrow 7$. Trivial.

$5 \Rightarrow 9$. R is a domain with gcd's, since R is a semilocal Prüfer domain. The result follows from 3.4.

$2 \Rightarrow 10$. Trivial.

$5 \Rightarrow 12$. 2.9. Q.E.D.

Condition 12 implies the first seven conditions if and only if R is h-local by 2.9. It is not known whether any of the other four conditions 2, 7, 9 and 10 imply the first seven conditions.

4.9 **Theorem.** *Let R be an h-local Prüfer domain. The following three statements are equivalent*:

3. *Every proper R-homomorphic image of Q is linearly compact.*
4. *K is linearly compact.*
11. *H is a maximal ring.*

Moreover, these three statements imply that R is semilocal and hence imply the following one statement:

9. *Every finitely generated R-module is a direct sum of cyclic R-modules.*

Moreover, this one statement implies the following eight statements and the following eight statements are equivalent:

1. *Every R-homomorphic image of Q is injective.*

2. *K is injective.*

5. *R is an almost maximal ring.*

6. *Every R-homomorphic image of Q is algebraically compact.*

7. *K is algebraically compact.*

8. *Every proper homomorphic image of R is algebraically compact.*

10. $H \cong \mathrm{Hom}_R(C, C)$ *for some injective R-module C.*

12. *R is a locally almost maximal ring.*

Proof. $3 \Leftrightarrow 4$. 4.1.

$3 \Rightarrow 11$. By 4.1, R is semilocal, and so this follows from 4.8.

$11 \Rightarrow 3$. By 4.7, R is semilocal, and so this follows from 4.8.

So the first three conditions are equivalent and these imply that R is semilocal.

$3 \Rightarrow 9$. R is semilocal, and so this follows from 4.8.

$9 \Rightarrow 5$. By 3.5, every finitely generated R_M-module is a direct sum of cyclic R_M-modules. By 2.3 (9) $\Rightarrow$ (5), R_M is an almost maximal ring. Thus R is an almost maximal ring by 2.9.

It remains to show that the last eight conditions are equivalent.

$1 \Rightarrow 2$. Trivial.

$2 \Rightarrow 1$. Let A be a nonzero module of Q. By [12, Theorem 3.3] $0 = \mathrm{inj\,dim}_R K = \sup\{\mathrm{inj\,dim}_{R_M} K_M : M \in \Omega\}$, since R is b-local. Thus K_M is R_M-injective for all $M \in \Omega$. By the local case 2.3 (2) $\Rightarrow$ (1), $Q/A_M \cong (Q/A)_M$ is an injective R_M-module. Thus by [12, Theorem 3.3] again, Q/A is an injective R-module.

$1 \Rightarrow 6$ and $2 \Rightarrow 7$. Every injective module is algebraically compact.

$7 \Rightarrow 2$ and $6 \Rightarrow 1$. Repeat the argument given in 2.3.

$2 \Leftrightarrow 5$. By [12, Theorem 3.3] K in an injective R-module if and only if K_M is an injective R_M-module for all $M \in \Omega$. By the local case 2.3 (2) $\Leftrightarrow$ (5), K_M is an injective R_M-module if and only if R_M is an almost maximal ring. R_M is an almost maximal ring for all $M \in \Omega$ if and only if R is an almost maximal ring by 2.9.

$5 \Rightarrow 8$. Every linearly compact module is algebraically compact.

$8 \Rightarrow 5$. Let I be a nonzero ideal of R. Since R is b-local, $R/I \cong \bigoplus \Sigma_{M \in \Omega} R_M/I_M$. Direct summands of algebraically compact modules are algebraically compact, so R_M/I_M is an algebraically compact R-module. It follows from the definition that R_M/I_M is then an algebraically compact R_M-module. By 1.1, R_M/I_M is a linearly compact R_M-module, and so R_M/I_M is a linearly compact R-module by 2.7.

R/I is a finite direct sum of lineary compact R-modules, and thus is linearly compact.

$2 \Rightarrow 10$. Trivial.

$10 \Rightarrow 2$. Suppose $H \cong \mathrm{Hom}_R(C, C)$ with C injective. H is h-reduced implies C is a torsion R-module. Since R is h-local, $C \cong \bigoplus \Sigma_{M \in \Omega} C_M$, and we will consider each C_M as a submodule of C.

Suppose $f: C_M \to C$ is an R-homomorphism. Let $N \in \Omega$, $N \neq M$, and let p_N: $C \to C_N$ be the projection map. By 2.7, the R-submodules of C_N are R_N-modules. Thus $p_N(f(C_M))$ is a torsion R_M- and R_N-module, and hence must be zero since $R_M \otimes_R R_N \cong Q$ [11, Lemma 8.1]. Thus $\mathrm{Im}\, f \subseteq C_M$. We have shown that $\mathrm{Hom}_R(C_M, C) \cong \mathrm{Hom}_R(C_M, C_M)$. By a straightforward element argument, one can show that $\mathrm{Hom}_R(C_M, C_M) \cong \mathrm{Hom}_{R_M}(C_M, C_M)$.

By [11, Corollary 8.6] $H(M) \cong H^M$. Thus $H(M) \cong H^M = \mathrm{Hom}_R(R_M, H) \cong \mathrm{Hom}_R(R_M, \mathrm{Hom}_R(C, C)) \cong \mathrm{Hom}_R(R_M \otimes_R C, C) \cong \mathrm{Hom}_R(C_M, C) \cong \mathrm{Hom}_R(C_M, C_M) \cong \mathrm{Hom}_{R_M}(C_M, C_M)$. $H(M)$ is the completion of R_M in the R_M-topology, and C_M is an injective R_M-module by [12, Theorem 3.3]. By the local case [12, Proposition 4.8] K_M is an injective R_M-module. Thus K is an injective R-module by [12, Theorem 3.3].

$5 \Leftrightarrow 12$. 2.9. Q.E.D.

In the previous theorem, the last eight conditions do not imply the one condition 9 as a Dedekind domain which is not a principal ideal domain shows. The one condition 9 does not imply the first three conditions as the ring of integers shows.

4.10 **Theorem.** *Let R be a semilocal h-local Prüfer domain. The following statements are equivalent:*

1. *Every R-homomorphic image of Q is injective.*
2. *K is injective.*
3. *Every proper R-homomorphic image of Q is linearly compact.*
4. *K is linearly compact.*
5. *R is an almost maximal ring.*
6. *Every R-homomorphic image of Q is algebraically compact.*
7. *K is algebraically compact.*
8. *Every proper homomorphic image of R is algebraically compact.*
9. *Every finitely generated R-module is a direct sum of cyclic R-modules.*
10. *$H \cong \mathrm{Hom}_R(C, C)$ for some injective R-module C.*
11. *H is a maximal ring.*
12. *R is a locally almost maximal ring.*

Proof. This follows directly from 4.8 and 4.9. Q.E.D.

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