ON ARBITRARY SEQUENCES OF ISOMORPHISMS IN $R^n \rightarrow R^n$

BY

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ABSTRACT. In this paper a new, clean proof of an algebraic theorem needed in ordinary differential equations is presented. The theorem involves the existence and uniqueness of a "complete splitting" for some subsequence of an arbitrary sequence of isomorphisms of Euclidean $m$-space. In the positive-definite case, a complete splitting is a limit condition on eigenspaces and eigenvalues.

1. Introduction. One of the hardest parts of the $C^1$ closing lemma [2] is the proof of the algebraic "subsequence and decomposition theorem". As remarked there, it is hard because vector techniques are used in place of subspace techniques. In this paper, a simpler proof of the theorem is given. A uniqueness theorem, new here, is also proved. Thanks are due to G. Mostow and D. Anosov for useful conversations.

We now state the main theorems, see §2 for the definitions of the terms employed and §§5, 6 for the proofs.

Let $H$ be an inner product space and let $\{T_k\}$ be an arbitrary sequence of monomorphisms $R^n \rightarrow H$.

Existence theorem. There exists a subsequence $\{T_{k_n}\} \subset \{T_k\}$ having a complete splitting.

Uniqueness theorem. If $R^n = \bigoplus Y^j$ is a complete splitting for $\{T_k\}$ and $R^n = \bigoplus W^j$ then $\bigoplus Y^j$ is a complete splitting for $\{T_{k_n}\}$ iff the flags of $\bigoplus Y^j$ and $\bigoplus W^j$ are equal.

2. Definitions and notations. Let $G, H$ be Euclidean spaces and call $M(G, H)$ the set of all monomorphisms $G \rightarrow H$.

Definition. For $T \in M(G, H)$, $m(T) = \min \|Tx\|: x \in G, \|x\| = 1$, $\|T\| = \max \|Tx\|: x \in G, \|x\| = 1$, and $\text{bol}(T) = \|T\|/m(T)$. These numbers are called the minimum norm of $T$, the norm of $T$, and the bolicity of $T$.

Clearly $m(T) \neq 0$ for $T \in M(G, H)$. The bolicity measures how much $T$ distorts the unit ball of $G$. The larger $\text{bol}(T)$ is, the greater the distortion. Families

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of \( T \in M(G, H) \) having uniformly bounded bolicity are easy to deal with, though not precompact.

**Notation.** If \( G = \bigoplus G^i \) then \( iG = \bigoplus_{i \neq j} G^i \). The symbol \( \bigoplus \) denotes the usual direct sum, \( \bigoplus \) denotes orthogonal direct sum.

**Definition.** The flag of \( G = \bigoplus G^i \) is the ascending sequence of subspaces: \( G^1, G^1 \oplus G^2, \ldots, G^1 \oplus \cdots \oplus G^{L-1}, G \).

**Definition.** If \( T \in M(G, H) \) and \( \bigoplus G^i = G \), the perpendicular transform of \( T \) is \( \tilde{T} : G \to H \) by \( \tilde{T} = \Sigma A^i \) where \( A^i : G^i \to [T(iG)]^i \) is defined by the commutativity of

\[
\begin{array}{ccc}
G & \xrightarrow{T} & H \\
\downarrow & & \downarrow \\
G^i & \xrightarrow{A^i} & [T(iG)]^i
\end{array}
\]

\( \pi^i \) being the orthogonal projection.

**Definition.** The maps \( A^i \) are called the altitudes of \( T \) respecting \( \bigoplus G^i \).

It is an open problem to investigate the algebraic properties of \( \tilde{T} \). It is clear that \( \tilde{T} \in M(G, H) \) but that \( \tilde{T} \) is not linear.

**Definition.** Let \( \{T_n\} \) be a sequence in \( M(G, H) \). Then \( G^- \oplus G^+ = G \) is a primary splitting for \( \{T_n\} \) iff the altitudes \( A_n^-, A_n^+ \) of \( T_n \) respecting \( G^- \oplus G^+ \) satisfy

\[
\text{bol}(A_n^-) \text{ is bounded as } n \to \infty, \\
m(A_n^+)/\|A_n^-\| \to \infty \text{ as } n \to \infty.
\]

**Definition.** \( m(A^+)/\|A^-\| \) is called the hyperbolicity of \( T \) respecting \( G^- \oplus G^+ \).

**Definition.** Let \( \{T_n\} \) be a sequence in \( M(G, H) \). Then \( \bigoplus_{i=1}^L G^i \) is a complete splitting for \( \{T_n\} \) iff the altitudes \( A_n^i \) of \( T_n \) respecting \( \bigoplus G^i \) satisfy

\[
\text{bol}(A_n^i) \text{ is bounded as } n \to \infty, \quad 1 \leq j \leq L, \\
m(A_n^{j+1})/\|A_n^j\| \to \infty \text{ as } n \to \infty, \quad 1 \leq j \leq L - 1.
\]

**Definition.** Let \( F, F' \) be subspaces of \( G \); let \( f, f' \) be the orthogonal projections of \( G \) onto \( F, F' \). Then

\[
\angle(F, F') = \sup \{ \angle(p, f'p) : p \in F - 0 \}
\]

where the angle \( \langle \angle \rangle \) between two vectors means the acute angle between the lines they span in \( G \). If one of the vectors is zero, the angle is by definition \( \pi/2 \).

Thus, \( \angle(F, F') \) means: at most how far is \( F \) off \( F' \).

**Remark.** \( \angle(F, F') = \angle(F', F) \) if \( F \) and \( F' \) have the same dimension. It is
enough to show \( \angle(F, F') \geq \angle(F', F) \). By Euclidean geometry we see that, for \( p \in F \), \( \angle(f'p, f'p) \leq \angle(p, f'p) \). We can assume \( f'(F) = F' \). For otherwise \( F \) contains a vector perpendicular to \( F' \) so that \( \angle(F, F') = \pi/2 \), certainly \( > \angle(F', F) \).

Take any \( p' \in F' - 0 \). We know there is a (unique) \( p \in F - 0 \) with \( p' = f'p \). Hence \( \angle(p', f'p) = \angle(f'p, f'p) \) which by our initial observation is \( \leq \angle(p, f'p) \). Hence, \( \angle(F', F) \leq \angle(F, F') \), and so they are equal.

**Notation.** For any vector \( x \in G \), we write \( \angle(x, F) \) instead of \( \angle(\text{span}(x), F) \).

For trigonometric functions we shall omit the notation \( \angle \). That is, we write \( \sin(F, F') \) instead of \( \sin(\angle(F, F')) \), etc.

The following lemma simplifies much algebra.

**Lemma 0.** Any isomorphism \( T: G \rightarrow H \) factors uniquely as \( T = OP = P'O' \) where \( O, O' \) are orthogonal and \( P, P' \) are positive definite symmetric.

**Proof [1].** Let \( P = (TT)^{1/2} \), \( P' = (T'T)^{1/2} \).

3. Some estimates on altitudes. We need several geometric inequalities so that we can estimate the norms and bolicities of the altitudes of a monomorphism \( T: G \rightarrow H \).

**Lemma 1.** Suppose \( G = F \oplus X \) and \( Y \) is a subspace of \( G \) with \( Y \cap F = 0 \). Let \( \sigma: Y \rightarrow X \) by projection along \( F \). Then

\[
\|\sigma\| \leq 1/\cos(F, X^\perp), \quad m(\sigma) \geq \cos(Y, F^\perp).
\]

**Proof.** For any \( y \in Y \), \( y = f + x \) uniquely, \( f \in F \), \( x \in X \), and \( \sigma y = x \). If \( |y| = 1 \), trigonometry shows that

\[
|f| = \frac{\sin(y, x)}{\sin(f, x)}, \quad |x| = \frac{\sin(y, f)}{\sin(f, x)}
\]

so that \( |\sigma y| = \sin(y, f)/\sin(f, x) \). Hence \( |\sigma y| \geq \sin(y, F) \geq \min(\sin(y, F)) \), \( y \in Y \) = \( \min(\cos(y, F^\perp)) \), \( y \in Y \) = \( \cos(Y, F^\perp) \) since the minimum of the cosine is achieved for the maximum angle. Similarly \( |\sigma y| \leq 1/\cos(F, X^\perp) \). Q.E.D.

**Lemma 2.** Let \( T: R^2 \rightarrow H \) be a monomorphism. Then

\[
\frac{\sin(Tu, Tu')}{\sin(u, u')} = \frac{\det(T)}{|Tu|} = \frac{m(T)}{|Tu|}
\]

for any independent unit vectors \( u, u' \in R^2 \).

**Proof.** This follows at once from the facts: \( |\det(T)| = \text{area of the parallelogram spanned by } (Tu, Tu') \), and \( |\det(T)| = m(T)|T| \). Q.E.D.
Lemma 3. Let $G = F \oplus X$ and $T: G \rightarrow H$ be a monomorphism. Then the altitude $A: X \rightarrow (TF)^\perp$ satisfies

$$\|A\| \leq \|T\|, \quad m(A) \geq m(T) \cos(X, F^\perp).$$

Proof. Take any $x \in X$, $|x| = 1$. Then $|Ax| = |Tx| \sin(Tx, TF) = |Tx| \sin(Tx, Tf)$ for some $f \in F$, $|f| = 1$. Call $T' = T|\text{span}(x, f)$. By Lemma 2,

$$|Tx| \sin(Tx, Tf) = \frac{|Tx| m(T') \|T'\| \sin(x, f)}{|Tx| \|Tf|}.$$

Since $\min \sin(x, f) \geq \min \{\sin(x, F)\} = \cos(X, F^\perp)$, this fraction is $\geq m(T') \cos(X, F^\perp) \geq m(T) \cos(X, F^\perp)$ as claimed. Also $\max |\sin(x, f)| \leq 1$ so that $|Ax| \leq \|T'\| \leq \|T\|$ as claimed. Q.E.D.

Definition. Let $G \subseteq H$ be a subspace and $0 < \phi < \pi/2$ be given. The cone of angle $\phi$ around $G$ in $H$ is

$$C_{\phi}(G; H) = \{h \in H: h = 0 \text{ or } (h, G) < \phi\}.$$

and the unit sphere in $G$ is $SG = \{g \in G: |g| = 1\}$.

Lemma 4. Suppose $P: H \rightarrow H$ is an isomorphism, $H = E^- \oplus E^+$, $PE^\pm = E^\pm$ and $\|P\| \leq m(P^*)$. Then for all $0 < \phi \leq \pi/2$,

$$\text{dist}(P(SE^+), P(C_{\phi}E^-)) \geq \|P^-\|/2^{1/2}[(1/b)^2 + (\text{bol}(P^-) \tan \phi)^2]^{1/2}$$

where $P^\pm = P|E^\pm$, $b = \text{hyperbolicity of } P$ respecting $E^- \oplus E^+ = H$.

See Figure 1. The factor $2^{1/2}$ may be superfluous.

Proof of Lemma 4. For $\phi = \pi/2$ both sides of (3.1) vanish, so we may suppose that $0 < \phi < \pi/2$. Choose any $g \in SE^+$ and any $c \in C_{\phi}E^-$, $c \neq 0$. Then $c = e^- + e^+ = e^\pm \in E^\pm$, and $e^- \neq 0$ since $\phi < \pi/2$ implies $c \notin E^+$. If $e^+ = 0$
then \( \text{dist} (P_g, \text{span} (P_c)) = |P_g| \geq m(P^+) \geq \|P^+\| > \) the r.h.s. of (3.1) since \( m(P^+) \geq \|P^+\| \) implies \( b \geq 1 \). Thus we may suppose that \( e^+ \neq 0 \) and consider \( v = e^+/|e^+| \in E^+ \).

Let \( G = \text{span} (g, v) \), \( H' = e^- \oplus G \), and consider \( SG \subset SE^+ \), \( C_\phi' = C_\phi (\text{span} (e^-); H') \).

It suffices to prove

\[
(3.2) \quad \text{dist} (P(SG), PC_\phi') \geq \mu/2^{(\mu^2/m^2 + \tan^2 \phi)}^{3/2}
\]

where \( \mu = |Pe^-|/|e^-| \), \( m = m(P|G) \). For

\[
\mu/[\mu^2m^2 + \tan^2 \phi]^{3/2} = 2^{-\nu(\mu^2/m^2 + \mu^2\tan^2 \phi)}^{3/2} \geq 2^{-\nu(\mu^2m^2 + \mu^2\tan^2 \phi)}^{3/2} = \) the r.h.s. of (3.1).

The cone \( C_\phi E^- \) and the sphere \( SE^+ \) are invariant by orthogonal isomorphisms of \( H \) leaving \( E^- \) invariant. Thus it is no loss of generality to suppose \( Pe^- = \mu e^- \), \( PG = G \), and \( P|G \) is positive definite symmetric since according to Lemma 0 there is an orthogonal \( O: H \to H \) having \( PO(H') = H' \) and \( PO|H' \) positive definite symmetric.

Let \( e_1, e_2 \) be unit eigenvectors of \( P|G \) having eigenvalues \( 0 < m \leq M \). If \( \dim (G) = 1 \) then \( e_1 = e_2 \) and \( m = M \). Otherwise \( e_1 \perp e_2 \). Call \( e_3 = e^-/|e^-| \).

Inscribe a square \( \Sigma \) in \( SG \) with edges parallel to \( e_1, e_2 \). Let \( F^+ = e_1 \oplus [e_3 \pm (\tan \phi)e_2], F^- = e_2 \oplus [e_3 \pm (\tan \phi)e_1] \). These four planes are tangent to \( C_\phi' \).

Their union contains a square cone \( \Sigma_\phi' \) in which \( C_\phi' \) is inscribed. Thus it is clear that \( \text{dist} (P(SG), PC_\phi') \geq \text{dist} (P\Sigma, P\Sigma_\phi') \).

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Take any point \( s = \pm 2^{-\frac{1}{2}} e_1 + ye_2, \ y \in \mathbb{R} \). Then \( \text{dist} (Ps, \Sigma_\phi) = \text{dist} (Ps, PF_2^-) = \text{dist} (Ps, PF_2^+) \) and
\[
\text{dist} (Ps, PF_2^+) = \text{dist} (\pm Pe_1, PF_2^+)/2^{1/2} = |Pe_1| \sin (Pe_1, PF_2^+)/2^{1/2} = m \mu \cos \phi/2^{1/2}[\mu^2 \cos^2 \phi + m^2 \sin^2 \phi]^{1/2}
\]
which equals the r.h.s. of (3.2). Similarly, for \( s = xe_1 \pm 2^{1/2} e_2, \ x \in \mathbb{R} \), we have
\[
\text{dist} (Ps, PF_1^+) \geq \text{dist} (Ps, PF_2^+)
\]
and
\[
\text{dist} (Ps, PF_1^+) = \mu/2^{1/2}[\mu^2/M^2 + \tan^2 \phi]^{1/2}
\]
which is greater than or equal to the r.h.s. of (3.2) since \( m \leq M \). All points of \( \Sigma \) are of the form \( \pm 2^{-\frac{1}{2}} e_1 + ye_2 \) or \( xe_1 \pm 2^{1/2} e_2 \) so this completes the proof of (3.2) and hence of (3.1). Q.E.D.

Theorem 1. Let \( P: G \rightarrow H \) be a monomorphism, \( G = E^- \oplus E^+ = G^- \oplus G^+ \). Let \( \Lambda^\perp: G^\perp \rightarrow (PG^\perp)^\perp \) be the altitudes. Let \( b_E, b_G \) be the hyperbolicities of \( P \) respecting \( E^- \oplus E^+ \), \( G^- \oplus G^+ \). Suppose \( PE^- = PE^+, \ m(P^-) = m(P), \) and \( \|P-\| \leq m(P^+) \) for \( P^+ = PE^+ \). Then
\[
E^- \subset \{ x \in G: \sin (x, G^-) \leq \cotl (P^-)/b_G \} = C_\phi (G^-; G)
\]
for \( \phi = \sin^{-1} (\cotl (P^-)/b_G) \). Whenever \( \sigma: E^- \rightarrow G^- \) by projection along \( G^+ \) is a well defined monomorphism,
\[
\|\Lambda^-Y\| \leq \|P^-\|/\cos (G^+, E^+),
\]
\[
|\Lambda^-x| \geq m(P^+) \cos (x \oplus G^+, E^+) \cos (G^-, G^+)\]
for \( Y = \sigma E^- \) and \( x \in G^- \). Furthermore if \( Y = G^- \) then
\[
m(\Lambda^+) \geq \frac{\|P^-\|/\cos (G^-, E^-)}{2^{1/2}[b_G^{-2} + (\cotl (P^-) \tan (G^-, E^-))^2]^{1/2}}.
\]
Proof. Take $x \in G$, $|x| = 1$, $x = g^+ + g^-$, $g^\pm \in G^\pm$. Then $|g^\pm| = \sin(x, g^\pm)/\sin(g^-, g^+)$ and $A^+g^+ = \pi^+(Pg^+) = \pi^+(Px)$, for $\pi^+: H \to (PG^-)$ by orthogonal projection. Since $\|A^+\| = 1$, we have

$$|P_x| \geq |\pi^+(P_x)| = |A^+g^+| \geq m(A^+) \sin(x, g^-)/\sin(g^-, g^+)$$

and hence $\{x \in G: |P_x|/|x|m(P) \leq K\} \subset \{x \in G: \sin(x, G^-) \leq K/B_G\}$ for any constant $K$. Since $m(P) = m(P^\pm)$ and $\sup\{P_x/|x|m(P^-): x \in E^-\} = \text{bol}(P^-)$ this means that $E^- \subset \{x \in G: |P_x|/|x|m(P) \leq \text{bol}(P^-)\} \subset \{x \in G: \sin(x, G^-) \leq \text{bol}(P^-)/B_G\}$ proving (3.3).

That $\|\sigma^{-1}\| \leq 1/\cos(G^+, E^+)$ is Lemma 1. For $y \in Y = \sigma E^-, |y| = 1$, $A^-y = \pi^-((P\sigma^{-1})y)$. Hence $|A^-y| \leq \|\pi^-\|\|P\|\|\sigma^{-1}\| \leq \|P\|/\cos(G^+, E^+)$, proving (3.4).

For $x \in G^+$, $|x| = 1$, Lemma 3 says that $|A^+x| \geq m(P|x \otimes G^+|) \cos(x, G^+)$. For any $g \in x \oplus G^+$, $g = e^- + e^+$, $e^- \in E^-$, and so

$$|P_g| \geq |P(e^+g)| \geq m(P^+) \min\{\cos(g, E^+): g \in E^+, E^+ \} = m(P^+) \cos(x \oplus G^+, E^+)$$

as claimed in (3.5).

Suppose $Y = G^-$. Let $\rho: G^+ \to E^+$ by projection along $G^-$. $\rho G^+ = E^+$ since $Y = G^-$. By Lemma 1, $m(\rho) \geq 1/\cos(G^-, E^-)$. For $x \in G^+$, $A^+x = \pi^+Px = \pi^+Px$ so that $m(A^+) \geq m(\pi^+m(P))$. But

$$m(\pi^+P^+) = \text{dist}(P(SE^+), PG^-) \geq \text{dist}(P(SE^+), PC_L(G^-, E^-)(E^-)),$$

so Lemma 4 implies (3.6.) Q.E.D.

4. Existence and uniqueness of primary splittings in the positive definite symmetric case.

Theorem 2. Let $P_k: R^m \to R^m$ be positive definite symmetric, $k = 1, 2, \ldots$. Then there exists a subsequence $\{P_k\} \subset \{P_k\}$ having a primary splitting $V^- \oplus V^+ = R^m$. That is,

(a) $\text{bol}(A^-) \text{ is uniformly bounded as } n \to \infty$,

(b) $m(A^-)/\|A^-\| \to \infty$ as $n \to \infty$,

where $A^\pm_k: V^\pm \to (P_k V^\pm)^\perp$ are the altitudes.

Proof. For each $P_k$, there are $m$ orthonormal eigenvectors $e_1^k, \ldots, e_m^k$ and $m$ corresponding positive eigenvalues $\lambda_1^k, \ldots, \lambda_m^k$ which we assume to be indexed by increasing size: $0 < \lambda_1^k \leq \ldots \leq \lambda_m^k$.

By abuse of notation we write subscripts $n$ instead of $k_n$ to denote a subsequence.

Select a subsequence $\{P_n\}$ of $\{P_k\}$ so that the ratios $\lambda_i^k/\lambda_j^k$ have limits $\gamma_j \in [0, \infty]$ as $n \to \infty$. Put

$$E^-_n = \text{span}\{e_n^j: \gamma_j > 0\}, \quad E^+_n = \text{span}\{e_n^j: \gamma_j = 0\}.$$
Thus \( R^m = E^+ \oplus E^+_n \) and we can choose a subsequence (not bothering to relabel it) such that \( E^+ \to V^\pm \) for some splitting \( R^m = V^- \oplus V^+ \). This is just the compactness of the Grassmann manifolds and the continuity of the dot product.

Call \( P^\pm = P_n | E^\pm_n \) and \( B = \sup \| \text{bol}(P^\pm) \| \). \( B \) is finite by our choice of \( E^\pm_n \) and the fact that \( P^\pm_n \) is positive definite symmetric. Clearly \( m(P^\pm_n) = m(P^\pm_n) \) and so \( m(A^\pm_n) \geq m(P^\pm_n) \) by Lemma 3. By (3.4), \( \| A^\pm_n \| \leq \| P^\pm_n \| / \cos(E^+, V^+) \). Hence \( \text{bol}(A^\pm_n) = \| A^\pm_n \| / m(A^\pm_n) \leq \text{bol}(P^\pm_n) \| / \cos(E^+, V^+) \) is bounded since \( \text{bol}(P^\pm_n) \| \leq B \) and \( \cos(E^+, V^+) \to \cos(V^+, V^+) = 1 \). This proves (4.1a).

By (3.4), (3.6), and the definition of \( B \),
\[
\frac{m(A^\pm_n)}{\| A^\pm_n \|} = \frac{m(A^\pm_n)}{\| P^\pm_n \|} \| A^\pm_n \| \geq \frac{\cos(E^+, V^+)}{[h_n^{-2} + B^2 \tan^2(E^+, V^-)]^{\frac{1}{2}}}
\]
for \( b_n = \) the hyperbolicity of \( P_n \) respecting \( E^-_n \oplus E^+_n = R^m \). This fraction tends to \( 1/0 = \infty \) as \( n \to \infty \) since \( b_n \to \infty \) and \( E^\pm_n \to V^\pm \). This proves (4.1b). Q.E.D.

Theorem 3. Let \( P_k : R^m \to R^m \) be positive definite symmetric, \( k = 1, 2, \ldots \).
Let \( G^- \oplus G^+ = R^m = W^- \oplus W^+ \). Suppose that \( W^- \oplus W^+ \) is a primary splitting for \( \{ P_k \} \). Then \( G^- \oplus G^+ \) is a primary splitting for \( \{ P_k \} \) for \( G^- = W^- \).

Proof. To show \( \Rightarrow \) it is enough to prove that \( G^- = V^- \) where \( V^\pm \) is as constructed in Theorem 2, \( E^\pm \to V^\pm \). Let \( P^\pm_n = P_n | E^\pm_n \) and let \( A^\pm_n : G^\pm \to (P^\pm_n)^\perp \) be the altitudes. Observe that \( \text{bol}(P^\pm_n)/b_{G_n} \to 0 \) since \( \text{bol}(P^\pm_n) \| \leq B \) and \( b_{G_n} \) is the hyperbolicity of \( P_n \) respecting \( G^- \oplus G^+ \), \( b_{G_n} \to \infty \) by assumption. Hence (3.3) implies that \( \lim(E^\pm) = V^- \subset G^- \).

If \( V^- \not\subset G^- \) then \( G^- \cap E^+_n \neq 0 \) because \( \dim(G^-) + \dim(E^+_n) > \dim(V^-) + \dim(E^+_n) = m \). Take \( g_n \in G^- \cap E^+_n \), \( |g_n| = 1 \), \( g_n \to g \in G^- \cap V^+ \). Applying (3.5) yields
\[
\| A^\pm_n \| \geq \| A_g g_n \| \geq m(P^\pm_n) \cos(g \oplus G^+, E^+_n) \cos(G^-, G^\perp) \cos(G^+, G^\perp),
\]
By (3.4), \( m(A^\pm_n) \leq \| P^\pm_n \| / \cos(G^+, E^+_n) \). Hence
\[
\frac{\| A^\pm_n \|}{m(A^\pm_n)} \geq \frac{m(P^\pm_n)}{\| P^\pm_n \|} \cos(g \oplus G^+, E^+_n) \cos(G^-, G^\perp) \cos(G^+, E^+_n).\]
The product of the three cosines tends to \( \cos(g \oplus G^+, V^\perp) \cos(G^-, G^\perp) \cos(G^+, V^\perp) \) as \( n \to \infty \). This is not zero since it is easily seen that \( g \oplus G^+ \) is independent of \( V^- \), \( G^- \cap G^+ = 0 \), and \( G^+ \cap V^- = 0 \). So we have \( \| A^\pm_n \| / m(A^\pm_n) \to \infty \) which contradicts the assumption that \( \{ P_k \} \) obeyed (4.1a) respecting \( G^- \oplus G^+ \). Hence \( V^- = G^- \), completing the proof of \( \Rightarrow \).

Now suppose that \( G^- = W^- \). By \( \Rightarrow \) \( W^- = V^- \) for \( E^\pm_n \to V^- \). Again let \( A^\pm_n : G^\perp \to (P^\perp_n)^\perp \) be the altitudes. By (3.4) and Lemma 3
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\[
\frac{\|A_n\|}{m(A_n)} \leq \frac{\|P_n^{-}\|}{m(P_n)\cos(G^+, E_n^{-})} = \frac{\|P_n^{-}\|}{m(P_n)\cos(G^+, E_n^{-})} = \frac{\text{bol}(P_n^{-})}{\cos(G^+, E_n^{-})}.
\]

But $\text{bol}(P_n^{-}) \leq B$ and $\cos(G^+, E_n^{-}) \rightarrow \cos(G^+, V^+) \neq 0$ since $V^- \oplus V^+ = R^m = G^- \oplus G^+$ and $G^- = V^-$. Hence $\{A_n^{-}\}$ obeys (4.1a).

As for (4.1b), by (3.4) and (3.6) we have

\[
\frac{m(A_n^+)}{\|A_n^+\|} \leq \frac{m(A_n^+)}{\|A_n^+\|} \leq \frac{\cos(G^+, E_n^+)/\cos(G^-, E_n^-)}{[b_{E_n}^2 + (\text{bol}(P_n^-)\tan(G^-, E_n^-))^2]^{1/2}}
\]

where $b_{E_n} = \text{hyperbolicity of } P_n$ respecting $E_n^- \oplus E_n^+$. But $b_{E_n} \rightarrow \infty$, $\text{bol}(P_n^-) \leq B$, and $\tan^2(G^-, E_n^-) \rightarrow 0$ while $\cos(G^+, E_n^+) \rightarrow \cos(G^+, V^+) \neq 0$ and $\cos(G^-, E_n^-) \rightarrow 1$. This verifies (4.1b) for $\{A_n^+\}$. Q.E.D.

5. Existence of complete splittings.

Existence theorem. Let $T : R^m \rightarrow H$ be a monomorphism, $k = 1, 2, \ldots$. Then there exists a subsequence $\{T_{k_n} \subset [T_i]_i$ having a complete splitting: $\bigoplus_{j=1}^L V^j$.

That is,

(a) $\text{bol}(A_n^j)$ is bounded as $n \rightarrow \infty$, $1 \leq j \leq L$,

(b) $m(A_n^{j+1})/\|A_n^j\| \rightarrow \infty$ as $n \rightarrow \infty$, $1 \leq j \leq L - 1$,

where $A_n^j : V^j \rightarrow [T_{k_n} iV]^j$ are the altitudes respecting $\bigoplus_{j=1}^L V^j$ and $iV = \bigoplus_{j=1}^L V^j$.

In the proof of the above we need the following lemma.

Lemma 5. If $E \oplus F = G \subset H$ then

\[
\begin{array}{ccc}
H & \xrightarrow{\pi} & G^L \\
\downarrow & \downarrow \chi & \downarrow \\
E^\perp & \xrightarrow{\omega} & F^\perp
\end{array}
\]

commutes, all the maps except the inclusion being orthogonal projections.

Proof. This is clear. Any $x \in H$ is $x = y + e + f$ for some $y \in G^\perp$, $e \in E$, and $f \in F$. Then $r(x) = y + f$, $\omega(rx) = y$, $\chi(rx) = y$, while $\pi(x) = y$.

Proof of the existence theorem. We use induction on $m$. For $m = 1$, the theorem is trivial since $L$ must be 1 and $m(A_n^i) = \|A_n^i\|$. Next we observe that if $\bigoplus_{j=1}^L V^j = R^m$ is a complete splitting for $\{T_{k_n} \}$ then it is also one for $\{S_{n_k} T_{n_k} \}$ where $\{S_n \}$ is any sequence of orthogonal maps of $H$. For the altitudes of $S_{n_k} T_{n_k}$ are $B_n^j$ where
\[ \pi^i_n \text{ and } r^j_n \text{ being the orthogonal projections. This commutes because } S_n \text{ is orthogonal. Thus } B^i_n = S_n A^i_n \text{ and so (5.1) is clear for } \{B^i_n\}. \]

By Lemma 0, we may factor each \( T_k \) in the given sequence as \( T_k = O_k P_k^i \), \( 1, 2, \ldots \), where \( O_k \) is orthogonal and \( P_k: R^m \to R^m \) is positive definite symmetric. According to the preceding paragraph, we need only prove the theorem for the arbitrary sequence \( \{P_k\} \).

Again we use \( n \) instead of \( k \) for the subscripts of the subsequence. According to Theorem 2 there is a subsequence \( \{P_n\} \subset \{P_k\} \) having a primary splitting \( V^- \oplus V^+ \) with altitudes \( A^+_n: V^+ \to [P_n(V^+)]^\perp \). If \( V^+ = 0 \) then (4.1a) implies (5.1a) for \( R^m = V^- \oplus V^1 \) while (5.1b) is vacuous. \( V^- \neq 0 \) by construction. Thus we may apply the induction hypothesis to the sequence of monomorphisms \( A^+_n: V^+ \to H \). This gives a subsequence (unlabeled) of \( \{A^+_n\} \) having a complete splitting \( \bigoplus_{j=1}^L V^j = V^+ \). Let \( C^j_n \) be the altitudes of \( A^+_n \) respecting this splitting, \( 2 \leq j \leq L \).

With \( V^- = V^1 \), consider the splitting \( R^m = \bigoplus_{j=1}^L V^j \) having the altitudes \( A^i_n: V^j \to [P_n(iv)]^\perp \). Thus \( A^i_n = A^-_n \). We claim that \( C^j_n = A^i_n, j \geq 2 \). This holds because the following diagram commutes.
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$p_n^+, p_n^-, \omega_n^+$ being the orthogonal projections. That there is an inclusion for the rightmost map is true because $A_n^+V^i$ is just $P_n V^i$ made orthogonal to $P_n V^1$ and so $A_n^+V^i \subset P_n V^1 \oplus P_n V^i$, $i = 2, \ldots, L$. Thus $\bigoplus_{2 \leq i \neq j} A_n^+V^i \subset P_n (V^i)$ and the reverse is true for their orthogonal complements.

That the leftmost triangle commutes is trivial. That the first square commutes is the definition of $A_n^+$. That the second square commutes is Lemma 5 applied to $E = P_n V^-, F = \bigoplus_{2 \leq i \neq j} A_n^+V^i$, $G = E \oplus F = P_n (V^i)$, $H = \mathbb{R}^m$.

Thus, $C_n^j = A_n^j$, $j \geq 2$, and so (5.1) holds for $|A_n^j|$, $j \geq 2$, while (4.1a) implies (5.1a) for $j = 1$. As for (5.1b) in the case $j = 1$, we have

$$\frac{\|A_n^1\|}{\|A_n^2\|} = \frac{m(A_n^+)}{m(A_n^+)} \frac{m(A_n^2)}{\|A_n^2\|}.$$  

The first factor tends to 0 by (4.1b). The second factor is $\leq 1$ by Lemma 3. The last factor is always $\leq 1$. Hence (5.1b) is verified for $j = 1$, also, completing the proof of the existence theorem. Q.E.D.

6. Uniqueness of complete splittings.

Uniqueness theorem. Let $T_k: \mathbb{R}^m \to H$ be a monomorphism, $k = 1, 2, \ldots$. Suppose that $\{T_k\}$ has the complete splitting $\bigoplus W^i = \mathbb{R}^m$. Then $\bigoplus G^i$ is a complete splitting for $\{T_k\} \Rightarrow$ the flags of $\bigoplus G^i$ and $\bigoplus W^i$ are equal.

Again we need a lemma.

Lemma 6. If $\bigoplus G^i$ is a complete splitting of a sequence of monomorphisms $G \to H$, $\{T_n\}$, and $S: F \to G$ is an isomorphism then $\bigoplus S^{-1}G^i$ is a complete splitting of $\{T_n S\}$.

Proof. Let the altitudes of $\{T_n\}$ respecting $\bigoplus G^i$ be $A_n^i$ and of $\{T_n S\}$ respecting $\bigoplus S^{-1}G^i$ be $B_n^i$. It is clear that $B_n^i = A_n^i S$. Hence $\bigoplus S^{-1}G^i$ is a complete splitting for $\{T_n S\}$. Q.E.D.

Proof of the uniqueness theorem. To make the induction easier, we also assert

$$\frac{m(D_n^1)}{m(T_n)}$$  

for $D_n^1 = \text{the first altitude of any complete splitting of } \{T_n\}$. As before, the case $m = 1$ is trivial and it is no loss of generality to replace $\{T_k\}$ with $\{P_n\}$, $P_n: \mathbb{R}^m \to \mathbb{R}^m$ being positive definite symmetric. Let $\bigoplus V^i = \mathbb{R}^m$ be the complete splitting of (an unlabeled subsequence of) $\{P_n\}$ as constructed in §5. Thus $V^- \oplus V^+$ is a primary splitting for $\{P_n\}$, $V^- = \bigoplus_{j \geq 2} V^j$. To prove “$\Rightarrow$” it is enough to prove flag (\bigoplus G^i) = flag (\bigoplus V^i).

Let $G^- = G^1$, $G^+ = \bigoplus_{j \geq 2} G^j$, and $A_n^+, A_n^-, B_n^+, B_n^-$ the altitudes of $P_n$ respecting $V^- \oplus V^+$, $\bigoplus V^j$, $G^- \oplus G^+$, $\bigoplus G^j$. Thus, $A_n^+ = A_n^1$, $B_n^- = B_n^1$. As in
the altitudes of $|B|^j$ respecting $G^+ = \bigoplus_{j \geq 2} G^j$ are $B_n^j$, $j \geq 2$. This makes $\bigoplus_{j \geq 2} G^j$ a complete splitting of $|B|^j_n$ and so by induction on (6.1), $m(B^2_n)/m(B^j_n)$ is bounded as $n \to \infty$. Hence

$$\frac{m(B^2_n)}{\|B^2_n\|} = \frac{m(B^2_n)}{m(B^j_n)} \frac{m(B^j_n)}{\|B^j_n\|} \to \infty \text{ as } n \to \infty.$$  

Thus, $G^- \oplus G^+$ is a primary splitting of $|P|^j_n$ and so by Theorem 3, $G^- = V^-$, i.e. $G^1 = V^1$.

Let $\sigma : R^m \to V^+$ by orthogonal projection (i.e., projection along $V^+$). Since $G^- = V^-$, it is clear that $A^+\sigma = B^+$. By Lemma 6, $\bigoplus_{j \geq 2} \sigma^{-1}V^j$ is a complete splitting of $|A^+\sigma| = |B^+|$. By induction, flag($\bigoplus_{j \geq 2} G^j$) = flag($\bigoplus_{j \geq 2} \sigma^{-1}V^j$) and hence flag($G^j$) = flag($\bigoplus V^j$).

Now assume that flag($\bigoplus V^j$) = flag($\bigoplus W^j$). By "$\Rightarrow$" flag($\bigoplus V^j$) = flag($\bigoplus V^j$) for $\bigoplus V^j$ as above. In particular, $V^1 = G^1$ so that $\sigma : G^+ \to V^+$ as in the proof of "$\Rightarrow$" is well defined and flag($\bigoplus_{j \geq 2} \sigma^{-1}V^j$) = flag($\bigoplus_{j \geq 2} G^j$). By Lemma 6, $\bigoplus_{j \geq 2} \sigma^{-1}V^j$ is a complete splitting of $|A^+\sigma| = |B^+|$. By induction, $\bigoplus_{j \geq 2} G^j$ is a complete splitting of $|B^j_n|$. The altitudes of $B_n^j$, $j \geq 2$, as shown above. Also as above,

$$(6.2) \quad \frac{m(B^2_n)}{m(B^j_n)} \text{ is bounded as } n \to \infty.$$  

Thus, (5.1a) holds for $B_n^j$, $j \geq 2$, and (5.1b) holds for $j \geq 2$. $V^- \oplus V^+$ is a primary splitting of $|P|^j_n$ by construction. By Theorem 3 and $G^- = V^-$, $G^- \oplus G^+$ is also a primary splitting of $|P|^j_n$. Hence (5.1a) also holds for $B_n^j = B_n^1$ and $m(B^j_n)/\|B_n^j\| \to 0$ as $n \to \infty$. With (6.2) this implies that

$$\frac{m(B^2_n)}{\|B^2_n\|} = \frac{m(B^2_n)}{m(B^j_n)} \frac{m(B^j_n)}{\|B^j_n\|} \to \infty \text{ as } n \to \infty.$$  

That is, (5.1b) also holds for $j = 1$, which completes the proof of "$\Leftarrow$".

All that remains to prove is (6.1) in dimension $m$. Again it is no loss of generality to replace $|T|^j_n$ with $|P|^j_n$, $P : R^m \to R^m$ being positive definite symmetric. For $m(D^1_n)/m(T^j_n)$ is unchanged when we multiply $T^j_n$ on the left by an orthogonal isomorphism of $H$. So let $P^j$ be any complete splitting of $|P|^j_n$, $D^j : P^j \to (P^j)^T$ the altitudes.

By the first half of the proof of this theorem, $F^1 = V^1$ where $\bigoplus V^j = R^m$ is as above. By construction of $\bigoplus V^j$, $V^1 = \lim(E_n^-)$ where $E_n^- \oplus E_n^+ = R^m$, $P_n E_n^-$, $P_n E_n^+$, and $m(P_n^-) = m(P_n^+)$ for $P_n^\pm = P_n^-|E_n^\pm$. Thus, Theorem 3 applies.

By (3.4), $\|D^1_n\| \leq \|P_n^-\|/\cos (P^\pm, E^\pm)$ and so
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$$\frac{m(D^1_n)}{m(P_n)} \leq \frac{\|D^1_n\|}{m(P^n_n)} \leq \frac{1}{\cos (F^+, E^+)} \to \frac{1}{\cos (F^+, V^+)} < \infty$$

completing the proof of (6.1) in dimension $m$, also. Q.E.D.

7. An example. Let $\{P_i\}$ be a sequence of positive definite symmetric isomorphisms $\mathbb{R}^m \to \mathbb{R}^m$ whose eigenvalues are $0 < \epsilon_1 < \cdots < \epsilon^m_n$. Let $\epsilon^n_1, \cdots, \epsilon^n_m$ be a corresponding orthonormal set of eigenvectors. Suppose $\mathcal{E}_j = \lim_n \epsilon^n_i/\epsilon^n_j$ exist, for each $i, j$, $\mathcal{E}_j \in [0, \infty]$. Because we ordered the $\epsilon_i$ by increasing size, we have: $\mathcal{E}_i$ increases as $i$ increases, $\mathcal{E}_j$ decreases as $j$ increases, $\mathcal{E}_i = 1$. Consequently, the finite nonzero entries of the matrix $\mathcal{E}_i$ give a sequence of blocks down the diagonal. Partition the eigenvectors according to these blocks. That is, we say that $\epsilon^n_i$ is equivalent to $\epsilon^n_j$ iff $\mathcal{E}_i$ is nonzero, finite. Then we look at the equivalence classes of the $\epsilon^n_i$ and the corresponding classes of the $\epsilon^n_j$. There are say $L$ of them, $L \leq m$. We look at the space spanned by all the $\epsilon^n_i$ equivalent to a given one. Letting $j = 1, \cdots, m$, this gives $L$ distinct orthogonal subspaces, $E^1_n, \cdots, E^L_n$. Their dimensions are independent of $n$.

Suppose $E^j_n \to V^j_i$, $1 < j < L$. (This could be arranged by a subsequence of course.) It would be natural to expect that $\bigoplus V^j = \mathbb{R}^m$ is a complete splitting of $\{P_i\}$. This is not true. It was really necessary to go first to $\mathbb{R}^m = V^- \bigoplus V^+$, then to $V^+ = \bigoplus_{j=2}^L V^j$ as in the proof of the existence theorem.

In fact, consider, for positive parameters $\epsilon$ and $\delta$, the $3 \times 3$ matrix

$$P = \begin{pmatrix} 
\delta + (1 - \delta) \sin^2(\epsilon) & (1 - \delta) \sin(2\epsilon)/2 & 0 \\
(1 - \delta) \sin(2\epsilon)/2 & \delta + (1 - \delta) \cos^2(\epsilon) & 0 \\
0 & 0 & 1 
\end{pmatrix}.$$ 

This makes $P = R_{-\epsilon}Q_\delta R_\epsilon$ where

$$Q_\delta = \begin{pmatrix} 
\delta & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{pmatrix}, \quad R_\epsilon = \begin{pmatrix} 
\cos(\epsilon) & -\sin(\epsilon) & 0 \\
\sin(\epsilon) & \cos(\epsilon) & 0 \\
0 & 0 & 1 
\end{pmatrix}.$$ 

$R_\epsilon$ is the rotation through angle $\epsilon$ in the $(x, y)$-plane. The eigenspaces of $P$ are the images by $R_{-\epsilon}$ of those of $Q_\delta$. Thus they are $E^1 = R_{-\epsilon}(x\text{-axis})$, $E^2 = R_{-\epsilon}(y, z)$-plane). As $\epsilon \to 0$, $E^1 \to V^1 = x\text{-axis}$ and $E^2 \to V^2 = (y, z)$-plane.

We let $\epsilon$ and $\delta$ tend to zero. This makes $E^1$ and $E^2$ spaces of equivalent vectors under the notion of equivalence defined above. We show, however, that if $\epsilon/\delta \to \infty$ while $\epsilon, \delta \to 0$, then $A^2(z) \equiv z$ and $A^2(y) \to 0$ for $z = (0, 0, 1)$, $y = (0, 1, 0)$. This means that (5.1a) is violated for the splitting $R^3 = V^1 \bigoplus V^2$. That $A^2(z) \equiv z$ is clear.

Call $x = (1, 0, 0)$. Then $|A^2y| = |Py| |\sin(Px, Py)|$. By Lemma 2 and inspection
(\delta \leq 1), \sin(P_x, P_y) = m(P')\|P'\|/\|P_x\|\|P_y\| = \delta/|P_x||P_y|, where \( P' = P|\text{span}(x, y) \). Hence
\[
|A^2y| = \delta/|P_x| = \delta/|(\delta + (1 - \delta)\sin^2(\epsilon), (1 - \delta)\sin(2\epsilon)/2)|
\]
\[
= |(1 + ((1 - \delta)/\delta)\sin^2(\epsilon), ((1 - \delta)/2\delta)\sin(2\epsilon)|^{-1}
\]
which tends to zero as \( \epsilon, \delta \to 0 \) and \( \epsilon/\delta \to \infty \). The correct splitting is (x-axis) \( \Theta \) (y-axis) \( \Theta \) (z-axis) as is easily checked.

REFERENCES

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