FINITE GROUPS WITH NICE SUPPLEMENTED SYLOW NORMALIZERS

BY

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ABSTRACT. This paper considers finite groups G whose Sylow normalizers are supplemented by groups D having a cyclic Hall 2'-subgroup. G is solvable and all odd order composition factors of G are cyclic. If S ∈ Syl_2^*(D) is cyclic, dihedral, semidihedral, or generalized quaternion, then G is almost supersolvable.

Let \( \mathcal{D} \) denote the class of finite groups D which satisfy:
\[(\ast) \quad D = ST, \quad \text{where } S \in \text{Syl}_2(D) \text{ and } T \text{ is cyclic group of odd order.}\]

We say G is \( \mathcal{D} \)-supplemented if G is finite and every Sylow normalizer in G has a supplement \( D \in \mathcal{D} \).

Theorem 1. \( \mathcal{D} \)-supplemented groups are solvable.

Proof. Assume the theorem is false, and let G be a counterexample of minimal order. Since any homomorphic image of G is \( \mathcal{D} \)-supplemented, G/N is solvable for any 1 ≠ N ◁ G. Thus, G has a unique minimal normal subgroup M. M is nonsolvable, and so 2 divides |M| by the Feit-Thompson Theorem. Choose \( P \in \text{Syl}_2(M) \) and \( Q \in \text{Syl}_2(G) \), \( P ≤ Q \). By the Frattini argument \( G = MN_P \). Let \( D ∈ \mathcal{D} \) be a supplement for \( N_P \). Since \( Q ∈ \text{Syl}_2(G) \), we can assume \( D \) is cyclic of odd order. Choose a subgroup \( H ≥ N(P) \) which is maximal in G. Since \( N(P) ≥ N(Q) \), \( D \) is a supplement for \( H \). \((D ∩ H)^G = (D ∩ H)^H ≤ H\). If \( D ∩ H ≠ 1 \), then \( M ≤ (D ∩ H)^G ≤ H \), a contradiction. Consequently, \( N(P) = N(Q) \) is maximal in G, and \( D \) is a complement for \( N(P) \). G has a faithful primitive representation on the \( d = |D| \) cosets of \( N(P) \), and \( D \) is regularly represented. If \( d \) is not prime, then \( D \) is a B-group \([8, 25.2] \), and so G is 2-transitive. Otherwise \( d \) is prime, and G is 2-transitive by a theorem of Burnside.

Recent results of Shult and O'Nan classify 2-transitive groups H in which \( H_a \) is a 2-local subgroup. If \( T = O_2(H_a) \) is semiregular on \( Ω = |Ω| \), then Shult's Fusion Theorem (see [5]) implies that H has a regular normal subgroup, or \( N ≤ H ≤ Aut(N) \), where N is isomorphic to PSL_2(2^a), PSU_3(2^a), or Sz(2^{2a+1}) in its standard 2-transitive permutation representation. (We need Shult’s result only in the case \( O_2(G_a) ∈ \text{Syl}_2(G) \). This special case follows from Suzuki’s work.

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on finite groups with independent Sylow 2-subgroups [7].) If \( T \) is not semiregular, then work of O'Nan [6] implies that \( H \) has a regular normal subgroup or \( N \trianglelefteq H \leq \text{Aut}(N) \), where \( N \simeq \text{PSL}_n(2^a) \). Since \( G \) has no regular normal subgroup and 
\( O_2(G) \) is a Sylow 2-subgroup of \( G \), the only possibility is \( N \trianglelefteq G \leq \text{Aut}(N) \), where \( N \simeq \text{PSL}_2(2^a), \text{PSU}_3(2^a), \text{or Sz}(2^{2a+1}) \). In these cases one easily finds a prime \( p \) and \( S \in \text{Syl}_p(G) \) so that \( N(S) \) has no supplement \( D \in \mathcal{D} \). For example, if \( G \simeq \text{PSL}_2(4) \) take \( p = 3 \), and if \( G \simeq \text{PGL}_2(4) \) take \( p = 2 \).

Remark. If \( G \simeq \text{PSL}_2(2^a) \) and \( S \in \text{Syl}_2(G) \), then \( N(S) \) has a cyclic complement of odd order.

**Theorem 2.** If \( G \) is \( \mathcal{D} \)-supplemented then every chief factor of \( G \) of odd order is cyclic.

**Proof.** Let \( G \) be a counterexample of minimal order. A result of Huppert [4, VI. 8.6] implies that \( \Phi(G) = 1 \). \( G \) has a unique minimal normal subgroup \( M \). Since \( G \) is solvable, \( M \) is an elementary abelian \( p \)-group. \( p \) is odd. Set \( P = O_p(G) \). \( P \) is elementary abelian since \( \Phi(P) \leq \Phi(G) = 1 \).

There is a prime \( q \neq p \) and a \( q \)-group \( 1 \neq Q \subset G \) so that \( PQ \trianglelefteq G \). \( P = [P, Q] \times C_P(Q) \). Since \( [P, Q] \neq 1 \) and \( C_P(Q) \) and \( [P, Q] \) are normal in \( G \), \( C_P(Q) = 1 \). \( G \) is a split extension of \( P \) by \( N(Q) \). If \( Q \trianglelefteq Q_1 \in \text{Syl}_q(G) \), then 
\( N(Q) \geq N(Q_1) \). Consequently, \( N(Q) \) has a supplement \( D \in \mathcal{D} \). \( D \) contains an element \( x \) of order \( p^m = |P| \). The image \( \bar{x} \) of \( x \) in \( \bar{G} = G/P \) has order at least \( p^{m-1} \). Since \( \bar{G} \) is isomorphic to a subgroup of \( \text{GL}_m(p) \), \( pm > p^{m-1} \). Hence, \( m = 2 \). \( G \) contains an element of order \( p^2 \), and so \( p \) divides \( |\bar{G}| \). But \( O_p(\bar{G}) = 1 \) and \( \bar{G} \) is solvable. The only possibility is \( p = 3 \) and \( \bar{G} \simeq \text{SL}_2(3) \) or \( \text{GL}_2(3) \). Then the normalizer of \( S \in \text{Syl}_2(G) \) has index 9 or 27 in \( G \). However, \( G \) contains no elements of order 9, a final contradiction.

Let \( \mathcal{D}^* \) denote the class of finite groups \( D \) which are the product of a cyclic group \( T \) of odd order and a cyclic, dihedral, semidihedral, or generalized quaternion 2-group \( S \). \( T \) is a Hall 2'-subgroup of \( D \) and \( S \in \text{Syl}_2(D) \). \( D \in \mathcal{D}^* \) implies \( D \in \mathcal{D} \), so that \( \mathcal{D}^* \)-supplemented groups are solvable. Buchthal [1] has shown that certain solvable \( \mathcal{D}^* \)-supplemented groups are either supersolvable or have \( \Sigma_4 \) as a homomorphic image.

**Theorem 3.** If \( G \) is \( \mathcal{D}^* \)-supplemented, then \( G \) contains a normal subgroup \( N \) such that every \( G \)-composition factor of \( N \) is cyclic and \( G/N \) is isomorphic to 1, \( A_4, \Sigma_4, \) or one of the groups \( \Gamma_1, \Gamma_2, \Gamma_3 \) defined below.

The group \( \Gamma_1 \) is defined as follows. Let \( W \) be an elementary abelian group of order 16. Choose \( g \in \text{Aut}(W) \) so that \( |g| = 3 \) and \( C_w(g) = 1 \). Let \( S \) be a Sylow 2-subgroup of \( N_{\text{Aut}(W)}(\langle g \rangle) \simeq \Gamma L_2(4) \). \( S \) and \( g \) generate a group \( X \) of order 24.
Define $\Gamma_1$ to be the split extension of $W$ by $X$. The normalizer $N(R)$ of $R \in \text{Syl}_3(\Gamma_1)$ has index 16 in $\Gamma_1$. The only supplements $D \in \mathcal{D}^*$ for $N(R)$ are semidihedral or generalized quaternion groups of order 16. (These facts are established in the proof of Theorem 3.)

Suppose $W \cong \mathbb{Z}_4 \times \mathbb{Z}_4$. Let $a$ and $b$ be generators of $W$. Define automorphisms $g$, $x$, $z$, and $s$ of $W$ as follows.

1. $a^g = b^{-1}$, $b^g = ab^{-1}$,
2. $a^z = a^{-1}$, $b^z = b^{-1}$,
3. $a^x = ab^2$, $b^x = a^2b^{-1}$,
4. $a^s = b$, $b^s = a$.

The element $g \in \text{Aut}(W)$ has order 3, while $x$, $z$, and $s$ are involutions.

$C_{\text{Aut}(W)}(g) = \langle g, x, z \rangle$ and $N_{\text{Aut}(W)}(\langle g \rangle) = \langle g, x, z, s \rangle = X$. $\Gamma_2$ is the split extension of $W$ by $X$. $S = \langle a, b, x, z, s \rangle$ is a Sylow 2-subgroup of $\Gamma_2$. $S$ contains no elements of order 16, and every element of order 8 in $S$ is conjugate to $sa$.

$N_S(\langle sa \rangle)$ is a split extension of $(sa)$ by the 4-group $(zb, a^2)$. $(sa, zb)$ and $(sa, a^2)$ are complements for $N_S(\langle g \rangle)$ in $S$, while $(sa, zba) \cap N_S(\langle g \rangle) = \langle sz \rangle$. Also, $(sa, zb)$ is semidihedral, and $(sa, a^2)$ is neither dihedral nor semidihedral. These facts yield the following result.

**Lemma 1.** $\Gamma_2$ is $\mathcal{D}^*$-supplemented. Any proper subgroup of $\Gamma_2$ which contains $(a, b, g)$ and is $\mathcal{D}^*$-supplemented is conjugate in $\Gamma_2$ to $\langle a, b, g, z, s \rangle$. Moreover, if $\Gamma = \Gamma_2$ or $\Gamma_3$ and $R \in \text{Syl}_3(\Gamma)$, then $N(R)$ has index 16 in $\Gamma$ and the only supplements $D \in \mathcal{D}^*$ for $N(R)$ are semidihedral groups of order 16.

**Proof of Theorem 3.** In the following discussion, $\Gamma$ denotes any one of the groups $\Gamma_1$, $\Gamma_2$, or $\Gamma_3$.

Let $G$ be a counterexample of minimal order. Choose $N \trianglelefteq G$ of minimal order so that $G/N \cong 1$, $A_4$, $\Sigma_q$, or $\Gamma$. (E.g., if $G$ has both $\Sigma_4$ and $\Gamma$ as homomorphic images, choose $N$ such that $G/N \cong \Gamma$.) $N$ contains a unique minimal normal subgroup $M$ of $G$. $M$ is not cyclic. Theorem 2 implies that $M$ is a 2-group. Set $P = O_2(N)$. $C_N(P) \subseteq P$ and $O_2(M) = 1$. Suppose $\Phi(P) \neq 1$. Then by induction each $G$-composition factor of $P/\Phi(P)$ is cyclic, and so $G/C_{G}(P/\Phi(P))$ is a 2-group. Hence, $G/C_{G}(P)$ is a 2-group [3, 5.1.4], in which case $P \cap Z(G) \neq 1$. This contradiction implies that $P$ is elementary abelian.

Assume $P \neq N$. Then there is a prime $q \neq 2$ and a $q$-group $1 \neq Q < N$ so that $QP$ is normal in $G$. $C_{G}(Q) = 1$. Let $|P| = 2^m$. If $C_{G}(P) = P$, then the proof of Theorem 2 shows that $2^{m-2} < 2m$, or $m \leq 5$. If $m < 4$, there is no choice for $q$. If $m = 5$, then $q = 31$. But the normalizer in $GL_2(2)$ of a group of order 31 has order $31 \cdot 5$. It follows that $P \in \text{Syl}_2(G)$, and so $N(Q)$ is not $\mathcal{D}^*$-supplemented. Thus, the only possibility is $m = 4$ and $q = 3$ or $5$. $N(Q)$ has a supplement $D$.

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which is cyclic, dihedral, semidihedral, or generalized quaternion of order at least $\left| P \right| = 16$. $D$ has no normal elementary abelian subgroup of order 4, and so $\left| D \cap P \right| \leq 2$. Thus, $DP/P$ has order at least 8. The normalizer in $GL_4(2)$ of a cyclic group of order 5 is metacyclic group of order 60. Consequently, $q = 3$. Since $C_P(Q) = 1$, the normalizer in $GL_4(2)$ of $Q$ is $\Gamma L_2(4)$. Since $G$ is solvable, the only possibility is that $N(Q)$ is a split extension of $Q$ by $D_8$ or $\Sigma_4$. In either case $O_2(G/P) \cong Z_2 \times Z_2 \cong C_p(O_2(G/P))$. By induction $G/P$ acts reducibly on $P/C_G(O_2(G/P))$, which is not the case. Therefore, $C_G(P)$ properly contains $P$, whence $N \neq G$. There is a group $P < K \leq G$ so that $K/P \cong Z_2 \times Z_2$, $C_P(Q) = 1$ and $[K,Q] \leq P$ imply $C_K(Q) \cong Z_2 \times Z_2$ [3, 5.3.15]. $C_K(Q) = C_K(PQ)$ is normal in $G$. Then $K \leq C(P)$ implies $K = P \times C_K(Q)$, and so $K$ is elementary abelian.

If $X \leq G$ let $\bar{X}$ denote the image of $X$ in $\bar{G} = G/C_K(Q)$. By induction $G$ has a normal subgroup $H \supset C_K(Q)$ so that $\bar{G}/\bar{H} \cong 1$, $A_4$, $\Sigma_4$ or $\Gamma$, and each $G$ composition factor of $\bar{H}$ is cyclic. From the facts that $M$ is noncyclic, $M \cap C_K(Q) = 1$, and $M$ is the only minimal normal subgroup of $G$ contained in $N$, it follows that $N$ is isomorphic to a subgroup of $\bar{G}/\bar{H}$. Thus, $Q \cong Z_3$. Choose $S \in Syl_3(G)$.

Suppose $G/N \cong \Gamma$. Then $G : N(S) \geq 64$. Consequently, there is a cyclic, dihedral, semidihedral, or generalized quaternion group $D$ of order at least 64 which is a supplement for $N(S)$. For $X \leq G$ let $\bar{X}$ denote the image of $X$ in $\bar{G} = G/N \cong \Gamma$. Then $S \in Syl_3(G)$ and $N_G(S) = N_G(S)$. Thus, $\bar{D} \cong D/D \cap N$ is a supplement for $N_G(S)$. But $D \cap N \neq 1$ since $\Gamma$ has exponent 24. Hence, $\bar{D}$ is cyclic or dihedral, whereas a $D^*$-supplement for $N_G(S)$ in $\bar{G} \cong \Gamma$ must be semidihedral or generalized quaternion of order 16. This contradiction implies $G/N \cong A_4$ or $\Sigma_4$, whence $N \cong A_4$, or $\Sigma_4$. Then $S \cong Z_3 \times Z_3$, $G : N(S) = 16$, and $|G/K|$ divides 36. A Sylow 2-subgroup of $G/K$ is not cyclic of order 4, and so the exponent of $G$ divides 12. Hence, $G(N(S))$ does not have a supplement $D \in D^*$.

The only remaining case is $P = N$. Then $G \neq N$ and there is an element $g \in G$ of order 3. Assume $G/N \cong \Gamma$. $C_G(N)$ does not contain $g$, for otherwise $G/C_G(N)$ is a 2-group, and $N \cap Z(G) \neq 1$. Hence, $N : C_N(g) \geq 4$, and so $G : N(\langle g \rangle) \geq 64$. Since $G$ has exponent 24 or 48, $N(\langle g \rangle)$ has no supplement $D \in D^*$.

Thus, $G/N \cong A_4$ or $\Sigma_4$. Set $K = O_2(G)$. $K/N \cong Z_2 \times Z_2$. Suppose $[N,K] = 1$. Then $C_N(g) = 1$, and by induction $|N| = 4$. A supplement $D \in D^*$ for $N(\langle g \rangle)$ has order at least 16, whence $|D \cap K| \geq 8$. Since $K$ has exponent 2 or 4, $D \cap K$ is dihedral or quaternion. Thus, $K$ is a nonabelian group of order 16 and exponent 4. According to Burnside [2, p. 146] $|K'| = 2$, and so $N$ contains a subgroup of order 2 which is normal in $G$. This contradiction implies $[N,K] = U \neq 1$.

By induction each $G$-composition factor of $N/M$ is cyclic of order 2. Consequently, $g$ centralizes $N/M$. Then $K = N[K, g]$ also centralizes $N/M$, so $U \leq M$. Thus, $U = M$ and $K$ centralizes $U$. Then $C_U(g) = 1$. By induction
$|U| = 4$. Set $V = C_N(g)$, so that $N = V \times U$. Choose $H < K$ such that $H : N = 2$. 
$C_N(H)$ contains $U$ and therefore is normalized by $g$. Then $C_N(H) = C_N(H^g) = C_N(H^u) = C_N(K)$. Since $C_N(K) \cap C_N(g) = 1$, $C_N(H) = U$. Consequently, $|N| = 8$ or 16.

Suppose there is an element $x \in K$ so that the image $x$ of $x$ in $K = K/U$ has order 4. $x^2 \in N$ since $K/N \simeq Z_2 \times Z_2$, but $x \notin N$ since $N$ is elementary. Hence, $C_K(x^2)$ properly contains $N$, which is not the case. Consequently, $K/U$ is elementary abelian, and so $K/U = C_K(U(g) \times [K/U, g])$. Since $U = [U, g] < W = [K, g]$ and $C_U(g) = 1$, $K$ is a split extension of $W$ by $V$. $W$ has order 16 and is normal in $G$. Since $W$ has exponent at most 4 and $|W'| \neq 2$, $W$ is abelian.

Let $R = (g)$. $G$ is a split extension of $W$ by $N(R)$. $N(R)$ acts faithfully on $W$. $N(R)$ has a supplement $D$ which is cyclic, dihedral, semidihedral, or generalized quaternion. Since $K$ has exponent 4, the only possibility is $G/N \simeq \Sigma_4$. $D$ is a complement for $N(R)$, and $D$ is dihedral, semidihedral, or generalized quaternion of order 16. Suppose $W \simeq Z_4 \times Z_4$. Then Lemma 1 yields $G \simeq \Gamma_2$ or $\Gamma_3$. This contradiction implies that $W$ is elementary abelian of order 16, and $G$ is isomorphic to a subgroup of $\Gamma_1$. The nonidentity elements of $N \cup W$ have order 2, while all elements in $K - (N \cup W)$ have order 4. Since $D$ has no normal 4-group, $D \cap N = D \cap W \simeq Z_2$. Hence, $D \cap K$ is a quaternion group, and $G$ is semidihedral or generalized quaternion of order 16. Moreover, $|DW| = |D||W|/|D \cap W| = 2^7$, so that $|G| = 3 \cdot 2^7 = |\Gamma_1|$. Then $G \simeq \Gamma_1$, a final contradiction.

Thus, $G$ has a normal subgroup $N$ so that every $G$-composition factor of $N$ is cyclic and $G/N \simeq 1, A_4, \Sigma_4, \Gamma_1, \Gamma_2,$ or $\Gamma_3$. $N$ is the join of all groups $H \subseteq G$ which are supersolvably embedded in $G$, and so $N$ is unique.

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