LATTE POINTS AND LIE GROUPS. I

BY

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ABSTRACT. Assume that $G$ is a compact semisimple Lie group and $\mathfrak{g}$ its associated Lie algebra. It is shown that the number of irreducible representations of $G$ of dimension less than or equal to $n$ is asymptotic to $k n^{a/b}$, where $a = \text{the rank of } \mathfrak{g}$ and $b = \text{the number of positive roots of } \mathfrak{g}$.

Let $G$ be a simple, compact or complex, simply connected Lie group and $\mathfrak{g}$ its associated Lie algebra. If $G$ is compact a representation is a real analytic group homomorphism $f: G \rightarrow \text{GL}(V)$ where $V$ is a complex vector space. If $G$ is complex a representation is a complex analytic group homomorphism $f: G \rightarrow \text{GL}(V)$. In either case $f$ will be called irreducible if $V$ has no nontrivial invariant subspaces under the action of $f(G)$. A homomorphism of Lie groups induces a homomorphism of the associated Lie algebras,

$$f^*: \mathfrak{g} \rightarrow \mathfrak{gl}(V),$$

a Lie algebra representation, and $f^*$ will be called irreducible if $V$ has no nontrivial invariant subspaces under the action of $f^*(\mathfrak{g})$. It is seen from this definition that $f$ is irreducible $\iff f^*$ is irreducible. If $G$ is simply connected a Lie algebra representation of $\mathfrak{g}$ induces a group representation of $G$ and we thus have a bijection between irreducible representations of $G$ and $\mathfrak{g}$. By the dimension of a representation we mean the dimension of $V$. Identifying conjugate representations we ask, "How many irreducible representations of $G$ (or equivalently $\mathfrak{g}$) are of dimension $\leq T$?" The question is simpler when asked of Lie algebras since the structure of the representations is less complex.

$\mathfrak{g}$ is a complex simple Lie algebra if $G$ is a complex simple Lie group or a compact real form of a complex simple Lie algebra when $G$ is a compact simple Lie group. In the latter case there is a bijection between the complex representations of $\mathfrak{g}$ defined over $\mathbb{R}$ and the complex representations of its complexifications, $\mathfrak{g} \otimes \mathbb{C}$, a complex simple Lie algebra so that we need only consider the case of $\mathfrak{g}$ complex and simple.

The root space decomposition of a simple complex Lie algebra is well known...
and is found in [1] and [2]. We let \( \mathfrak{g} \) be a Cartan subalgebra, \( \mathfrak{g}^* \) its dual and
\[ \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{z} \] be the canonical root space decomposition of \( \mathfrak{g} \),
\[ \mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, H \in \mathfrak{g} \}. \]

\( R = \{ \alpha \in \mathfrak{g}^* \mid \mathfrak{g}_\alpha \neq 0 \} \) is called the set of roots. A subset of \( R \), \( \{ \alpha_1, \ldots, \alpha_a \} \), will be called simple if they are linearly independent, span \( \mathfrak{g}^* \) and form an integer basis for \( R \). The dimension of \( \mathfrak{g} = \alpha_0 \) is the rank of \( \mathfrak{g} \).

The Killing form is defined by \( (X, Y) = \text{Tr}(\text{Ad}X \circ \text{Ad}Y) \). Restricted to \( \mathfrak{g} \) it is symmetric and nondegenerate. \( (, \) induces a dual form on \( \mathfrak{g}^* \) so we may speak of \( (\alpha, \beta) \) when \( \alpha \) and \( \beta \) are roots. Further, there are unique vectors \( H_\alpha, H_\beta \in \mathfrak{g} \) such that \( (\alpha, \beta) = \alpha(H_\beta) = \beta(H_\alpha) = (H_\alpha, H_\beta) \).

If \( f^*: \mathfrak{g} \rightarrow \mathfrak{g}(V) \) is a representation it has a weight space decomposition,
\[ V = \bigoplus \lambda \Lambda \lambda, \]
where
\[ V_\lambda = \{ v \neq 0 \mid f^*(H)v = \lambda(H)v, \text{any } H \in \mathfrak{g} \}. \]

If \( f^* \) is finite dimensional it is necessary that
\[ \lambda(H_i) = \lambda(2H_{a_i}/(\alpha_i, \alpha_i)) = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z} \]
for any \( \alpha_i, i = 1, \ldots, a \). If \( f^* \) is irreducible there exists a weight \( \lambda \), called the dominant weight, such that \( \lambda \geq \lambda' \) for any other \( \lambda' \) in \( f^* \) and \( \lambda(H_i) \in \mathbb{Z}^+ \), \( i = 1, \ldots, a \). Furthermore, if \( f^{**} \) is another irreducible representation with \( \lambda \) as dominant weight then \( f^* \) is conjugate to \( f^{**} \). Thus we may identify \( f^* \) with its dominant weight and we will write \( \pi_\lambda \) for \( f^* \). The lattice of dominant weights is \( \mathbb{Z}^+ \lambda_1 \oplus \cdots \oplus \mathbb{Z}^+ \lambda_a \) where \( \lambda_i(H_i) = \delta_{ij} \). The interest of this is that the dimension of \( \pi_\lambda \) is a polynomial in \( \lambda \). By the Weyl character formula
\[ f^*_\mathfrak{g} = \dim \pi_\lambda = \prod_{\alpha > 0} (\lambda + \delta, \alpha)/\prod_{\alpha > 0} (\delta, \alpha) \]
where \( \delta = \frac{1}{2} \Sigma_{\alpha > 0} \alpha \cdot \delta = \Sigma \lambda_i \) [1, p. 257], so if \( \lambda \) belongs to the lattice of dominant weights then \( \lambda + \delta \) belongs to the lattice of dominant weights. If we change coordinates to \( \Lambda = \lambda + \delta = \Sigma \Lambda_i \lambda_i \), where \( \Lambda_i \in \mathbb{R} \), then
\[ \dim \pi_\Lambda = f^*_\mathfrak{g}(\Lambda) = \prod_{\alpha > 0} (\Lambda, \alpha)/\prod_{\alpha > 0} (\delta, \alpha). \]

The number of irreducible representations of \( \mathfrak{g} \) of dimension \( \leq n \) is then equal to the number of lattice points, \( \Lambda \), such that \( \Lambda_i > 0 \) and \( f^*_\mathfrak{g}(\Lambda) \leq n \). We now state

**Theorem.** Let \( G \) be a simply connected, simple, complex or compact Lie group. The number of irreducible representations of \( G \) of dimension \( \leq n \) is asymptotic to \( kn^{a_0/b_0} \), \( b_0 = \) the number of positive roots of \( \mathfrak{g} \).

**Proof.** We first note that \( (\Lambda, \alpha) \) is a linear homogeneous polynomial in the coefficients of \( \Lambda \) since
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\[
\left( \sum_{i=1}^{a} X_i \lambda_i, \sum_{i=1}^{a} m_i \alpha_i \right) = \sum_{i=1}^{a} m_i (\lambda_i, \alpha_i) X_i.
\]

If \( e_1, \cdots, e_a \) is an orthonormal basis of \( \mathbb{R}^a \) and if \( M: \lambda_i \rightarrow e_i \), then if \( M^t \) is the transpose of \( M \) with respect to \( (, ) \)

\[
(\Lambda, \alpha) = (M^{-1} M \Lambda, \alpha) = (M \Lambda, (M^{-1})^t \alpha)
\]

and \( M \Lambda \) lies in the regular integer lattice in \( \mathbb{R}^a \). Thus if \( L = \sum_{i=1}^{a} X_i e_i, \ X_i > 0, \)

and

\[
\frac{\alpha(L)}{\alpha} = \frac{\alpha(M^{-1} !a \alpha)}{\alpha(M \delta, M^{-1})^t \alpha}
\]

then \( f(L) = f(\sum_{i=1}^{a} X_i e_i) = f(\sum_{i=1}^{a} X_i \lambda_i) \) so we may regard \( f \) as having asymptotes \( e_i = 0 \) and the lattice of weights as the ordinary integer lattice. We now prove a lemma on homogeneous functions.

**Lemma 1.** Let \( f \) be a homogeneous function on \( \mathbb{R}^a \) of degree \( b \) which is the product of linear forms \( \sum m_i x_i, \ m_i \geq 0 \). If \( f = 0 \) on the planes \( x_i = 0, \ i = 1, \cdots, a \), and if

\[
S(1) = \{ x \in \mathbb{R}^a | f(x) \leq 1, x_i \geq 0 \}
\]

has finite volume then the number of lattice points in

\[
S(r) = \{ x \in \mathbb{R}^a | f(x) \leq r, x_i \geq 0 \}
\]

is asymptotic to \( \text{Vol}(S(1)) r^{a/b} \).

**Proof.** It is clear that the volume of \( S(r) = \text{Vol}(S(1)) r^{a/b} \). If \( x \in S(r) \) then

\[
f(x/(r^{1/b})) = (r^{-1/b} x)(x) = r^{-1} f(x) \leq 1.
\]

Since we are in \( \mathbb{R}^a \) the Jacobian of the coordinate change \( x \rightarrow ax \) is \( a^a \) so \( \text{Vol}(S(r)) = r^{a/b} \text{Vol}(S(1)) \). We will be done if the number of lattice points in \( S(r) \sim \text{Vol}(S(r)) \). To see this, draw a unit \( a \)-cube at every lattice point of \( S(r), w, \) with vertices at \( w, w + e_i \) any \( i \). Call the union of these cubes \( L(r) \); a set which will contain \( S(r) \cap \{ x_i \geq 1 \ \text{all} \ i \} \) since \( f \) will be increasing in each coordinate.

Now at each lattice point, \( w \), draw a unit cube with vertices \( w, w - e_i \) any \( i \). Call the union of these cubes \( L(r) \). \( L(r) \subset S(r) \) and \( \text{Vol} L(r) = \text{Vol} L(r) \). Call \( E(r) = S(r) \cap \{ x_i \leq 1 \ \text{some} \ i \} \). Then

\[
L(r) \subset S(r) \subset L(r) \cup E(r)
\]

which implies \( |\text{Vol} S(r) - \text{the number of lattice points}| \leq \text{Vol} E(r) \). However

\[
\text{Vol} E(r) = r^{a/b} \text{Vol}\{ x \in S(1) | x_i \leq r^{-1/b} \ \text{some} \ i \}
\]

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and since $\text{Vol } S(1) < \infty$, the volume of this latter set $\to 0$ by dominated convergence. Thus $\text{Vol } E(r)$ is $o(\text{Vol } S(r))$ and the number of lattice points in $S(r)$ is asymptotic to $\text{Vol } S(r)$.

We now have a criterion we would like to apply to the polynomials $f_{ij}$. A canonical example is the algebra $A_2$. The positive roots of $A_2$ are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ and the polynomial $f_{A_2}^0(x, y) = kxy(x + y)$. We wish to show

$$\text{Vol} \{x, y | x > 0, y > 0, kxy(x + y) \leq 1\} < \infty$$

or equivalently $\text{Vol } A < \infty$ where

$$A = \{x, y | x > 0, y > 0, xy(x + y) \leq 1\}.$$

We divide $A$ into two subsets, $A_x = A \cap \{x \geq y\}$, $A_y = A \cap \{x \leq y\}$. If $(x, y) \in A_x$, $xy(x + y) \leq 1$ which implies $x^2y \leq 1$.

$$A_x \subset \{(x, y) | x > y > 0, x^2y \leq 1\}.$$

$\text{Vol } A_x \cap \{x \in [0, 1]\} \leq \frac{1}{2}$ so $\text{Vol } A_x$ is finite if

$$\text{Vol} \{(x, y) | x > y, x > 1, x^2y \leq 1\} < \infty.$$

The volume of this set is $\int_1^\infty x^{-2} \, dx = 1$ so $\text{Vol } A_x \leq 3/2$. Similarly, $\text{Vol } A_y \leq 3/2$ so $\text{Vol } A < 3$ and the theorem is true for the algebra $A_2$. We now extend this method to higher dimensions.

**Lemma 2.** In $\mathbb{R}^a$ let $f(x)$ be a sum of monomials of degree $b$. If for every permutation $i$ of $\{1, \ldots, a\}$ there exists in $f(x)$ a monomial $x_{i(1)}^{s_1} \cdots x_{i(a)}^{s_a}$ where $s_1 > \ldots > s_a > 0$, then the volume of the set $S(1) = \{x | f(x) \leq 1, x_i \geq 0\}$ is finite.

**Remark.** From Lemma 1 this implies $\text{Vol } S(r) = \text{Vol } S(1) r^{n/b}$.

**Proof of Lemma 2.** We proceed by induction. If $a = 2$ we have monomials $x_1^{s_1}x_2^{s_2}$ and $x_1^{s_1'}x_2^{s_2'}, s_1 > s_2, s_1' > s_2'$. Again partitioning $S(1)$ into $A_x$ and $A_y$ we see

$$\text{Vol } A_x \leq \frac{1}{2} + \int_1^\infty x^{-s_1/s_2}.$$  

$$= \frac{1}{2} + (s_1/s_2 - 1)^{-1} < \infty$$  

since $s_1 > s_2$.

Similarly $\text{Vol } A_y \leq \frac{1}{2} + (s_1'/s_2' - 1)^{-1}$.

Now assume the lemma true for $a - 1$. Partition $S(1)$ into the sets

$$A_{i_1, \ldots, i_a} = S(1) \cap \{x_i \geq \cdots \geq x_1\}.$$

We wish to show $\text{Vol } A_{i_1, \ldots, i_a} < \infty$ for any $i$. As before
A_1, \ldots, A_n \subseteq \{ x \mid x_1 \geq \cdots \geq x_{i_1}, x_{i_1}^{s_1} \cdots x_{i_a}^{s_a} \leq 1 \}.

If x_{i_1} \geq 1 a cross-section of this set at x_{i_1} is the set

\{(x_{i_2}, \ldots, x_{i_a}) \mid x_{i_2} \geq \cdots \geq x_{i_a} \geq 0, x_{i_2}^{s_2} \cdots x_{i_a}^{s_a} \leq 1/x_{i_1}^{s_1} \}.

By induction and the previous remark the volume of the cross-section = kx_{i_1}^{-y}

where \( y = s_1(a-1)/(\sum_{i=2}^{a} s_i) \). The volume of

\[ A_1, \ldots, A_n \subseteq \text{Vol}(A_1, \ldots, A_n \cap \{ x_1 \in [0, 1] \}) + \int_1^\infty y^{-y} \, dy. \]

The first set is contained in the unit cube so it has volume \( \leq 1 \) and the integral is

finite as long as \( y \geq 1 \). But \( s_1 > s_i \forall i > 1 \) so \( (a-1)s_1 > \sum_{i=2}^{a} s_i \Rightarrow y > 1 \). □

The proof of Theorem 1 will be complete if we show the criterion of Lemma

2 applies to the polynomials \( f_0 \) for all simple complex Lie algebras.

If \( \Lambda = \sum_{i=1}^{a} x_i \lambda_i \) then for each \( \alpha = \sum_{i=1}^{a} m_i \alpha_i \)

\( (\Lambda, \alpha) = \sum_{i=1}^{a} m_i (\lambda_i, \alpha_i) x_i. \)

Thus to determine \( f \) we must list all the positive roots of \( \mathfrak{g} \) in terms of the

simple roots. We begin with the \( A_n \) algebras.

**Lemma 3.** The monomial \( x_1^{s(1)} \cdots x_n^{s(n)} \) is found in the expansion of \( f_{A_n} \)

for every permutation \( s \) of \( (1, \ldots, n) \).

**Proof.** By referring to Serre [2] the positive roots of \( A_n \) are \( \alpha_1, \ldots, \alpha_n; \alpha_1 + \alpha_2, \ldots, \alpha_{n-1} + \alpha_n, \ldots; \alpha_1 + \cdots + \alpha_n \). Since \( (\lambda_1, \alpha_i) = c_i f_{A_n} = kx_1 \cdots x_n (x_1 + x_2) \cdots (x_{n-1} + x_n) \cdots (x_1 + \cdots + x_n) \). We now apply induction. If \( n = 2 \), \( f_{A_2} = x_1^2 x_2 + x_1 x_2^2 \). Now assume the lemma for \( n - 1 \). We write

\[ f_{A_n} = x_n (x_n + x_{n-1}) \cdots (x_1 + \cdots + x_n) f_{A_{n-1}}. \]

Pick an arbitrary permutation \( s \).

Then \( s(n) = j \). By induction \( x_1^{s(1)} \cdots x_{n-1}^{s(n-1)} \) occurs in \( f_{A_{n-1}} \) where

\[ s(i)' = \begin{cases} s(i) & \text{if } s(i) < j, \\ s(i) - 1 & \text{if } s(i) > j. \end{cases} \]

Multiply this monomial by \( x_n \) in the first \( j \) factors \( x_n, \ldots, (x_n + \cdots + x_{n+i-1}). \)

Now pick the least \( i \) such that \( s(i)' < s(i) \). Multiply the monomial by \( x_i \) in

\( (x_1 + \cdots + x_n) \). Then pick the next \( i' \) such that \( s(i') < s(i) \) and multiply

by \( x_{i'} \) in \( (x_2 + \cdots + x_n) \). Since \( i' > i \Rightarrow i' \geq 2, X_{i'} \) is found in

\( (x_2 + \cdots + x_n) \). We may thus continue until we have \( X_1^{s(1)} \cdots x_n^{s(n)} \). □

**Remark.** The degree of \( f_{A_n} \) is minimal such that we may find monomials
\[ X_{s(1)}^{s_1} \cdots X_{s(n)}^{s_n} \text{ where } s_1 > \cdots > s_n > 0 \text{ since } s_n \geq 1, s_{n-1} \geq 2, \ldots, s_1 \geq n \text{ so that the degree of } f = \sum_{i=1}^{n} s_i > \sum_{i=1}^{n} i = \text{the degree of } f_{A_n} \]

**Lemma 4.** The monomial \( X_{s(1)}^{2s(1)-1} \cdots X_{s(n)}^{2s(n)-1} \) is found in the polynomials \( f_{B_n} \) and \( f_{C_n} \) for any permutation \( s \).

**Proof.** The positive roots of \( B_n \) are \( \alpha_1, \ldots, \alpha_n; \alpha_1 + \alpha_2, \ldots, \alpha_{n-1} + \alpha_n; \ldots; \alpha_1 + \cdots + \alpha_n \) and \( \alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n \) where \( i < j \leq n \) [2]. \( f_{B_n} = k_f A_n \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (X_i + \cdots + X_{j-1} + 2X_j + \cdots + 2X_n) \). From Lemma 3 we know the monomial \( X_{s(1)}^{s(1)} \cdots X_{s(n)}^{s(n)} \) is in \( f_{A_n} \). We wish then to show that \( X_{s(1)}^{2s(1)-1} \cdots X_{s(n)}^{2s(n)-1} \) where \( s(j) = 1 \) lies in

\[
\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (X_i + \cdots + X_{j-1} + 2X_j + \cdots + 2X_n).
\]

We proceed as follows. There are \( n-1 \) factors containing \( X_1 \). \( s(1) - 1 \leq n - 1 \) so we may choose \( X_1 \) in \( s(1) - 1 \) of these factors. There are \((n-1)+(n-2)\) factors containing \( X_2 \) and

\[
(s(1) - 1) + (s(2) - 1) \leq (n-1) + (n-2)
\]

so choose \( X_2 \) in the next \( s(2) - 1 \) factors. Thus we may proceed at each stage being able to choose \( s(i) - 1 \) \( X_i \)'s. Multiplying we have the monomial \( X_{s(1)}^{2s(1)-1}X_{s(2)}^{2s(2)-1} \cdots X_{s(n)}^{2s(n)-1} \).

For \( C_n \) the positive roots are \( \alpha_1, \ldots, \alpha_n; \alpha_1 + \alpha_2, \ldots, \alpha_{n-1} + \alpha_n; \cdots; \alpha_1 + \cdots + \alpha_n; \alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n, i < n, i \leq j \leq n - 1. \)

The roots are different from \( B_n \) but contain the same \( \alpha_i \) so the argument is the same. □

**Lemma 5.** \( f_{D_n} \) contains monomials of descending degrees for \( n \geq 6 \).

**Proof.** Referring to Serre the positive roots of \( D_n \) are \( \alpha_1, \ldots, \alpha_{n-1}; \alpha_1 + \alpha_2, \ldots, \alpha_{n-2} + \alpha_{n-1}, \cdots; \alpha_1 + \cdots + \alpha_{n-1}; \alpha_{n-2} + \alpha_{n-1}, \cdots; \alpha_1 + \cdots + \alpha_n; \alpha_1 + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_n \). We may write

\[
f_{D_n} = k f_{A_n} (X_{n-2} + X_n)(X_{n-3} + X_{n-2} + X_n) \cdots (X_1 + \cdots + X_{n-2} + X_n)
\]

\[ \cdots X_n (X_{n-2} + X_{n-1} + X_n) \cdots (X_1 + \cdots + X_n) \]

\[ \cdot \prod_{i=1}^{n-3} \prod_{j=i+1}^{n-2} (X_i + \cdots + 2X_j + \cdots + 2X_{n-2} + X_{n-1} + X_n), \]

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The bracketed expression is what is needed along with $f_{A_{n-1}}$ to create $f_{A_n}$ except for the missing factor $(X_{n-1} + X_n)$. We compensate by adding the term $(X_{n-3} + 2X_{n-2} + X_{n-1} + X_n)$ to create a function containing every monomial of $f_{A_n}$. The remaining terms we write as

$$g_{D_n} = \prod_{i=1}^{n-2} (X_i + \cdots + X_{n-2} + X_n)$$

$$\prod_{i=1}^{n-4} \prod_{j=1}^{n-2} (X_i + \cdots + 2X_j + \cdots + 2X_{n-2} + X_{n-1} + X_n).$$

We know from Lemma 3 that $X_{s(1)}^1 \cdots X_{s(n)}^n$ is found in $f_{A_n}$ for any permutation $s$. We wish to produce a monomial with descending degrees in the $X_{s(i)}$ in $g_{D_n}$ for any permutation $s$. There are two cases. First assume that $s(1) \neq n - 1$.

Then we will be done if the monomial

$$X_{s(n)}^{n-2} \cdots X_{s(6)}^4 X_{s(5)}^2 X_{s(4)} X_{s(3)} X_{s(2)}$$

is in $g_{D_n}$. First choose $n - 2$ different $X_i$ from $\sum_{i=1}^{n-2} (X_i + \cdots + X_{n-2} + X_n), \quad i \neq n - 1, s(1)$.

We then proceed to the second factor. There are $n - 3$ terms containing $X_1$ so if $s(j) = 1$ we may pick $X_1$ in $j - 3$ terms. Mimicking Lemma 4 we may continue by picking $j' - 3 X_{n-1}$'s; we have our starting point. The sole difference in the procedure will be that if $j \in (1, 2, 3, 4)$ we choose no $X_{s(j)}$'s. After $X_{n-2}$ every term contains $X_{n-1}$ and $X_n$ so we may arbitrarily choose $k - 2 X_{n-1}$'s and $k' - 3 X_n$'s; $s(k) = n - 1, s(k') = n$. We have thus produced the desired monomial belonging to $g_{D_n}$ and multiplying by $X_{s(1)}^1 \cdots X_{s(n)}^n$ we have a monomial with strictly decreasing degrees.

If $n - 1 = s(1)$ we will be done if

$$X_{s(n)}^{n-3} X_{s(n-1)}^{n-3} \cdots X_{s(6)}^4 X_{s(5)}^2 X_{s(4)} X_{s(3)} X_{s(2)} X_{s(1)}$$

is in $g_{D_n}$. First pick $\{X_{s(n-1)}, \ldots, X_{s(2)}\}$ in $\prod_{i=1}^{n-2} (X_i + \cdots + X_{n-2} + X_n)$. Then proceed as before choosing $j - 3 X_1$'s, $j' - 3 X_2$'s and so on again skipping $X_{s(1)}$, $\ldots, X_{s(4)}$. Proceed to $X_{n-2}$ and then to $X_n$. There will be one remaining term which a priori contains $X_{n-1}$. Multiplying by $X_{n-1}$ from this factor we produce our monomial.

We have proved Theorem 1 for $A_n, B_n, C_n$ and $D_n$ for $n \geq 6$. These are all the complex simple Lie algebras except for the algebras $G_2, F_4, D_4, D_5, E_6, E_7$, and $E_8$. In these cases the conditions of Lemma 2 may be verified directly.

We now summarize the results:
We now extend our results to semisimple Lie algebras.

**Corollary.** Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, with $\mathfrak{g}_i$ the simple components. If $c_{\mathfrak{g}_1} = \cdots = c_{\mathfrak{g}_s} > c_{\mathfrak{g}_{s+1}} \geq \cdots \geq c_{\mathfrak{g}_n}$, then the number of irreducible representations of $\mathfrak{g}$ of dimension less than or equal to $T$ is asymptotic to $kT^{c_{\mathfrak{g}_1}} \log^{s-1} T$.

**Proof.** We first assume that $\mathfrak{g}$ has two simple factors, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The irreducible representations of $\mathfrak{g}$ are tensor products of irreducible representations of the simple factors and the dimension of the tensor representation is a product of the dimensions of the factor representations. The number of irreducible representations of $\mathfrak{g}$ of dimension $\leq r$ is $b(r) = \sum_{m,n \in \mathbb{Z}^+} M_1(m)M_2(n)$, where $M_i(x)$ is the number of irreducible representations of $\mathfrak{g}_i$ of dimension $x$.

We partition $S = \{x, y | xy \leq r, x, y \geq 0\}$ into $S_x = S \cap \{x \in [0, r^{1/2}]\}$, $S_y = S \cap \{y \in [0, r^{1/2}]\}$. $S = S_x \cup S_y$, so if we estimate both $b_x(r) = \sum_{m,n \in \mathbb{Z}^+} M_1(m)M_2(n)$ and $b_y(r) = \sum_{m,n \in \mathbb{Z}^+} M_1(m)M_2(n)$ asymptotically, then $b(r) \sim \max(b_x(r), b_y(r))$.

Assume $c_{\mathfrak{g}_1} > c_{\mathfrak{g}_2}$ (for brevity); we will deal with $c_1 = c_2$ later. Theorem 1 states $\sum_{j=1}^{n} M_1(j) \sim \mu_n c_i$. Thus

$$b_x(r) = \sum_{i=1}^{[r/2]} M_1(i) \sum_{j=1}^{[r/i]} M_2(j).$$

For any $\epsilon$ there exists $r_2$ such that

$$\left| \left( \sum_{j=1}^{L} M_2(j) - \mu_2 L^{c_2} \right) / \left( \sum_{j=1}^{L} M_2(j) \right) \right| < \epsilon \text{ any } L \geq r_2.$$
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Then

\[ b_x(r) = \mu^2 \epsilon^2 \sum_{i=1}^{[\epsilon^{1/2}]} M_1(i)/i^2 \epsilon^2 + \epsilon' b_x(r) \]

where \(|\epsilon'| < \epsilon\) if \(r > r_2^2\). Thus

\[ b_x(r) \sim \mu^2 \epsilon^2 \sum_{i=1}^{[\epsilon^{1/2}]} M_1(i)/i^2 \epsilon^2. \]

By the Abel summation formula

\[
\sum_{i=1}^{[\epsilon^{1/2}]} M_1(i)/i^2 \epsilon^2 = \sum_{i=1}^{[\epsilon^{1/2}]} \left( \sum_{j=1}^{i} M_1(j) \right) \left( 1/i^2 \epsilon^2 - 1/(i+1)^2 \epsilon^2 \right) + \sum_{i=1}^{[\epsilon^{1/2}]} M_1(i) \cdot r^{-c_2/2}.
\]

Now \(c_2/i^3 + 1 > 1/i^3 - 1/(i+1)^3 > c_2/(i+1)^3 + 1\), so

\[
\sum_{i=1}^{[\epsilon^{1/2}]} \left( \sum_{j=1}^{i} M_1(j) \right) \left( 1/i^3 - 1/(i+1)^3 \right) > \sum_{i=1}^{[\epsilon^{1/2}]} \left( \sum_{j=1}^{i} M_1(j) \right) \cdot c_2/(i+1)^3 + 1.
\]

For any \(\epsilon > 0\) there exists \(r_1\) such that

\[
\left| \left( \sum_{j=1}^{L} M_1(j) - \mu_1 L \epsilon^3 \right)/\sum_{j=1}^{L} M_1(j) \right| < \epsilon \quad \text{any } L \geq r_1.
\]

If \(r \gg r_1^2, r_2^2\)

\[
\sum_{i=1}^{[\epsilon^{1/2}]} \left( \sum_{j=1}^{i} M_1(j) \right) \cdot c_2/i^3 + 1 = \sum_{i=1}^{[\epsilon^{1/2}]} \mu_1 c_2^i c_1 - c_2^{-1} + E + A
\]

where

\[
|E| < \epsilon \sum_{i=r_1}^{[\epsilon^{1/2}]} \left( \sum_{j=1}^{i} M_1(j) \right) \cdot c_2/i^3 + 1
\]

and

\[
A = \sum_{i=1}^{r_1} \left( \sum_{j=1}^{i} M_1(j) - \mu_1 i \epsilon^3 \right) \cdot c_2/i^3 + 1.
\]

\[
\sum_{i=1}^{[\epsilon^{1/2}]} \mu_1 c_2^i c_1 - c_2^{-1} \sim \mu_1 c_2 \int_{1}^{[\epsilon^{1/2}]} x c_1 - c_2^{-1} \, dx
\]

\[
= \mu_1 c_2/(c_1 - c_2) x c_1 - c_2 \left|_{1}^{[\epsilon^{1/2}]} \right. - k r^{(c_1 - c_2)/2} + k'.
\]
Also
\[ \sum_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_1(i) \cdot r^{-c_2/2} = k_0 r^{(c_1-c_2)/2} + E' \]
where \(|E'| < \epsilon \sum_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_1(i) \cdot r^{-c_2/2} \). Thus
\[ \sum_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_1(i)/i^{c_2} = (k + k_0) r^{c_1-c_2/2} + (k' + A) + (E + E'). \]

From this
\[ (1 + 2e) b_x(r) > (k + k') r^{c_1+c_2/2} + (k' + A) r^{c_2} > (1 - 2e) h(r). \]

Thus \( b_x(r) \sim c_2 r^{c_2} + c_r r^{c_1+c_2} \). Similarly \( b_y(r) \sim \mu_1 r^{c_1} \sum_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_2(i)/i^{c_1} \). But in this case \( \sum_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_2(i)/i^{c_1} \) is asymptotic to a constant. To see this
\[ \sum_{j=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_2(j)/i^{c_1} = \sum_{j=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} \left( \sum_{j=1}^{i} M_2(j) \right) \left( 1/i^{c_1} - 1/(i+1)^{c_1} \right) + \sum_{j=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_2(j) r^{-c_1/2}. \]

\[ \Sigma_j M_2(j) \text{ is } O(x^{c_2}) \text{ and } (1/i^{c_1} - 1/(i+1)^{c_1}) < c_1/i^{c_1+1}, \text{ so} \]
\[ \sum_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_2(i)/i^{c_2} \leq k \int_1^{\frac{\sqrt{y}}{2}} x^{c_2-c_1-1} dx + k_0 r^{c_2-c_1/2} \]
\[ = k/(c_1 - c_2) (1 - r^{c_2-c_1/2}) + k_0 r^{c_2-c_1/2}. \]

But \( c_2 - c_1 < 0 \) so the above sum is \( \leq 2k/(c_1 - c_2) \) if \( r \) is sufficiently large and \( \lim_{r \to \infty} \Sigma_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_2(i)/i^{c_1} \text{ exists and is equal to } k'. \) Thus \( b_y(r) \sim k' r^{c_1} \) and \( b(r) \sim b_y(r) \).

This settles the case of \( \mathfrak{c} = \bigoplus_{i=1}^n \mathfrak{c}_i \) where \( c_1 > c_i, i > 1 \). By the above argument \( \mathfrak{c}_1 \oplus \mathfrak{c}_2 \) has asymptotically \( k' r^{c_1} \) irreducible representations of dimension \( \leq n \). By iteration \( (\mathfrak{c}_1 \oplus \mathfrak{c}_2) \oplus \mathfrak{c}_3 \) still has \( \sim k' r^{c_1} \) irreducible representations and so on. This leaves the case of \( c_1 = \cdots = c_s \). Let \( \mathfrak{c} = \mathfrak{c}_1 \oplus \mathfrak{c}_2 \). Tracing the argument for \( c_1 \neq c_2 \) nothing is changed until we arrive at
\[ \int_1^{\lfloor \frac{\sqrt{y}}{2} \rfloor} x^{c_1-c_2-1} dx \text{. This integral now equals } \int_1^{\lfloor \frac{\sqrt{y}}{2} \rfloor} x^{-1} dx = \frac{1}{2} \log r \text{ so that} \]
\[ b_x(r) \sim k r^{c_1} \log r \text{ and } b_y(r) \sim k' r^{c_1} \log r \text{. Now} \]
\[ b(r) = b_x(r) + b_y(r) = \sum_{i,j \in S_x \cap S_y} M_1(i) M_2(j) \]
and the latter sum equals
\[ \sum_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_1(i) M_2(j) = \sum_{i=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_1(i) \cdot \sum_{j=1}^{\lfloor \frac{\sqrt{y}}{2} \rfloor} M_2(j). \]
which is $O(r^{c_1})$ so that $b(r) \sim kr^{c_1} \log r$. Taking $\mathfrak{g} = (\mathfrak{g}_1 \oplus \mathfrak{g}_2) \oplus \mathfrak{g}_3$ we arrive at the integral

$$\int_1^{r^{\frac{1}{2}}} \frac{\log x}{x} \, dx = \frac{1}{8} \log^2 r.$$

So $b_x(r) \sim kr^{c_1} \log^2 r$, $b_y(r) \sim k' r^{c_1} \log^2 r$, $\sum_{i,j \in S_x \cap S_y} M_1(i) M_2(j)$ is $O(r^{c_1} \log r)$ and $b(r) \sim k_0 r^{c_1} \log^2 r$. Continuing to the case $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i$ we have $b(r) \sim kr^{c_1} \log^{s-1} r$ and our corollary is proven. □

BIBLIOGRAPHY


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