

## COMMUTATIVE TORSION THEORY

BY

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**ABSTRACT.** This paper links several notions of torsion theory with commutative concepts. The notion of dominant dimension [H. H. Storrer, *Torsion theories and dominant dimensions*, Appendix to Lecture Notes in Math., vol. 177, Springer-Verlag, Berlin and New York, 1971. MR 44 #1685.] is shown to be very close to the notion of depth. For a commutative ring  $A$  and a torsion theory such that the primes of  $A$ , whose residue field is torsion-free, form an open set  $U$  of the spectrum of  $A$ ,  $\text{Spec } A$ , a concrete interpretation of the module of quotients is given: if  $M$  is an  $A$ -module, its module of quotients  $Q(M)$  is isomorphic to the module of sections  $\tilde{M}(U)$ , of the quasi-coherent module  $\tilde{M}$  canonically associated to  $M$ . In the last part it is proved that the (T)-condition of Goldman is satisfied [O. Goldman, *Rings and modules of quotients*, J. Algebra 13 (1969), 10–47. MR 39 #6914.] if and only if the set of primes, whose residue field is torsion-free, is an affine subset of  $\text{Spec } A$ , together with an extra condition. The extra, more technical, condition is always satisfied over a Noetherian ring, in this case also it is classical that the (T)-condition of Goldman means that the localization functor  $Q$  is exact. This gives a new proof to Serre's theorem [J.-P. Serre, *Sur la cohomologie des variétés algébriques*, J. Math. Pures Appl. (9) 36 (1957), 1–16. MR 18, 765.]. As an application, the affine open sets of a regular Noetherian ring are also characterized.

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0. Well-centered torsion theory. All the rings are supposed commutative and unitary. We denote by  $(\mathcal{T}, \mathcal{F})$  a torsion theory over a ring  $A$ . We recall that if  $\mathfrak{p}$  is a prime of  $A$ ,  $A/\mathfrak{p}$  is either a torsion module, and we say that  $\mathfrak{p}$  is a torsion-prime, or a torsion-free module, and we say that  $\mathfrak{p}$  is a free-prime. Therefore the spectrum  $\text{Spec } A$  is partitioned into two subsets  $T$  and  $F$ ; the set  $T$ , set of torsion-primes is closed under specialization (and its complement  $F$  is, of course, closed under generization). We denote by  $\text{Ass}_A(M)$  the set of primes weakly associated to an  $A$ -module  $M$  [I, Chapter IV, Exercise 17]; if  $\text{Ass}_A(M) \subset F$  then  $M$  is torsion-free, whereas if  $M$  is torsion  $\text{Ass}_A(M) \subset T$ . The converses of these two statements do not hold in general. When they do hold we say that  $(\mathcal{T}, \mathcal{F})$  is well centered. In fact, for  $(\mathcal{T}, \mathcal{F})$  to be well centered it is enough that  $\text{Ass}_A(M)$

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$\subset \mathbf{F}$ , for every torsion-free module  $M$ . If the ring  $A$  is Noetherian, every torsion theory is well-centered; in this case there is a bijection between the set of torsion theories over  $A$ , and the subsets  $\mathbf{F}$  of  $\text{Spec } A$ , closed under generization. (For all these preliminaries cf. [2].)

#### A. DEPTH

1. **Dominant dimension.** Let  $A$  be a ring with a torsion theory  $(\mathcal{T}, \mathcal{F})$ , following Tachikawa [18] and Storrer [16] we introduce the definition:

**Definition 1.1.** A module  $M$  is said to have  $(\mathcal{T}, \mathcal{F})$ -dominant dimension  $\geq n$ , if there exists an injective resolution of  $M$  whose first  $n$ -terms are torsion-free.

We denote the  $(\mathcal{T}, \mathcal{F})$ -dominant dimension of  $M$  by  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M)$ ; it is an integer or the symbol  $\infty$ .

The following results are immediate [16]:

1.2.  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \geq n$  if and only if the first  $n$  terms of the minimal injective resolution of  $M$  are torsion-free.

1.3.  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \geq 1$  if and only if  $M$  is torsion-free.

1.4.  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \geq 2$  if and only if  $M$  is torsion-free and divisible.

We give now another approach to dominant dimension. To every module corresponds its torsion radical  $T(M)$ . Clearly, if  $f: M \rightarrow N$  is a module morphism, the restriction  $T(f)$  of  $f$  to  $T(M)$  takes  $T(M)$  into  $T(N)$ , hence  $T$  can be regarded as a functor. It is easy to show that  $T$  is left exact as a functor from the category of  $A$ -modules to itself. If  $0 \rightarrow M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_n \rightarrow \dots$  is an injective resolution of  $M$ , it is classical to define the  $n$ th derived functor  $T_n$  of  $T$  as the  $n$ th homology group of the complex  $0 \rightarrow T(M_0) \rightarrow T(M_1) \rightarrow \dots \rightarrow T(M_n) \rightarrow \dots$  and then  $T_0$  is functorially isomorphic to  $T$  [14].

**Proposition 1.5.**  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) \geq n + 1$  if and only if  $T_k(M) = 0, \forall k \leq n$ .

**Proof.** From 1.3.  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = 0$  if and only if  $M$  is not torsion-free, that is  $T(M) = T_0(M) \neq 0$ . Conversely if  $M$  is torsion-free, we denote the injective hull of  $M$  by  $I(M)$  and the quotient  $I(M)/M$  by  $M'$ ; we consider the exact sequence  $0 \rightarrow M \rightarrow I(M) \rightarrow M' \rightarrow 0$ , it gives rise to a long exact sequence

$$\begin{aligned} 0 \rightarrow T_0(M) \rightarrow T_0(I(M)) \rightarrow T_0(M') \rightarrow T_1(M) \rightarrow \dots \\ \rightarrow T_k(M) \rightarrow T_k(I(M)) \rightarrow T_k(M') \rightarrow T_{k+1}(M) \rightarrow \dots \end{aligned}$$

and clearly  $T_k(M) = 0 \forall k \leq n$  if and only if  $T_k(M') = 0 \forall k \leq n - 1$ ; it is also clear from the definition that  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = (\mathcal{T}, \mathcal{F})\text{-}d_A(M') + 1$ . Hence, the proposition would follow from an easy induction on the dominant dimension of  $M$ , going from  $M'$  to  $M$ .

The following proposition is trivial but worth noting:

**Proposition 1.6.** *If  $(\mathcal{F}', \mathcal{F}')$  is a torsion theory smaller than  $(\mathcal{F}, \mathcal{F})$  (that is to say that every torsion-free module for  $(\mathcal{F}, \mathcal{F})$  is also torsion-free for  $(\mathcal{F}', \mathcal{F}')$ ), then, for every module  $M$ ,  $(\mathcal{F}', \mathcal{F}')$ - $d_A(M) \geq (\mathcal{F}, \mathcal{F})$ - $d_A(M)$ .*

**2. Depth.** We let now  $A$  be a Noetherian ring. If  $M$  is an  $A$ -module, and  $I$  an ideal of  $A$ , a sequence  $f_1, f_2, \dots, f_n$  of elements of  $I$  is called an  $M$ -regular sequence in  $I$  if  $f_1$  is not a zero divisor in  $M$ , and  $f_{i+1}$  is not a zero divisor in  $M/f_1M + \dots + f_iM$ ,  $1 \leq i \leq n-1$ . The  $I$ -Depth of  $M$ , denoted by  $I$ -Depth $_A(M)$  is the maximal length of an  $M$ -regular sequence in  $I$ ; if  $A$  is a local ring and if  $\mathfrak{m}$  denotes its maximal ideal; one writes Depth $_A(M)$  for  $\mathfrak{m}$ -Depth $_A(M)$  and calls it simply the depth of  $M$  [11, Chapter 6]. On the other hand there exists a torsion theory  $(\mathcal{F}_I, \mathcal{F}_I')$  corresponding to the partition  $\mathbf{T}, \mathbf{F}$ , where  $\mathbf{T} = V(I) = \{\mathfrak{p} \in \text{Spec } A \mid I \subset \mathfrak{p}\}$ , since  $V(I)$  is closed under specialization [§ 0]. Here is the main theorem of this section.

**Theorem 2.1.** *Let  $A$  be a Noetherian ring and  $I$  be an ideal of  $A$ . For every  $A$ -module  $M$  of finite type:  $(\mathcal{F}_I, \mathcal{F}_I')$ - $d_A(M) = I$ -Depth $_A(M)$ .*

**Proof.** The proof is in every respect similar to Matsumara's proof [11, 15.B, Theorem 26], that  $\text{Ext}_A^i(A/I, M) = 0$  for any  $i < n$  if and only if there exists an  $M$ -regular sequence  $f_1, \dots, f_n$  in  $I$ . Show by induction that there exists such an  $M$ -regular sequence if and only if  $T_i(M) = 0$ , for any  $i < n$  (where  $T$  is the torsion radical functor relative to  $(\mathcal{F}_I, \mathcal{F}_I')$ ,  $T_1, T_2, \dots, T_n, \dots$  its derived functors). We show only the first step:

Suppose that  $T_0(M) = T(M) = 0$ , hence that  $M$  is torsion-free for  $(\mathcal{F}_I, \mathcal{F}_I')$ . Then  $\text{Ass}_A(M)$  does not meet  $\mathbf{T} = V(I)$ . No prime associated to  $M$  contains  $I$ . So  $I$  is not included in the finite union of these primes and there is an element  $f_1$  in  $I$  which is not a zero divisor in  $M$ .

Conversely suppose that there is such an element  $f_1$  in  $I$ .  $f_1$  cannot be contained in any prime associated to  $M$ ,  $\text{Ass}_A(M)$  must be included in  $\mathbf{F}$ ,  $M$  must be torsion-free.

**3. Change of rings.** Let  $A$  be a ring with a torsion theory  $(\mathcal{F}, \mathcal{F})$  and  $\phi: A \rightarrow B$  a ring morphism. The direct image of  $(\mathcal{F}, \mathcal{F})$  by  $\phi$  is a torsion theory  $(\mathcal{F}^\phi, \mathcal{F}^\phi)$  for  $B$ -modules. It is the theory such that a  $B$ -module is regarded as a torsion module if and only if its inverse image  $\phi_*(M)$  is a torsion module for the theory  $(\mathcal{F}, \mathcal{F})$  over  $A$ . It is easy to check that the  $B$ -torsion free modules are the modules  $M$  such that  $\phi_*(M)$  is torsion-free for  $(\mathcal{F}, \mathcal{F})$ . We state the two following results without proof:

An ideal  $\mathfrak{b}$  of  $B$  is in the idempotent filter associated to  $(\mathcal{F}^\phi, \mathcal{F}^\phi)$  [9, Proposition 0.4] if and only if  $\phi^{-1}(\mathfrak{b})$  is in the idempotent filter associated to  $(\mathcal{F}, \mathcal{F})$ .

If  $(\mathcal{T}, \mathcal{F})$  is the partition of  $\text{Spec } A$  corresponding to  $(\mathcal{J}, \mathcal{F})$ , and  $(\mathcal{T}^\phi, \mathcal{F}^\phi)$  the partition of  $\text{Spec } B$  corresponding to  $(\mathcal{J}^\phi, \mathcal{F}^\phi)$ , then  $q \in \mathcal{T}^\phi \Leftrightarrow \phi^{-1}(q) \in \mathcal{T}$ , and  $q \in \mathcal{F}^\phi \Leftrightarrow \phi^{-1}(q) \in \mathcal{F}$ .

**Proposition 3.1.** *Let  $A$  be a ring with a torsion theory  $(\mathcal{J}, \mathcal{F})$ ,  $T$  the corresponding torsion radical functor and  $T_1, T_2, \dots, T_n, \dots$  its derived functors. Let  $\phi: A \rightarrow B$  be a ring morphism such that  $B$  is a flat  $A$ -module, and  $T^\phi$  the torsion radical functor corresponding to  $(\mathcal{J}^\phi, \mathcal{F}^\phi)$ ,  $T_1^\phi, T_2^\phi, \dots, T_n^\phi, \dots$  its derived functor. Let  $M$  be a  $B$ -module and  $\phi_*(M)$  its inverse image; then*

$$\phi_*(T_n^\phi(M)) \cong T_n(\phi_*(M)) \quad \text{for all } n$$

and in particular  $(\mathcal{J}, \mathcal{F})\text{-}d_A(\phi_*(M)) = (\mathcal{J}^\phi, \mathcal{F}^\phi)\text{-}d_B(M)$ .

**Proof.** Very easily  $\phi_*(T^\phi(M)) \cong T(\phi_*(M))$ , and since the inverse image of an injective resolution of  $M$  is an injective resolution of  $\phi_*(M)$  (since  $\phi$  is flat) the isomorphism holds for all  $n$ .

If  $S$  is a multiplicative set of a Noetherian ring  $A$ , and  $I$  is an injective  $A$ -module,  $S^{-1}I$  is an injective  $S^{-1}A$ -module [16, Proposition 7.17 and Corollary 7.14], then

**Proposition 3.2.** *Let  $A$  be a Noetherian ring with a torsion theory  $(\mathcal{J}, \mathcal{F})$ ,  $T$  the corresponding torsion radical functor and  $T_1, T_2, \dots, T_n, \dots$  its derived functors. Let  $S$  be a multiplicative set of  $A$  and  $(S^{-1}\mathcal{J}, S^{-1}\mathcal{F})$  be the direct image of  $(\mathcal{J}, \mathcal{F})$  under the canonical morphism  $A \rightarrow S^{-1}A$ . Let  $(S^{-1}T)$  be the torsion radical functor associated to this theory,  $(S^{-1}T_1), (S^{-1}T_2), \dots, (S^{-1}T_n), \dots$  its derived functors. Let  $M$  be an  $A$ -module; then*

$$S^{-1}[T_n(M)] \cong T_n(S^{-1}M) \cong (S^{-1}T_n)(S^{-1}M) \quad \text{for all } n$$

and in particular

$$(\mathcal{J}, \mathcal{F})\text{-}d_A(M) \leq (\mathcal{J}, \mathcal{F})\text{-}d_A(S^{-1}M) = (S^{-1}\mathcal{J}, S^{-1}\mathcal{F})\text{-}d_{S^{-1}A}(S^{-1}M).$$

**4. Local formulae.** For any prime  $\mathfrak{p}$  of  $A$ , we denote the direct image of a torsion theory  $(\mathcal{J}, \mathcal{F})$  under the canonical morphism  $A \rightarrow A_{\mathfrak{p}}$ , by  $(\mathcal{J}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$ . A direct consequence of Proposition 3.2 is

**Proposition 4.1.** *Let  $A$  be a Noetherian ring with a torsion theory  $(\mathcal{J}, \mathcal{F})$  and  $M$  an  $A$ -module; then*

$$(1) \quad (\mathcal{J}, \mathcal{F})\text{-}d_A(M) = \text{Inf}_{\mathfrak{p} \in \text{Spec } A} \{(\mathcal{J}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}.$$

However, it is not necessary to look at all the primes of  $\text{Spec } A$ .

**Corollary 4.2.** *Let  $A$  be a Noetherian ring with a torsion theory  $(\mathcal{T}, \mathcal{F})$  and  $M$  an  $A$ -module, then*

(2)  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \text{Inf}_{\mathfrak{p} \in \mathbf{T}} \{(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}$  (where  $(\mathbf{T}, \mathbf{F})$  is the partition of  $\text{Spec } A$  associated to  $(\mathcal{T}, \mathcal{F})$ ).

(3)  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \text{Inf}_{\mathfrak{p} \in \text{Max } A} \{(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}$  (where  $\text{Max } A$  is the set of maximal ideals of  $A$ ).

(4)  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \text{Inf}_{\mathfrak{p} \in \text{Supp}_A(M)} \{(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}$  (where  $\text{Supp}_A(M)$  is the support of the  $A$ -module  $M$ ).

**Proof.** (2) If  $\mathfrak{p} \in \mathbf{F}$  it is clear that  $(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$  is the torsion theory such that every  $A$ -module is torsion-free, hence  $(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \infty$ .

(3) and (4) are equally easy.

The set of minimal primes of  $(\mathcal{T}, \mathcal{F})$  is the set of primes which belong to  $\mathbf{T}$  and are minimal among the elements of  $\mathbf{T}$ . We denote this set  $\mathbf{T}_0$ . As a generalization of Theorem 2.1 we get

**Theorem 4.3.** *Let  $A$  be a Noetherian ring with a torsion theory  $(\mathcal{T}, \mathcal{F})$  and let  $\mathbf{T}_0$  be the set of minimal primes of  $(\mathcal{T}, \mathcal{F})$  and  $M$  an  $A$ -module of finite type; then*

$$(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \text{Inf}_{\mathfrak{p} \in \mathbf{T}_0} \{\mathfrak{p}\text{-Depth}_A(M)\}.$$

**Proof.**  $\mathfrak{p}\text{-Depth}_A(M)$  is the depth of  $M$  relative to the torsion theory whose partition  $(\mathbf{T}_{\mathfrak{p}}, \mathbf{F}_{\mathfrak{p}})$  is such that  $\mathbf{T}_{\mathfrak{p}} = V(\mathfrak{p})$  [Theorem 2.1], hence this theory is smaller than  $(\mathcal{T}, \mathcal{F})$  and:

$$(\mathcal{T}, \mathcal{F})\text{-Depth}_A(M) \leq \mathfrak{p}\text{-Depth}_A(M) \quad \forall \mathfrak{p} \in \mathbf{T}_0 \quad [\text{Proposition 1.6}].$$

The reverse inequality is shown by induction. We just prove the first step, that is, if  $\text{Inf}_{\mathfrak{p} \in \mathbf{T}_0} \{\mathfrak{p}\text{-Depth}_A(M)\} > 0$  then  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) > 0$ . Indeed, if this infimum is greater than 0, then  $M$  is torsion-free for any of the theories corresponding to the partitions  $(\mathbf{T}_{\mathfrak{p}}, \mathbf{F}_{\mathfrak{p}})$ , so

$$\text{Ass}_A(M) \cap V(\mathfrak{p}) = \emptyset \quad \forall \mathfrak{p} \in \mathbf{T}_0.$$

Since  $\mathbf{T} = \bigcup_{\mathfrak{p} \in \mathbf{T}_0} V(\mathfrak{p})$ ,  $\text{Ass}_A(M) \subset \mathbf{F}$ , and  $M$  is torsion-free for  $(\mathcal{T}, \mathcal{F})$ , which is equivalent to  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) > 0$ .

From the formula  $l\text{-Depth}_A(M) = \text{Inf}_{\mathfrak{p} \in \mathbf{T}} \{\text{Depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}$  [11, 15.6]. We get

**Theorem 4.4.** *Let  $A$  be a Noetherian ring with a torsion theory  $(\mathcal{T}, \mathcal{F})$  and  $M$  an  $A$ -module of finite type; then*

$$(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = \text{Inf}_{\mathfrak{p} \in \mathbf{T}} \{\text{Depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})\}.$$

B. MODULE OF QUOTIENTS

We let  $A$  be a Noetherian ring with a torsion theory  $(\mathcal{T}, \mathcal{F})$ , and  $T, F$  be the corresponding partition of  $\text{Spec } A$ .

5. An exact sequence.

**Proposition 5.1.** *Let  $M$  be an  $A$ -module and  $f: (M) \rightarrow Q(M)$  be the canonical morphism of  $M$  into its module of quotients, then  $\text{Ker } f$  is functorially isomorphic to the torsion radical  $T(M)$  of  $M$ , and  $\text{Coker } f$  is functorially isomorphic to  $T_1(M)$ , where  $T_1$  is the first derived functor of  $T$ .*

*In other words if  $g: M \rightarrow N$  is a module homomorphism the diagram below is commutative and its lines are exact:*

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & T(M) & \longrightarrow & M & \xrightarrow{f(M)} & Q(M) & \longrightarrow & T_1(M) & \longrightarrow & 0 \\
 & & \downarrow T(g) & & \downarrow g & & \downarrow Q(g) & & \downarrow T_1(g) & & \\
 0 & \longrightarrow & T(N) & \longrightarrow & N & \xrightarrow{f(N)} & Q(N) & \longrightarrow & T_1(N) & \longrightarrow & 0
 \end{array}$$

**Proof.** By definition of  $Q(M)$  the kernel of  $f(M)$  is the torsion radical  $T(M)$  of  $M$ , and its cokernel  $C$  is a torsion module. If  $M'$  denotes the image of  $M$  in  $Q(M)$ , we have two short exact sequences:

- (a)  $0 \rightarrow T(M) \rightarrow M \rightarrow M' \rightarrow 0$ ,
- (b)  $0 \rightarrow M' \rightarrow Q(M) \rightarrow C \rightarrow 0$

from which we get two long exact sequences:

- (c)  $\dots \rightarrow T_1(T(M)) \rightarrow T_1(M) \rightarrow T_1(M') \rightarrow T_2(T(M)) \rightarrow \dots$
- (d)  $\dots \rightarrow T(Q(M)) \rightarrow T(C) \rightarrow T_1(M') \rightarrow T_1(Q(M)) \rightarrow \dots$

Since  $Q(M)$  is torsion-free and divisible, then  $T(Q(M))$  and  $T_1(Q(M))$  are both 0 [1.4], and since  $C$  is a torsion module  $T(C) = C$ , so from (d),  $C$  is isomorphic to  $T_1(M')$ . On the other hand  $T_1(T(M))$  and  $T_2(T(M))$  are also both 0, since  $T(M)$  is a torsion module (from the lemma below) and then  $T_1(M')$  is isomorphic to  $T_1(M)$ , which is in turn isomorphic to  $C$ . The isomorphisms are functorial since the exact sequences (c) and (d) are functorial.

**Lemma 5.2.** *If  $M$  is a torsion module then  $T_n(M) = 0$ ,  $n > 0$ .*

**Proof.** Since  $M$  is a torsion module, then  $\text{Ass}_A(M) \subset T$ . If  $M_0$  is an injective hull of  $M$ , then  $\text{Ass}_A(M_0) \subset T$ , since  $A$  is Noetherian, hence  $M_0$  is also a torsion module; the quotient  $M_0/M$  is a torsion module and, step by step, one can get an injective resolution of  $M$  by torsion modules. Applying  $T$  to this resolution does not change it, then the homology groups of the resulting complex are all trivial.

6. **Open sets.** When  $F$  is an open set  $U$  of  $\text{Spec } A$ , we can give an explicit description of  $Q(M)$ .

**Theorem 6.1.** *Let  $M$  be an  $A$ -module,  $\tilde{M}$  be the quasi-coherent module canonically associated to  $M$  over  $\text{Spec } A$ ,  $j$  the canonical morphism of  $M$  into the module of sections  $\tilde{M}(U)$ . Then,  $j: M \rightarrow \tilde{M}(U)$  can be identified with the canonical morphism of  $M$  into its module of quotients.*

**Proof.** It is enough to show that  $\text{Ker } g$  and  $\text{Coker } g$  are torsion modules whereas  $\tilde{M}(U)$  is torsion-free and divisible [9, Proposition 0.7]. By definition of  $\tilde{M}(U)$ , if  $\mathfrak{p} \in F = U$  then the localization  $j_{\mathfrak{p}}: M_{\mathfrak{p}} \rightarrow \tilde{M}(U)_{\mathfrak{p}}$  is an isomorphism, hence  $(\text{Ker } g)_{\mathfrak{p}} = (\text{Coker } g)_{\mathfrak{p}} = 0$  and then  $\text{Ass}_A(\text{Ker } g) \subset \text{Supp}_A(\text{Ker } g) \subset T$ ; also  $\text{Ass}_A(\text{Coker } g) \subset T$ , so the kernel and cokernel of  $g$  are torsion modules.  $U$  is quasi-compact, since  $A$  is Noetherian, and can be covered by finitely many special open sets of  $\text{Spec } A$ , say  $U_1 = D(f_1), \dots, U_n = D(f_n)$ , and  $U_i \cap U_j = D(f_i f_j), \forall i, j \in \{1, \dots, n\}$ . Since  $\tilde{M}$  is a sheaf, there is an exact sequence

$$0 \rightarrow \tilde{M}(U) \rightarrow \bigoplus_{i=1}^n \tilde{M}(U_i) \rightarrow \bigoplus_{i,j} \tilde{M}(U_i \cap U_j)$$

or also,

$$0 \rightarrow \tilde{M}(U) \rightarrow \bigoplus_i M_{f_i} \rightarrow \bigoplus_{i,j} M_{f_i f_j}.$$

Since  $U_i \subset U = F$ , then  $\text{Ass}_A(M_{f_i}) \subset U_i \subset F$  and  $M_{f_i}$  is torsion-free, the direct sum  $\bigoplus_i M_{f_i}$  is also torsion-free and so is its submodule  $\tilde{M}(U)$ . We can write  $T(\tilde{M}(U)) = 0$ . For the same reasons  $M_{f_i f_j}$  is torsion-free, as well as the direct sum  $\bigoplus_{i,j} M_{f_i f_j}$ . We denote by  $P$  the image of  $\bigoplus_i M_{f_i}$  into  $\bigoplus_{i,j} M_{f_i f_j}$ , then  $P$  is torsion-free. From the short exact sequence  $0 \rightarrow \tilde{M}(U) \rightarrow \bigoplus_i M_{f_i} \rightarrow P \rightarrow 0$ , we get the long exact sequence  $\dots \rightarrow T(P) \rightarrow T_1(\tilde{M}(U)) \rightarrow T_1(\bigoplus_i M_{f_i}) \rightarrow \dots$  (\*) where  $T(P) = 0$ . Since  $U_i \subset F$ , the torsion theory  $(\mathcal{F}_i, \mathcal{F}_i)$ , direct image of  $(\mathcal{F}, \mathcal{F})$  over  $A_{f_i}$  is clearly trivial [§ 3], every  $A_{f_i}$ -module is torsion-free for this theory, in other words, one check easily that

$$(\mathcal{F}_i, \mathcal{F}_i)\text{-}d_{A_{f_i}}(N) = \infty, \text{ for every } A_{f_i}\text{-module } N.$$

Then  $(\mathcal{F}, \mathcal{F})\text{-}d_A(N) = \infty$  [Proposition 3.2], and in particular  $T(M_{f_i}) = T_1(M_{f_i}) = 0$ , and easily  $T_1(\bigoplus_i M_{f_i}) = 0$ . From (\*), it results that  $T_1(\tilde{M}(U)) = 0$ , and since also  $T(\tilde{M}(U)) = 0$ ,  $\tilde{M}(U)$  is torsion-free and divisible.

7. **Localization.** We show here that over a Noetherian ring, the functor  $Q$  commutes with localization with respect to a multiplicative subset of  $A$ . The result is easy but useful for the following.

**Proposition 7.1.** *Let  $S$  be a multiplicative subset of  $A$ ,  $M$  an  $A$ -module and  $f: M \rightarrow Q(M)$  be the canonical morphism of  $M$  into its module of quotients. Then  $S^{-1}f: S^{-1}M \rightarrow S^{-1}Q(M)$  is the canonical morphism of the  $S^{-1}A$ -module  $S^{-1}M$  into its ring of quotients for the theory  $(S^{-1}\mathcal{J}, S^{-1}\mathcal{F})$ , direct image of  $(\mathcal{J}, \mathcal{F})$  under the morphism  $A \rightarrow S^{-1}A$ .*

**Proof.** Since  $Q = Q(M)$  is torsion-free and divisible then  $(\mathcal{J}, \mathcal{F})\text{-}d_A(Q) \geq 2$ , but then  $(S^{-1}\mathcal{J}, S^{-1}\mathcal{F})\text{-}d_{S^{-1}A}(S^{-1}Q) \geq 2$  [Proposition 3.2] and  $S^{-1}Q$  is a torsion-free and divisible  $S^{-1}A$ -module. The kernel and cokernel of  $f$  are torsion  $A$ -modules, their localization are torsion  $A$ -modules and then torsion  $S^{-1}A$ -modules, from the following lemma:

**Lemma 7.2.** *Let  $A$  be any commutative ring,  $S$  a multiplicative subset of  $A$ , and  $M$  a torsion  $A$ -module, then  $S^{-1}M$  is a torsion  $A$ -module.*

**Proof.** Every element of  $S^{-1}M$  can be written  $x/s$ , where  $x \in M$ , and  $s \in S$ , and the submodule  $Ax/s$  of  $S^{-1}M$  is an homomorphic image of the submodule  $Ax$  of  $M$ .

It is interesting to note that such a property does not always hold for torsion-free modules [3, § 1, Corollaire 2].

### C. AFFINE TORSION THEORY

**8. (T)-condition of Goldman.** In general  $Q(M)$  is not isomorphic to  $M \otimes_A Q(A)$ . If this holds for every module  $M$ , Goldman says that  $(\mathcal{J}, \mathcal{F})$  satisfied the (T) condition (T for tensor). We quote [5, Theorem 4.3]:

**Theorem 8.0.** *The following conditions are equivalent:*

- (i) (T) condition:  $Q(M) \cong M \otimes_A Q(A)$ ; for every  $A$ -module  $M$ ,
- (ii) every  $Q(A)$ -module is torsion-free,
- (iii)  $\mathfrak{b}Q(A) = Q(A)$ , for every ideal  $\mathfrak{b}$  of  $A$  such that  $A/\mathfrak{b}$  is a torsion module,
- (iv)  $Q$  is exact and commutes with direct sums.

We want to show that the (T) condition is related to the set of free-primes  $F$  being an affine subset of  $\text{Spec } A$ . If however  $F$  is not an open subset of  $\text{Spec } A$ , the notion affine subset would make sense only in the theory of "Espaces Étales" [4, II, § 1–2] but Lazard proved that  $F$  is affine if and only if there is a flat epimorphism of rings  $f: A \rightarrow B$ , such that  $F$  is the set of primes of  $A$  which are lifted in  $B$  [10, IV, Proposition 2.5]. We can take this characterization as a definition, for open sets it gives back the usual notion of affine open sets.

We denote by  $f$  the canonical morphism of  $A$  into its ring of quotients  $Q(A)$ , and by  ${}^a f$  the morphism  ${}^a f: \text{Spec } Q(A) \rightarrow \text{Spec } A$  canonically associated to  $f$ .

**Lemma 8.1.** *For every prime  $\mathfrak{p}$  of  $F$ , the localization  $f_{\mathfrak{p}}: A_{\mathfrak{p}} \rightarrow Q(A)_{\mathfrak{p}}$  of  $f$  is an isomorphism.*

**Proof.** Let  $K$  and  $C$  be the kernel and cokernel of  $f$ . Then  $K_{\mathfrak{p}}$  and  $C_{\mathfrak{p}}$  are the kernel and cokernel of  $f_{\mathfrak{p}}$ . Their associated primes are clearly included in  $\mathfrak{p}$ , then belong to  $F$ , since  $\mathfrak{p} \in F$  and  $F$  is closed under generalization:  $\text{Ass}_A(K_{\mathfrak{p}}) \subset F$ , and  $\text{Ass}_A(C_{\mathfrak{p}}) \subset F$  so  $K_{\mathfrak{p}}$  and  $C_{\mathfrak{p}}$  are torsion-free modules [§ 0]. But  $K$  and  $C$  are torsion modules, hence  $K_{\mathfrak{p}}$  and  $C_{\mathfrak{p}}$  are also torsion modules [Lemma 7.2]. Being both torsion and torsion-free they are trivial.

From this lemma it is clear that  $F \subset {}^a f[\text{Spec } Q(A)]$ .

**Theorem 8.2.** *The followings are equivalent:*

- (i)  $(\mathcal{T}, \mathcal{F})$  satisfies the (T)-condition of Goldman.
- (ii)  $F = {}^a f[\text{Spec } Q(A)]$ .
- (iii)  $F$  is affine and  $(\mathcal{T}, \mathcal{F})$  is well-centered.

**Proof.** (i)  $\Rightarrow$  (ii). If  $(\mathcal{T}, \mathcal{F})$  satisfies the (T)-condition of Goldman, and if  $\mathfrak{p} \in T$ , then  $A/\mathfrak{p}$  is torsion [§ 0], and  $\mathfrak{p}Q(A) = Q(A)$  [Theorem 8.0(iii)]. Hence there are no primes in  $Q(A)$  lifting  $\mathfrak{p}$ ,  ${}^a f[\text{Spec } Q(A)]$  is included in  $F$ , in fact is equal to  $F$ , since the reverse inclusion is true [Lemma 8.1].

(ii)  $\Rightarrow$  (iii). If  $F = {}^a f[\text{Spec } Q(A)]$ ; then  $\forall q \in \text{Spec } Q(A)$ ,  $f^{-1}(q) \in F$  and  $f \otimes_A A_{f^{-1}(\mathfrak{p})}$  is an isomorphism [Lemma 8.1], hence  $f$  is a flat epimorphism [10, IV, Proposition 2.4]. Since  $f$  is a flat epimorphism and  $F$  is the set of primes which are lifted in  $Q(A)$ ,  $F$  is affine. Now, if  $M$  is torsion-free,  $M$  is a submodule of  $Q(M)$  and  $Q(M)$  is a  $Q(A)$ -module, then

$$\text{Ass}_A(M) \subset \text{Ass}_A(Q(M)) \subset {}^a f[\text{Ass}_{Q(A)}(Q(M))] \subset {}^a f[\text{Spec } Q(A)] = F$$

(for the second inclusion [10, II, Proposition 3.1]). Then  $(\mathcal{T}, \mathcal{F})$  is well centered [§ 0].

(iii)  $\Rightarrow$  (i). If  $F$  is affine there is a flat epimorphism  $g: A \rightarrow B$  such that  $F = {}^a g[\text{Spec } B]$ . There is also a torsion theory  $(\mathcal{T}', \mathcal{F}')$  such that  $g$  is the canonical morphism of  $A$  into its ring of quotients  $Q'(A) = B$  [9, Proposition 2.7], and such that every  $B$ -module is torsion-free.  $(\mathcal{T}', \mathcal{F}')$  satisfies the (T)-condition of Goldman [Theorem 8.0(ii)], and then the set of free-primes of  $(\mathcal{T}', \mathcal{F}')$  is  ${}^a g[\text{Spec } B] = F$ , since (i)  $\Rightarrow$  (ii).  $(\mathcal{T}', \mathcal{F}')$  is well-centered since (ii)  $\Rightarrow$  (iii), then, in fact,  $(\mathcal{T}', \mathcal{F}')$  is the same torsion theory as  $(\mathcal{T}, \mathcal{F})$  since they both have the same set of free-primes  $F$ , and are both well-centered, hence both have the same torsion-free modules [§ 0].

**Definition 8.3.** If  $(\mathcal{T}, \mathcal{F})$  satisfies the (T)-condition of Goldman, we say also that  $(\mathcal{T}, \mathcal{F})$  is affine.

**Remarks.** If  $Q$  is exact,  $(\mathcal{T}, \mathcal{F})$  is not necessarily affine [1, II, § 2, Exercise 20].  $F$  may be affine, but  $(\mathcal{T}, \mathcal{F})$  not well-centered [(2)].  $f$  may be a flat

epimorphism, but  $(\mathcal{T}, \mathcal{F})$  not affine, even if the ring  $A$  is Noetherian: take  $A$  the ring of polynomials in two indeterminates  $X$  and  $Y$  over any field  $K$ . The torsion theory such that the maximal ideal  $(X, Y)$  is the only torsion prime of  $A$ , is such that  $Q(A) = A$ , however the complement of  $(X, Y)$  in  $\text{Spec } A$  is not affine.

**Proposition 8.4.** *If  $(\mathcal{T}, \mathcal{F})$  is affine the dominant dimension  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M)$  of any  $A$ -module  $M$  can take only the values 0, 1 or  $\infty$ .*

**Proof.** If  $(\mathcal{T}, \mathcal{F})$  is affine, the morphism  $f: A \rightarrow Q(A)$  is a flat epimorphism. We denote the direct image of  $(\mathcal{T}, \mathcal{F})$  under  $f$ , by  $(\mathcal{T}_f, \mathcal{F}_f)$ ; since every  $Q(A)$ -module is torsion-free, the  $(\mathcal{T}_f, \mathcal{F}_f)$ -dominant dimension of every such module is infinite. If  $N$  is an  $A$ -module such that  $(\mathcal{T}, \mathcal{F})\text{-}d_A(N) \geq 2$ , then  $N$  is torsion-free and divisible,  $N = Q(N)$  is a  $Q(A)$ -module and  $(\mathcal{T}_f, \mathcal{F}_f)\text{-}d_{Q(A)}(N) = \infty$ , but  $(\mathcal{T}, \mathcal{F})\text{-}d_A(N) = (\mathcal{T}_f, \mathcal{F}_f)\text{-}d_{Q(A)}(N)$ , since  $f$  is flat [Proposition 3.1].

**Remark.** Morita has shown independently that  $(\mathcal{T}, \mathcal{F})$ -dominant dimension can take only the values 0, 1 and  $\infty$  if and only if  $Q$  is exact [12].

9. Noetherian rings. We suppose now that  $A$  is Noetherian. For any prime  $\mathfrak{p}$  of  $A$ , we let  $(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$  be the direct image of the torsion theory  $(\mathcal{T}, \mathcal{F})$ , under the localization  $A \rightarrow A_{\mathfrak{p}}$  [§ 3], and  $Q_{\mathfrak{p}}$  the quotient functor in the category  $\text{Mod } A$  of  $A_{\mathfrak{p}}$ -modules, relative to this theory.

**Proposition 9.1.** *If  $A$  is Noetherian the following statements are equivalent.*

- (i)  $(\mathcal{T}, \mathcal{F})$  is affine.
- (ii) For every prime  $\mathfrak{p}$  of  $\text{Spec } A$ ,  $(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$  is affine.

**Proof.** The localization  $f_{\mathfrak{p}}$  can be identified with the canonical morphism of  $A_{\mathfrak{p}}$  into its ring of quotients  $Q_{\mathfrak{p}}(A_{\mathfrak{p}})$  for  $(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$  [Proposition 7.1] also the set of free-primes for  $(\mathcal{T}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$  in  $A_{\mathfrak{p}}$  is the set of primes  $F_{\mathfrak{p}}$ , corresponding to the primes of  $F$  contained in  $\mathfrak{p}$  [§ 3], hence  $F = {}^a f[\text{Spec } Q(A)]$  if and only if  $F_{\mathfrak{p}} = {}^a f_{\mathfrak{p}}[\text{Spec } Q_{\mathfrak{p}}(A_{\mathfrak{p}})]$  for every  $\mathfrak{p}$ .

Theorem 8.2 becomes also much simpler:

**Theorem 9.2.** *If  $A$  is Noetherian, the following statements are equivalent:*

- (i)  $(\mathcal{T}, \mathcal{F})$  is affine.
- (ii)  $Q$  is exact.
- (iii) For every module  $M$ ,  $T_2(M) = 0$ .
- (iv) For every module  $M$ ,  $T_n(M) = 0 \forall n \geq 2$ .
- (v) For every module  $M$ ,  $(\mathcal{T}, \mathcal{F})\text{-}d_A(M) = 0, 1$  or  $\infty$ .

**Proof.** (i)  $\Rightarrow$  (ii). [5, Theorem 4.4].

(ii)  $\Rightarrow$  (iii). For every module  $M$  there is a functorial exact sequence  $0 \rightarrow T(M) \rightarrow (M) \rightarrow Q(M) \rightarrow T_1(M) \rightarrow 0$  [Proposition 5.1]. Then if  $\phi: M \rightarrow P$  is an

onto map, there is a commutative square

$$\begin{array}{ccccc}
 & \rightarrow & Q(M) & \rightarrow & T_1(M) & \rightarrow & 0 \\
 (*) & & \downarrow & & Q(\phi) & \downarrow & T_1(\phi) \\
 & & & & \downarrow & & \\
 & \rightarrow & Q(P) & \rightarrow & T_1(P) & \rightarrow & 0
 \end{array}$$

where  $Q(\phi)$  is onto. Then  $T_1$  is right exact and it follows easily that  $T_2$  is trivial.

(iii)  $\Rightarrow$  (iv). Easy by induction on  $n$ , from  $n = 2$ .

(iv)  $\Rightarrow$  (v). Obvious.

(v)  $\Rightarrow$  (iv). If  $M$  is torsion-free and divisible, then  $T(M) = T_1(M) = 0$  and thus  $T_n(M) = 0 \ \forall n \geq 0$ . If  $M$  is a torsion module  $T_n(M) = 0 \ \forall n \geq 1$  [Lemma 5.2]. Consequently if  $M$  is torsion-free, using the exact sequence  $0 \rightarrow M \rightarrow Q(M) \rightarrow T_1(M) \rightarrow 0$  where  $Q(M)$  is torsion-free and divisible and where  $T_1(M)$  is torsion, one can prove that  $T_n(M) = 0 \ \forall n \geq 2$ . Finally if  $M$  is any  $A$ -module, using the exact sequence  $0 \rightarrow T(M) \rightarrow M \rightarrow M/T(M) \rightarrow 0$ , where  $T(M)$  is torsion and where  $M/T(M)$  is torsion-free, one can prove that  $T_n(M) = 0 \ \forall n \geq 2$ .

(iv)  $\Rightarrow$  (iii). Obvious.

(iii)  $\Rightarrow$  (ii). If  $T_1$  is right exact, then  $Q$  also is right exact, using again the commutative square (\*). Since  $Q$  is always left exact,  $Q$  is exact.

**10. Applications: affine open sets.** If  $A$  is Noetherian and if  $U$  is an open set of  $\text{Spec } A$ , there is a unique torsion theory over  $A$  such that  $U = F$ . Also if we denote  $\text{Spec } A$  by  $X$ , and the quasi-coherent  $\mathcal{O}_X$ -module canonically associated to an  $A$ -module  $M$  by  $\tilde{M}$  then  $\tilde{M}(U)$  is functorially isomorphic to  $Q(M)$  [Theorem 6.1]. The following proposition, a direct consequence of Theorem 9.2, is similar to Serre's theorem [15, Theorem 1].

**Proposition 10.1.** *Let  $A$  be a Noetherian ring and  $U$  an open set of  $\text{Spec } A$ , the following statements are equivalent:*

- (i)  $U$  is affine.
- (ii) For every module  $M$ ,  $\tilde{M}(U) \cong M \otimes_A \mathcal{O}_X(U)$ .
- (iii) The functor  $M \rightarrow \tilde{M}(U)$  is exact in  $\text{Mod } A$ .

We now characterize the affine open sets of a regular Noetherian ring, that is to say a Noetherian ring  $A$  such that  $A_{\mathfrak{p}}$  is a regular local ring at every prime  $\mathfrak{p}$ . If  $I$  is an ideal of  $A$ , we let  $V(I)$  be the set of primes containing  $I$  and  $b(I)$ , the height of  $I$ , be the maximum of the heights of the minimal primes of  $I$ . First we suppose only that  $A$  is Cohen-Macaulay, that is to say that  $\text{Depth}(A_{\mathfrak{p}}) = b(\mathfrak{p})$ , for every prime  $\mathfrak{p}$ .

**Proposition 10.2.** *Let  $A$  be a Noetherian Cohen-Macaulay ring and  $(\mathcal{J}, \mathcal{F})$  an affine torsion theory over  $A$ . If  $\mathfrak{p} \in \mathcal{T}_0$ , that is  $\mathfrak{p}$  is minimal among the torsion-primes, then  $b(\mathfrak{p}) \leq 1$ .*

**Proof.** Since  $\mathfrak{p} \in \mathcal{T}_0$ ,  $\mathfrak{p}A_{\mathfrak{p}}$  is the only torsion-prime of the theory  $(\mathcal{J}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$ , direct image of  $(\mathcal{J}, \mathcal{F})$  under the localization  $A \rightarrow A_{\mathfrak{p}}$ , hence  $(\mathcal{J}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \text{Depth}(A_{\mathfrak{p}})$  [Theorem 2.1]. But  $\text{Depth}(A_{\mathfrak{p}}) = b(\mathfrak{p}) < \infty$ , and since  $(\mathcal{J}, \mathcal{F})$  is affine,  $(\mathcal{J}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})$  also is affine [Proposition 9.1] and  $(\mathcal{J}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}})\text{-}d_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = b(\mathfrak{p}) \leq 1$  [Proposition 8.4].

**Theorem 10.3.** *Let  $A$  be a regular Noetherian ring,  $I$  an ideal of  $A$ , and  $U$  the complement of  $V(I)$  in  $\text{Spec } A$ , then*

(1)  *$U$  is an affine open set if and only if  $ht(I) \leq 1$ .*

(2) *If  $A$  is local and  $U$  is affine, then  $U$  is a special open set, that is  $U$  is the set of primes which do not contain an element  $f$  of  $A$ .*

**Proof.** A regular Noetherian ring is also Cohen-Macaulay. If  $U$  is affine, since the minimal primes of  $I$  are also the minimal torsion-primes of the theory such that  $F = U$ , then from the previous proposition  $b(I) \leq 1$ . Conversely if  $b(I) \leq 1$ , then, for every prime  $\mathfrak{p}$ ,  $b(I_{\mathfrak{p}}) \leq 1$ ; also to prove that  $U$  is affine it is enough to prove that for every  $\mathfrak{p}$ , the complement  $U_{\mathfrak{p}}$  of  $V(I_{\mathfrak{p}})$  in  $\text{Spec } A_{\mathfrak{p}}$  is affine [Proposition 9.1]. Then we can assume that  $A$  is local, so it is also a unique factorization domain. If  $ht(I) = 0$ , then  $U = \text{Spec } A$  is clearly a special open set. If  $ht(I) = 1$ , the minimal primes of  $I$  must all be of height 1, and thus be principal. If  $p_1, p_2, \dots, p_n$  are respective generators of those primes it is easy to check that  $U$  is the set of primes which do not contain  $f = p_1 \cdot p_2 \cdot \dots \cdot p_n$ .

**Remark.** This theorem is similar to the characterization of affine open sets, complement of a nonsingular variety in the projective space [7, II, Proposition 3.1].

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