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ON SEQUENCES CONTAINING AT MOST 3 PAIRWISE COPRIME INTEGERS

BY

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ABSTRACT. It has been conjectured by Erdős that the largest number of natural numbers not exceeding n from which one cannot select $k+1$ pairwise coprime integers, where $k \geq 1$ and $n \geq p_k$, with p_k denoting the kth prime, is equal to the number of natural numbers not exceeding n which are multiples of at least one of the first k primes. It is known that the conjecture holds for $k = 1$ and 2. In this paper we establish the truth of the conjecture for $k = 3$.

1. **Introduction.** Let $h_k(n)$ denote the largest number of natural numbers not exceeding n from which one cannot select $k+1$ integers which are pairwise coprime, where $k \geq 1$ and $n \geq 1$. Clearly $h_k(n)$ is at least equal to $A_k(n)$, which denotes the number of integers not exceeding n which are multiples of at least one of the first k primes $2, 3, \cdots, p_k$.

The question has been raised by Erdős (see [1, p. 183]) whether it is true that

$$h_k(n) = A_k(n), \qquad n \geq p_k, \tag{1}$$

for every $k \geq 1$. Erdős remarked in [1] that (1) could be proved without difficulty for the cases $k = 1$ and $k = 2$. In this paper our main objective is to establish (1) for the case $k = 3$.

We denote by $A(n_1, n_2)$ the number of integers in $[n_1, n_2]$ which are multiples of at least one of the primes 2, 3, 5, so that in this notation, the following theorem is a restatement of the case $k = 3$ of (1):

Theorem 1. *We have*

$$h_3(n) = A(1, n), \qquad n \geq 5. \tag{2}$$

We shall also prove the following

Theorem 2. *Provided $n \geq n_0$, any set of $A(1, n)$ integers not exceeding n, where at most three of these integers are pairwise coprime, necessarily consists of all the multiples of the primes* 2, 3, 5.

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2. **Some preliminary lemmas.** A set of integers is said to be good if it contains 4 pairwise coprime integers. Otherwise it is said to be bad.

Lemma 1. *Suppose* $\mathfrak{A}$ *is a set of* $A(1, 30)$ *integers in* $[1, 30]$. *Then* $\mathfrak{A}$ *is bad only if it coincides with the set of multiples of* 2, 3 *and* 5 *in* $[1, 30]$.

Proof. We note that $A(1, 30) = 22$ and 1, 7, 11, 13, 17, 19, 23, 29 are the only integers in $[1, 30]$ not divisible by 2, 3, or 5. Hence $\mathfrak{A}$ must contain at least all but 8 of the integers 2, 4, 8, 3, 9, 27, 5, 25, 1, 7, 11, 13, 17, 19, 23, 29. It is easy to see that unless the last eight integers are excluded, $\mathfrak{A}$ will be good. Thus $\mathfrak{A}$ coincides with the set of multiples of 2, 3 and 5 in $[1, 30]$.

Lemma 2. *Suppose* $5 \leq j \leq 15$ *and* $\mathfrak{A}$ *is a set of* $A(1, j) + 1$ *integers in* $[30k + 1, 30k + j]$. *Then* $\mathfrak{A}$ *is good.*

Proof. Denote $30k$ by M. Then $M + 1$, $M + 7$, $M + 11$, $M + 13$ are the only integers in $[M + 1, M + 15]$ not divisible by 2, 3 or 5. Clearly we need only establish the lemma for $j = 5, 6, 7, 11$ and 13. If $j = 5$ or 6 then $\mathfrak{A}$ contains all the integers in $[M + 1, M + j]$. Clearly $M + 1$, $M + 2$, $M + 3$, $M + 5$ are pairwise coprime. If $j = 7$, then $\mathfrak{A}$ contains at least all but one of the integers in $[30k + 1, 30k + 7]$. Thus $\mathfrak{A}$ must contain at least 4 of the pairwise coprime integers $M + 1$, $M + 2$, $M + 3$, $M + 5$, $M + 7$.

If $j = 11$, then $M + 2$, $M + 3$, $M + 5$, $M + 1$, $M + 7$, $M + 11$ are pairwise coprime and $\mathfrak{A}$ must contain at least 4 of these integers.

If $j = 13$, let x be an integer from $M + 4$, $M + 8$ not divisible by 7. Then x, $M + 3$, $M + 5$, $M + 1$, $M + 7$, $M + 11$, $M + 13$ are pairwise coprime and again $\mathfrak{A}$ must contain at least 4 of these. This concludes the proof of Lemma 2.

Lemma 3. *Suppose* $\mathfrak{A}$ *consists of* $A(16, 30) + 1$ *integers in* $[30k + 16, 30k + 30]$. *Then* $\mathfrak{A}$ *is good.*

Proof. Denote $30k$ by M. Let x be an integer from $M + 22$, $M + 26$ not divisible by 7. Then $\mathfrak{A}$ contains at least 4 of the 7 pairwise coprime integers x, $M + 25$, $M + 27$, $M + 17$, $M + 19$, $M + 23$, $M + 29$.

Lemma 4. *Suppose* $\mathfrak{A}$ *is a set of* $A(1, 30) + 1$ *integers in* $[30k + 1, 30k + 30]$. *Then* $\mathfrak{A}$ *is good.*

Proof. Since $\mathfrak{A}$ must contain at least $A(1, 15) + 1$ integers in $[30k + 1, 30k + 15]$ or at least $A(16, 30) + 1$ integers in $[30k + 16, 30k + 30]$, the lemma is an immediate consequence of Lemmas 2 and 3.

Lemma 5. *Suppose* $j \geq 1$ *and* $\mathfrak{A}$ *is a set of* $A(1, 30j)$ *integers in* $[1, 30j]$. *Then* $\mathfrak{A}$ *is bad only if it coincides with the set of multiples of* 2, 3 *and* 5 *in* $[1, 30j]$.

Proof. Suppose $\mathfrak{A}$ is bad. Then if follows from Lemma 4 that $\mathfrak{A}$ contains exactly $A(1, 30)$ integers in each of the intervals $[30k+1, 30k+30]$, $k = 0, 1, \cdots, j-1$. Hence, by Lemma 1, it contains the primes 2, 3 or 5.

Lemma 6. *Suppose* $1 \leq j \leq 30$ *and* $\mathfrak{A}$ *is a set of* $A(1, j)+2$ *integers in* $[30k+1, 30k+j]$. *Then* $\mathfrak{A}$ *is good.*

Proof. Since $A(1, j) = j-1$ for $1 \leq j \leq 6$, the assumption of the theorem cannot be satisfied unless $j \geq 7$. In view of Lemma 2 we may further assume that $j > 15$ and that $\mathfrak{A}$ contains precisely $A(1, 15)$ integers in $[30k+1, 30k+15]$ and $A(16, j)+2$ integers in $[30k+16, 30k+j]$. Thus $j \geq 19$ necessarily. Let M denote $30k$. Then $M+17$, $M+19$, $M+23$, $M+29$ are the only integers in $[30k+16, 30k+30]$ which are not divisible by 2, 3 or 5. Thus we need only establish the lemma for $j = 19$, 23 and 29.

If $j = 19$ then $\mathfrak{A}$ contains $M+16$, $M+17$, $M+18$, $M+19$ and all but four of the integers in $[30k+1, 30k+15]$. Since $M+16$, $M+17$, $M+19$ and $M+l$ are pairwise coprime for $l = 1$, 7, 11, or 13, we may assume that $\mathfrak{A}$ contains all other integers in $[30k+1, 30k+15]$. But then $M+9$, $M+14$, $M+17$, $M+19$ are pairwise coprime.

If $j = 23$, then $\mathfrak{A}$ contains at least all but one of the integers in $[30k+15, 30k+23]$. Then $M+22$, $M+21$, $M+17$, $M+19$, $M+23$ are pairwise coprime and $\mathfrak{A}$ contains at least 4 of these integers.

If $j = 29$, then $\mathfrak{A}$ contains at least 4 of the pairwise coprime integers $M+27$, $M+25$, $M+17$, $M+19$, $M+23$, $M+29$.

3. **Proofs of Theorems 1 and 2.** Let $\mathfrak{A}$ be a set of $A(1, n)+1$ integers in $[1, n]$ where $n \geq 5$. We shall show that $\mathfrak{A}$ is good. In view of Lemma 5 we may assume that n is not divisible by 30. Further, it is easily verified that any set of $A(1, n)+1$ integers in $[1, n]$ is good if $30 \geq n \geq 5$. Thus we may assume $n > 30$ so that $n = 30k+l$, where $1 \leq l < 30$. Let $\mathfrak{A}^*$ denote the subset of $\mathfrak{A}$ in $[30k+1, 30k+l]$. As the desired result follows from Lemma 6 if $|\mathfrak{A}^*| \geq A(1, l)+2$ we may assume that $|\mathfrak{A}^*| \leq A(1, l)+1$ and in view of Lemma 4 actually $|\mathfrak{A}^*| = A(1, l)+1$. Thus $\mathfrak{A}$ contains exactly $A(1, 30k)$ integers in $[1, 30k]$ and by Lemma 5, $\mathfrak{A}$ contains all the multiples of 2, 3 and 5 in $[1, 30k]$. In particular it contains 2, 3 and 5. Since $|\mathfrak{A}^*| = A(1, l)+1$, $\mathfrak{A}^*$ necessarily contains a number x not divisible by 2, 3 or 5. Then the integers 2, 3, 5 and x are pairwise coprime. This concludes the proof of (2).

We now proceed to show that if $\mathfrak{A}$ is a set of $\mathfrak{A}(1, n)$ integers in $[1, n]$, where $n \geq 150$, then $\mathfrak{A}$ is bad only if it coincides with the set of multiples of 2, 3 and 5 in $[1, n]$. As the desired result clearly follows from Lemma 5 if n is divisible by 30, we may assume $n = 30k+l$, where $k \geq 5$ and $1 \leq l < 30$.

Let $\mathfrak{A}_j$ denote the subset of $\mathfrak{A}$ in $[30j+1, 30j+30]$ $(j = 0, 1, \cdots, k-1)$ and let $\mathfrak{A}^*$ denote the subset of $\mathfrak{A}$ in $[30k+1, 30k+l]$. Suppose $\mathfrak{A}$ is bad. Then we have, by Lemmas 4 and 6, that $|\mathfrak{A}^*| \leq A(1, l)+1$ and $|\mathfrak{A}_j| \leq A(1, 30)$ $(j = 0, \cdots, k-1)$. We have thus two possibllities: either $|\mathfrak{A}^*| = A(1, l)+1$ and $\Sigma_{j=0}^{k-1}|\mathfrak{A}_j| = kA(1, 30)-1$ *or* $|\mathfrak{A}^*| = A(1, l)$ and $|\mathfrak{A}_j^*| = A(1, 30)$ for each $j = 0, \cdots, k-1$. The second possibility gives the desired result since Lemma 5 implies that $\mathfrak{A}$ contains the primes 2, 3 and 5. So we are left with the first possibility. If now $|\mathfrak{A}_0| = A(1, 30)$, then Lemma 1 implies that $\mathfrak{A}$ contains the primes 2, 3, 5. Hence we may assume that $|\mathfrak{A}_0| = A(1, 30)-1$. We shall establish that $\mathfrak{A}$ contains 2 or 3^2 or 2^3 or 2^4, $\mathfrak{A}$ contains 3 or 3^2 or 3^3 and $\mathfrak{A}$ contains 5 or 5^2 by showing that each of the following cases is impossible.

(a) $\mathfrak{A}_0$ does not contain 2 or 2^2 or 2^3 or 2^4.

(b) $\mathfrak{A}_0$ contains 2 or 2^2 or 2^3 or 2^4, but $\mathfrak{A}_0$ does not contain 3 or 3^2 or 3^3.

(c) $\mathfrak{A}_0$ contains 2 or 2^2 or 2^3 or 2^4, $\mathfrak{A}_0$ contains 3 or 3^2 or 3^3, but $\mathfrak{A}_0$ does not contain 5 or 5^2.

Cases (a) and (b) are comparatively trivial for in the former $\mathfrak{A}_0$ must contain four of the nine odd primes less that 30 while in the latter $\mathfrak{A}_0$ must contain four of the ten integers relatively prime to 6 and less than 30. (These ten integers are $1, 5, 5^2, 7, 11, 13, 17, 19, 23, 29$, and clearly it is possible to choose 4 pairwise coprime integers from the set consisting of one of $2, 2^2, 2^3, 2^4$, and any four of the ten integers.) If case (c) holds than $\mathfrak{A}_0$ contains one of the seven integers 1, 11, 13, 17, 19, 23, 29 or $\mathfrak{A}_0$ contains 7. In the first situation $\mathfrak{A}_1$ cannot contain the primes 31, 37, 41, 43, 47, 53, 59 or the integers $35 (= 5 \cdot 7)$ and $49 (= 7^2)$, a total of nine integers; this is not possible since $|\mathfrak{A}_1| = 22$. In the second situation, $\mathfrak{A}_4$ must exclude the primes 127, 131, 137, 139, 149 as well as the integers $125 (= 5^3)$, $145 (= 5 \cdot 29)$, $121 (= 11^2)$ and $143 (= 11 \cdot 13)$, nine integers in all; again this is impossible since $|\mathfrak{A}_4| = 22$.

To conclude we remark that it is possible to establish Theorem 2 for $n \geq 92$ but that the theorem fails to hold for $n = 91$.

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