RESTRICTING A SCHAUDER BASIS TO A SET OF POSITIVE MEASURE

BY

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ABSTRACT. Let \( \{f_n\} \) be an orthonormal system of functions on \([0, 1]\) containing a subsystem \( \{f_{n_k}\} \) for which (a) \( f_{n_k} \to 0 \) weakly in \( L_2 \), and (b) given \( E \subset [0, 1] \), \( |E| > 0 \), \( \liminf \int_E |f_{n_k}(x)| \, dx > 0 \). There then exists a subsystem \( \{g_n\} \) of \( \{f_n\} \) such that for any set \( E \) as above, the linear span of \( \{g_n\} \) in \( L_1(E) \) is not dense.

For every set \( E \) as above, there is an element of \( L_p(E), 1 < p < \infty \), whose Walsh series expansion converges conditionally and an element of \( L_1(E) \) whose Haar series expansion converges conditionally.

1. Gaposhkin, in a discussion of certain properties of "lacunary" systems of functions, see [G], makes extensive use of the following notion: A sequence \( \{f_n\} \) of real-valued functions on \([0, 1]\) is a "Riesz system" if the following estimates hold:

\[
A_1 \left( \sum_{n=1}^{N} C_n^2 \right)^{1/2} \leq \int_0^1 \left| \sum_{n=1}^{N} C_n f_n(x) \right| \, dx \leq A_2 \left( \sum_{n=1}^{N} C_n^2 \right)^{1/2}, \quad N \geq 1,
\]

where the constants \( 0 < A_1 \leq A_2 \) are independent of the choice of \( \{C_n\} \) and of \( N \). An inequality of this type is used as a definition of the term "lacunary" in [KP] in an examination of complemented subspaces of the \( L_p \) spaces. Moreover, the classical definition of a lacunary trigonometric system is used primarily to insure that such an inequality obtains [Zy, p. 203]. One difficulty in the application of this notion is finding enough Riesz systems in context. To this end, a sequence \( \{f_n\} \) is said to have "property (B)" if there is a subsequence \( \{f_{n_k}\} \) such that \( f_{n_k} \to 0 \) weakly in \( L_2 \) and \( \liminf \int_0^1 |f_{n_k}(x)| \, dx > 0 \).

A sequence \( \{f_n\} \) is said to have "property (B')" if there is a subsequence \( \{f_{n_k}\} \) such that \( f_{n_k} \to 0 \) weakly in \( L_2 \) and \( \liminf \int_E |f_{n_k}(x)| \, dx > 0 \) whenever \( E \subset (0, 1) \) and \( |E| > 0 \). The Lemma 1.2.6 of [G] shows that any sequence of functions having property (B) contains a subsequence that is a Riesz system. Lemma 1.2.6' of [G] shows that for any sequence of functions \( \{f_n\} \) having property (B') there is a subsequence \( \{f_{n_k}\} \) for which the following obtains: given \( E \subset [0, 1] \) and \( |E| > 0 \), there is a \( k_0 = k_0(E) \) such that \( f_{n_k} \to \text{something} \) is a Riesz system when the \( f_{n_k} \) are restricted to the set \( E \).

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In [PZ] Price and Zink, by taking advantage of the symmetry properties of the Rademacher functions, conclude that for any set \( E \subset (0, 1) \), \(|E| > 0\), the closed linear span of the Rademacher functions in \( L_2^2(E) \) is not all of \( L_2^2(E) \). We offer another explanation for this phenomenon by noting that the conclusion of Lemma 1.2.6 (cited above) applies to the Rademacher system. We also obtain a generalization of the Khintchine inequality [K, p. 130] that is of use in the next section.

Let \( \|f\| = \int_0^1 |f(t)| \, dt \) and \( \|f\|_E = \int_E |f(t)| \, dt \) when \(|E| > 0\). For each \( f \) in \( L_1(0, 1) \) define

\[
f^E(t) = \begin{cases} f(t) & \text{if } t \text{ is in } E, \\ 0 & \text{otherwise.} \end{cases}
\]

For \( X \subset L_1(0, 1) \) let \( X^E \) denote the closure in \( L_1(E) \) of \( \{f^E : f \text{ is in } X\} \).

A set \( S \subset L_1(0, 1) \) is "complete on the set \( E \)" where \(|E| > 0\) provided that \( S^E = L_1(E) \). Otherwise, \( S \) is "incomplete on the set \( E \)."

**Lemma 1.** Let \( X \) be a subspace of \( L_1(0, 1) \) having a separable topological dual \( X' \) and for which \( X^E \) has finite codimension in \( L_1(E) \), \(|E| > 0\).

There then exists an \( f \) in \( X^E \) such that if \( \{f_n\} \subset X \) and \( \|f - f_n\|_E = o(1) \), then \( \|f_n\| \to \infty \). Such functions comprise all of \( X^E \) except for a set of the first category in \( X^E \).

**Proof.** There can be no constant \( M > 0 \) such that given an \( \epsilon > 0 \) and an \( f \) in \( X^E \), there exists a \( g \) in \( X \) for which \( \|f - g\|_E \) and \( \|g\| \leq M\|f\|_E \), where \( M \) is independent of \( \epsilon \) and \( f \). For, suppose that such an \( M \) exists. Let

\[
\mathcal{R} = \{f^E : f \in X, \|f\|_E \leq 1, \text{ and } \|f\| \leq M\}, \quad S = \{f : f \in X^E, \text{ and } \|f\|_E \leq 1\},
\]

and let \( T \in (X^E)' \). \( T \) can be extended to an element of \( L_1'(0, 1) \), and so there exists a bounded measurable function \( b \) for which \( T(f) = \int_E f(t)b(t) \, dt \).

Let \( \|T\| \) denote the norm of \( T \) as an operator on \( X^E \), and let \( N(T) \) denote the norm of \( T \) as an operator on \( X \). Since \( R \) is by assumption dense in \( S \),

\[
MN(T) \geq \sup \{||T(f)|| : f \in \mathcal{R}\} = \sup \{||T(f)|| : f \in S\} = ||T||.
\]

The nonseparability of \( (X^E)' \) now implies the nonseparability of \( X^E \), a contradiction.

Now let \( K_n = \{f \in X^E : \|f\|_E \leq 1 \} \) and there is a sequence \( \{f_m\} \subset X \) for which \( \|f_m - f\|_E = o(1) \) and \( \|f_m\| \leq n \), for all \( m \).

Each \( K_n \) is closed, convex, symmetric with respect to the origin in \( X^E \). From what has been shown above, \( K_n \) is not all of the unit ball in \( X^E \) and is consequently nowhere dense. Thus \( \bigcup_n K_n \) is of the first category, which is precisely the conclusion desired.
A set \( S \subset L^1(0, 1) \) is "complete on a set \( E \)" where \(|E| > 0\) provided that for any \( f \) in \( L^1(0, 1) \) and any \( \epsilon > 0 \), there is a \( g \) in \( S \) for which \( \|f - g\|_E < \epsilon \).

**Theorem 2.** Let \( \{f_n\} \) be an orthonormal sequence of functions in \( L^1(0, 1) \), \( E \subset (0, 1) \), and \(|E| > 0\), and let \( X \) denote the closed linear span of \( \{f_n\} \) in \( L^1(0, 1) \).

If \( X' \) is separable and \( \{f_n^E\} \) is a Riesz system, then \( \{f_n\} \) is incomplete on the set \( E \).

**Proof.** Suppose \( \{f_n\} \) is complete on the set \( E \). Let \( f \) be as in the conclusion of Lemma 1 and let \( g_n \subset X \) be such that \( \|f - g_n\|_E = o(1) \). Lemma 1 implies that \( \|g_n\| \to \infty \) but the hypothesized properties of \( \{f_n\} \) imply that \( \|g_n\| \) is bounded, a contradiction.

**Corollary 3.** Every orthonormal system \( \{f_n\} \) satisfying property (B') contains a subsystem that is incomplete on every set of positive measure.

**Proof.** Lemma 1.2.6' of [G] assures the existence of a subsystem \( \{f_{nk} \} \) having the following property. Given \( E \subset (0, 1) \), \(|E| > 0\), there is a \( k_0 = k_0(E) \) such that \( \{f_{nk} \} \) and \( \{f_{nk'} \} \) are both Riesz systems. Theorem 2 now implies that \( \{f_{nk} \} \) is incomplete on the set \( E \), proving the corollary.

As a special case, we consider the Walsh system [K, p. 132]. This system is easily seen to satisfy property (B').

**Lemma 4.** Let \( \{w_n\} \) be a subsystem of the orthonormal system of Walsh possessing the property (a):

(a) if \( b_1, \ldots, b_k, i_1, \ldots, i_k \) denote positive integers, then

\[
\int_0^1 w^1_{i_1}(t) \cdots w^k_{i_k}(t) dt = \begin{cases} 1, & \text{when all } b_i \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}
\]

Let \( 1 \leq p \leq 2, E \subseteq [0, 1], |E| > 0, 0 < \epsilon < |E|/5, \) and let \( K \subseteq E \) with \(|E \sim K| < \epsilon \). Then there are constants \( A, B, \) and \( N \) depending only on \( E \) such that for any \( n > m > N \), and any sequence \( a_m \cdots a_n \) of real numbers

\[
A \left( \sum_{m}^{n} a_i^2 \right)^{1/2} \leq \left( \int_E \left| \sum_{m}^{n} a_i w_i(t) \right|^p dt \right)^{1/p} \leq B \left( \sum_{m}^{n} a_i^2 \right)^{1/2}.
\]

**Proof.** For any measurable set \( S \subset [0, 1] \), define \( S_{ij} = \int_S w_i(t) w_j(t) dt \). Note that \( \{w_i w_j\}_{i < j} \) is a subsystem of the Walsh system and that \( S_{ij} \) is the \( i, j \)th Walsh-Fourier coefficient of the characteristic function of \( S \). Bessel's inequality implies that \( \sum_{i < j} S_{ij}^2 \leq |S| \).
Since \( \| \cdot \|_1 \leq \| \cdot \|_p \leq \| \cdot \|_2 \), it is sufficient to establish the conclusion of the lemma for \( p = 1 \) and \( p = 2 \). For \( p = 2 \)

\[
\int_K \left( \sum_{m} a_i w_i(t) \right)^2 dt = \int_K \left( \sum_{m} a_i^2 + 2 \sum_{m \leq i < j \leq n} a_i a_j w_i(t) w_j(t) \right) dt
\]

\[
\leq |K| \sum_{m} a_i^2 + 2 \left( \sum_{m \leq i < j \leq n} a_i^2 a_j^2 \right)^{1/2} \left( \sum_{m \leq i < j \leq n} K_{ij}^2 \right)^{1/2}.
\]

A direct calculation yields \( \sum_{m \leq i < j \leq n} a_i^2 a_j^2 \)^{1/2} = \( \sum_{m} a_i^2 \). Also,

\[
\left( \sum_{m \leq i < j \leq n} K_{ij}^2 \right)^{1/2} \leq \left( \sum_{m \leq i < j \leq n} E_{ij}^2 \right)^{1/2} + \left( \sum_{m \leq i < j \leq n} (E \sim K)_{ij}^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{m \leq i < j \leq n} E_{ij}^2 \right)^{1/2} + \epsilon.
\]

Thus, if \( N \) is chosen so that \( \sum_{N} E_{ij}^2 < \epsilon \), and \( n > m > N \), then

\[
\int_K \left( \sum_{m} a_i w_i(t) \right)^2 dt \leq |E| \sum_{m} a_i^2 + 2 \left( \sum_{m} a_i^2 \right) (2\epsilon)
\]

\[
= \left( |E| + 4\epsilon \right) \left( \sum_{m} a_i^2 \right).
\]

Similar considerations will establish the other half of the desired inequality. One obtains

\[
\int_K \left( \sum_{m} a_i w_i(t) \right)^2 dt \geq \left( |E| - 5\epsilon \right) \left( \sum_{m} a_i^2 \right),
\]

and the case \( p = 2 \) is established.

For \( p = 1 \)

\[
\int_K \left| \sum_{m} a_i w_i(t) \right| dt \leq |E|^{1/2} \left( \int_K \left( \sum_{m} a_i w_i(t) \right)^2 dt \right)^{1/2}
\]

\[
\leq |E|^{1/2} (|E| + 4\epsilon)^{1/2} \left( \sum_{m} a_i^2 \right)^{1/2}
\]

provided that \( n > m > N(\epsilon) \).

For the left-hand side we have:
An application of Hölder's inequality with $p = 3/2$ and $q = 3$ yields

$$
\left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} \leq \left( |E| - 5\varepsilon \right)^{-1/2} \left( \int_{K} \left( \sum_{m}^{n} a_i w_i(t) \right)^2 dt \right)^{1/2}
$$

$$
= A \left( \int_{K} \left( \sum_{m}^{n} a_i w_i(t) \right)^{2/3} \left( \sum_{m}^{n} a_i w_i(t) \right)^{4/3} dt \right)^{1/2}.
$$

An application of Hölder's inequality with $p = 3/2$ and $q = 3$ yields

$$
\left( \sum_{m}^{n} a_k^2 \right)^{1/2} \leq A \left( \int_{K} \left| \sum_{m}^{n} a_i w_i(t) \right| dt \right)^{1/3} \left( \int_{K} \left( \sum_{m}^{n} a_i w_i(t) \right)^4 dt \right)^{1/6}.
$$

Observe that the Khintchine inequality [K, p. 130] is valid for any system of Walsh functions which possesses property (\ast). Taking this into account, set $p = 4$ in the Khintchine inequality. The last term in line (+) above can now be replaced by $B(\Sigma_{m}^{n} a_i)^{1/3}$, where $B > 0$ depends only on the choice of $p = 4$, without disturbing the sense of the inequality. The desired result is obtained when both sides of the resultant expression are cubed.

**Corollary 5.** No system $\{w_k\}$ of Walsh functions possessing property (\ast) of Lemma 4 can be complete on any set of positive measure.

**Proof.** Let $E \subset (0, 1)$, $|E| > 0$. Lemma 4 implies the existence of $N = N(E)$ for which $\{w_n\}_{n > N}$ and $\{w^E_n\}_{n > N}$ are both Riesz systems. Theorem 2 now implies that $\{w_n\}$ is incomplete on the set $E$.

A "quasi-basis" for a Banach space $(X, \|\cdot\|)$ is a double sequence $\{x_n, X_n\}$ of elements of $X$ and of continuous linear functionals, respectively, such that the series $\sum_{n}^{\infty} X_n(x)x_n$ converges in norm to $x$ for each $x$ in $X$. Such a structure may arise in the following way.

Suppose $\{y_n, Y_n\}$ is a Schauder basis for $L_p(0, 1)$, $1 \leq p < \infty$. Let $|E| > 0$ and define for all $z$ in $L_p(0, 1)$

$$
z^E(t) = \begin{cases} 
x(t), & t \in E, \\
0, & \text{otherwise}.
\end{cases}
$$

For each given $\phi$ in $L_p'(0, 1)$, define $\phi^E(z) = \phi(z^E)$ for any $z$ in $L_p(0, 1)$.

It is easily verified that $\{y^E_n, Y^E_n\}$ is a quasi-basis for $L_p(E)$. This double sequence shall be called the "restriction of $\{y_n, Y_n\}$ to the set $E$".

The quasi-basis $\{x_n, X_n\}$ is "unconditional" if $\sum_{n}^{\infty} X_n(x)x_n$ converges unconditionally for every $x$ in $X$. Otherwise, the quasi-basis is "conditional."

The restriction of an unconditional basis for $L_p(0, 1)$ to any set $E$, $|E| > 0$, is clearly an unconditional quasi-basis for $L_p(E)$. A problem that arises in this context is the determination of the sets $E$, if there are any, for which a given conditional basis for $L_p(0, 1)$ restricts to an unconditional quasi-basis for $L_p(E)$.
Let \( (X, \| \cdot \|) \) denote a Banach space of equivalence classes of Lebesgue measurable functions on \([0, 1]\) (see [L, p. 66] and [Z, Chapter 15]). Such spaces may be defined by specifying a certain class \( H \) of elements of \( L_1(0, 1) \) and defining
\[
\| x \| = \sup \left\{ \int_0^1 |x(t)b(t)| \, dt : b \in H \right\} \quad \text{and} \quad X = \{ x : \| x \| < \infty \}.
\]
Various conditions may be imposed on \( H \) in order that \( (X, \| \cdot \|) \) turns out as a Banach space.

Theorem 6. Given a quasi-basis \( \{ x_n, X_n \} \) for \( (X, \| \cdot \|) \). Let \( \{ r_n \} \) denote the Rademacher system and define
\[
C = \left\{ \theta : \sum_{k=1}^{\infty} r_k(\theta)X_k(x)x_k \text{ converges for all } x \in X \right\}.
\]
Let \( Z_N = \{ x \in X : X_1(x) = \cdots = X_N(x) = 0 \} \) and let \( G(x) = \| (\sum_{n=1}^{\infty} X_n(x)x_n^2)X \| \).

Then \( C \) is a Borel set, and if \( |C| > 0 \) there is an \( N \) for which the identity mapping \( (Z_N, \| \cdot \|) \rightarrow (Z_N, G(\cdot)) \) is continuous.

Proof. Let \( x(n, m, \theta, t) = \sum_{k=1}^{m} r_k(\theta)X_k(x)x_k(t) \). For each \( (n, m) \),
\[
\| x(n, m, \theta, \cdot) \| \text{ is a step function of } \theta \text{ and}
\]
\[
C = \left\{ \theta : \limsup_{n,m \to \infty} \| x(n, m, \theta, \cdot) \| = 0 \right\}.
\]
\( C \) is therefore a Borel set.

For each \( \theta \) in \( C \), the sequence of linear operators \( T_{n\theta}(\cdot) \) defined by
\[
T_{n\theta}(x) = \sum_{k=1}^{n} r_k(\theta)X_k(x)x_k
\]
is uniformly bounded by virtue of the Banach-Steinhaus theorem. The set \( S_M = \{ \theta : \| T_{n\theta} \| \leq M \text{ for all } n \} \) is clearly a Borel set, and \( \bigcup (S_M \cap C) = C \). Since \( |C| > 0 \), there is a constant \( K > 0 \) for which \( |S_K \cap C| > 0 \). Define \( S = S_K \cap C \), and for each \( \theta \) in \( S \) let \( T_\theta(x) = \lim_n T_{n\theta}(x) \). Then,
\[
\| T_\theta(x) \| = \lim_n \| T_{n\theta}(x) \| \leq M \| x \| \quad \text{for all } \theta \in S,
\]
and \( S \) is seen to be a Borel set of positive measure.
For any \( \epsilon > 0 \) there is, by definition of \( \| \cdot \| \), an \( b \) in \( H \) such that

\[
G(x) \leq \int_0^1 \left( \sum_{k=1}^{\infty} X_k^2(x)x_k^2(t) \right)^{1/2} |b(t)| \, dt + \epsilon.
\]

By Lemma 4, if \( F \subset S \) is of sufficiently large measure, there is a \( B > 0 \) and an integer \( N > 0 \), depending only on \( S \), such that

\[
B \left( \sum_{n=N}^{\infty} X_n^2(x)x_n^2(t) \right)^{1/2} |b(t)| \leq \int_F |x(N, n, \theta, t)b(t)| \, d\theta
\]

for every \( x \) in \( X \), and for every \( n > N \).

For each \( \theta \) in \( S \),

\[
\lim_n \int_0^1 |x(N, n, \theta, t)b(t)| \, dt = \int_0^1 |x(n, \infty, \theta, t)b(t)| \, dt.
\]

By Egoroff's theorem there is \( F \subset S \) in which the convergence in (4) is uniform, and having measure large enough so that line (3) obtains. We may integrate (4) with respect to \( \theta \) over the set \( F \), and the order of integration on the left side may be reversed. This yields

\[
\lim \int_0^1 \left[ \int_F |x(N, n, \theta, t)b(t)| \, d\theta \right] dt = \int_F \left[ \int_0^1 |x(N, \infty, \theta, t)b(t)| \, dt \right] d\theta.
\]

Assume now that \( x \) is in \( Z \), and combine (1), (2), (3), and (5):

\[
G(x) \leq \frac{1}{B} \int_F \left[ \int_0^1 |x(N, \infty, \theta, t)b(t)| \, dt \right] d\theta + \epsilon \leq \frac{1}{B} \int_F \|T_\theta(x)\| \, d\theta + \epsilon \leq \frac{M}{B} \|x\| + \epsilon
\]

for all \( x \) in \( Z \), and for any \( \epsilon > 0 \).

The norm \( G(\cdot) \) and unconditional convergence have been discussed previously in [0], and in the case that \( \{x_n, X_n\} \) is an unconditional basis it has been shown in [SZ] that the norms \( G(\cdot) \) and \( \| \cdot \| \) are equivalent.

Theorem 6 can be used to partially extend Lemma 4 to all of the spaces \( L_p(0, 1), 1 < p < \infty \):

Corollary 7. Let \( \{w_n\} \) be a subsystem of the Walsh system possessing the property (\( \ast \)) described in Lemma 4, and let \( 1 \leq p < \infty \), \( |E| > 0 \). Then there are constants \( A, B, \) and \( N \) depending only on \( E \) such that for any \( n > m > N \) and any sequence \( a_m \ldots a_n \) of real numbers

\[
A \left( \sum_{m}^{n} a_i^2 \right)^{1/2} \leq \left( \int_E \left| \sum_{m}^{n} a_i w_i(t) \right|^p \, dt \right)^{1/p} \leq B \left( \sum_{m}^{n} a_i^2 \right)^{1/2}.
\]
Proof. For $2 \leq p < \infty$, let $N$ be determined by the set $E$ as in Lemma 4, and let $Y$ be the closed linear span of $\{w_i^E\}_{i \in \mathbb{N}}$ in $L_p(E)$. If continuous linear functionals $W_i$ on $Y$ can be found for which $\{w_i, W_i\}_{i \geq N}$ is an unconditional basis for $Y$, the desired conclusion will then follow from Lemma 4, Theorem 6 and the fact that $\|\cdot\|_p \geq \|\cdot\|_2$.

Lemma 4 implies that for each $x$ in $Y$, there is a unique sequence $\{a_i\}$ in $l_2$ for which the series $\sum_{i \geq N} a_i w_i^E$ converges unconditionally to $x$ in the norm of $L_2(0, 1)$. The Khintchine inequality with the functions $w_i$ substituted for Rademacher's functions [see proof of Lemma 4 above] implies that $\sum_{i \geq N} a_i w_i \equiv \bar{x}$ converges unconditionally in the norm of $L_p(0, 1)$ and that the mapping $x \rightarrow \bar{x}$ taking $Y \rightarrow L_p(0, 1)$ is continuous in that norm.

Clearly, the functionals $W_i$ defined by $W_i(x) = \int_0^1 w_i(t)x(t)dt$ are as desired.

The case $1 < p < 2$ is a special case of Lemma 1, proving the corollary.

It is known that the Walsh system is a conditional basis for $L_p(0, 1)$, $1 < p < \infty$, $p \neq 2$, [O] and [P]. Also, the Haar system is a conditional basis for $L^1(0, 1)$, [D] and [M]. The conditionality of these systems is preserved under restriction to a set of positive measure.

Theorem 8. Let $\{w_n, W_n\}$ denote the quasi-basis for $L_p(E)$, $1 < p < \infty$, obtained by restricting the Walsh system to a set $E$, $|E| > 0$.

If $p \neq 2$, then $\{w_n, W_n\}$ is a conditional quasi-basis for $L_p(E)$. If $p = 2$, then this quasi-basis is unconditional.

Proof. If $p = 2$, the Walsh system is an unconditional basis for $L_p(0, 1)$, and so $\{w_n, W_n\}$ is an unconditional quasi-basis for $L_p(E)$.

The remaining cases $1 < p < 2$ and $2 < p < \infty$ are considered separately. For $p < 2$, let $x \in L_p(E) \sim L_2(E)$. Then

$$G(x) = \left\| \left( \sum_{n=1}^{\infty} W_n^2(x)w_n^2 \right)^{1/2} \right\|_p = \left( \sum_{n=1}^{\infty} W_n^2(x) \right)^{1/2} |E| = \infty.$$ 

It follows that the series $\sum_{n=1}^{\infty} W_n(x)w_n$ is conditionally convergent [O].

For $p > 2$, let $y \in L_q(E)$ where $1/p + 1/q = 1$. Were $\{w_n, W_n\}$ an unconditional quasi-basis for $L_p(E)$, then

$$
\varepsilon(x) = \lim_{n} \sum_{k=1}^{n} W_k(x)e_k w_k
$$

would exist for every $x$ in $L_p(E)$ and for every sequence $\varepsilon = \{e_k\}$ where $e_k = \pm 1$. Then
would converge for all \( x \) and \( \epsilon \).

The series \( \sum_{k=1}^{\infty} W_k(y) \) would then be subseries convergent in the weak topology on \( L_q(E) \) and hence subseries convergent in the norm topology [D, p. 60].

\( \{w_k, W_k\} \) would then be an unconditional quasi-basis for \( L_q(E) \), which is impossible since \( 1 < q < 2 \).

**Theorem 9.** Let \( \{b_{np}^E, H_{np}^E\} \) denote the quasi-basis for \( L_1(E) \) obtained by restricting the Haar system to a set \( E, |E| > 0 \).

Then this system is a conditional quasi-basis for \( L_1(E) \).

**Proof.** Theorem 6 implies that if \( \{b_{np}^E, H_{np}^E\} \) were an unconditional quasi-basis for \( L_1(E) \), then the norm \( \|\cdot\| \) would dominate the norm \( G(.) \) on some space \( Z_N = \{x \in L_1(E): H_{np}^E(x) = 0 \text{ for } n < N, p = 0, 1, \ldots, 2^n - 1\} \). But this cannot be the case: a sequence \( \{y_p\} \) is constructed for which

- (A) \( \{y_p\} \subset Z_N \),
- (B) \( \{y_p\} \) is bounded in \( (L_1(E), \|\cdot\|) \),
- (C) \( \{y_p\} \) is unbounded in \( (L_1(E), G(.) ) \).

Let \( p \) be a fixed positive integer, \( p > 4 \), and let \( \epsilon < 1/p \). Since almost every point of \( E \) is a point of metric density 1, there exists a sequence \( \{ I_{kn} \}_{k=0}^{p} \) of dyadic intervals, each of which is the support of a Haar function which shall be denoted by \( b_k \), and for which

\[
I_{k+1} = \text{the left half of } I_k,
\]

\[
|I_{k+1}| = 2^{-n-k}, \quad k = 0, 1, \ldots, p,
\]

\[
0 < 1 - |E \cap I_k|/|I_k| < \epsilon, \quad k = 0, \ldots, p.
\]

Let \( E_k = E \cap I_k \) and split \( E_k \) into a left half \( L_k \) and a right half \( R_k \) of equal measure. Define, for \( k = 0, 1, \ldots, p \),

\[
Y_k(t) = \begin{cases} 
\sqrt{2^{n+k}} & \text{if } t \text{ is in } L_k, \\
-\sqrt{2^{n+k}} & \text{if } t \text{ is in } R_k, \\
0 & \text{otherwise,}
\end{cases}
\]

and let \( y_p = \sum_{k=0}^{p} \sqrt{2^{n+k}} Y_k \).

The proof shall be completed when it is demonstrated that the sequence \( \{y_p\} \) possesses properties (A), (B) and (C) above. This necessitates estimates on \( H_{i_p}^E(y_p) = \sum_{k=0}^{p} \sqrt{2^{n+k}} H_{i_k}^E(Y_k) \).

The function \( Y_k \) resembles the Haar function \( b_k \) restricted to the set \( E \). In particular, if \( e_k \) is defined by the formula

\[
Y_k(t) = b_k^E(t) + e_k(t)
\]
then
\[ |\text{supp } e_k| \leq \frac{1}{2}(|l_k| - |E_k|) \leq \epsilon 2^{-n-k-1}, \]
\[ |e_k(t)| \leq 2\sqrt{2^{n+k}}. \]

Estimation of \( H^E_i(Y_k) \) for \( k \geq i \):
\[ |H^E_i(b_i)| = 2^{n+i}|E_i| \]
and
\[ |H^E_i(e_i)| \leq \sqrt{2^{n+i}} \max |e_i(t)| |\text{supp } e_i| = 2^{n+i+1}|\text{supp } e_i| \leq \epsilon. \]

Thus,
\[ |H^E_i(Y_k)| \geq 2^{n+i}|E_i| - \epsilon \geq 1 - 3\epsilon. \]

If \( k > i \), then \( \text{supp } Y_k \) is contained in that part of \( E_i \) on which \( b_i \) is of constant sign. In this case, \( H^E_i(Y_k) = 0 \).

Estimation of \( H^E_i(Y_k) \) for \( k < i \):
\[ |H^E_i(b_k)| \leq \sqrt{2^{n+i}}\sqrt{2^{n+k}}(|l_i| - |E_i|) \leq \epsilon \sqrt{2^{n+i}}\sqrt{2^{n+k}}2^{-n-i} = \epsilon \sqrt{2^{k-i}}, \]
\[ |H^E_i(e_k)| \leq 2\sqrt{2^{n+i}}\sqrt{2^{n+k}}|\text{supp } e_k \cap E_i|. \]

If \( k < i - 1 \), \( \text{supp } e_k \cap E_i = \emptyset \) and then \( H^E_i(e_k) = 0 \). Otherwise, \( |H^E_i(e_{i-1})| \leq 2\sqrt{2^{n+i}}\sqrt{2^{n+i-1}}2^{-n-i-1} = \epsilon \sqrt{2}. \) This gives
\[ |H^E_i(Y_k)| \leq \epsilon \sqrt{2^{k-i}} \text{ if } k < i - 1, \]
\[ |H^E_i(Y_{i-1})| \leq \epsilon \sqrt{2} + \epsilon \sqrt{2} \leq 3\epsilon. \]

Upon combining the above estimates in \( H^E_i(Y_k) \),
\[
|H^E_i(y_p)| = \left| \sum_{k=0}^i \sqrt{2^{n+k}}H^E_i(Y_k) \right|
\geq \sqrt{2^{n+i}}H^E_i(Y_i) - \sqrt{2^{n+i-1}}|H^E_i(Y_{i-1})| \sum_{k=0}^{i-2} \sqrt{2^{n+k}}|H^E_i(Y_k)|
\geq \sqrt{2^{n+i}}(1 - 3\epsilon) - 3\epsilon \sqrt{2^{n+i-1}} - \epsilon \sum_{k=0}^{i-2} \sqrt{2^{n+k}}\sqrt{2^{k-i}}
\geq \sqrt{2^{n+i}}(1 - 6\epsilon) - \epsilon \sqrt{2^{n-i}} \sum_{k=0}^{i-2} 2^k
\geq \sqrt{2^{n+i}}(1 - 6\epsilon) - \epsilon \sqrt{2^{n+i}} \sqrt{2^{n+i-1} - 7\epsilon}. \]
Let $\|\cdot\|_E$ denote the $L_1$-norm of functions restricted to $E$. An application of the estimate above then yields

$$G(y_p) \geq \left\| \left( \sum_{i=0}^{p} \left[ H_i^E(y_p) \right]^2 b_i^2 \right)^{1/2} \right\|_E \geq A \left( \sum_{i=0}^{p} 2^{n+i} b_i^2 \right)^{1/2} \geq B \sum_{i=0}^{p} 2^{-n-i} \left( \sum_{k=0}^{i} 2^{2(n+k)} \right)^{1/2} \geq C_p.$$ 

Statement (C) is thereby established. Moreover,

$$\|y_p\| \leq \left\| \sum_{k=0}^{p} \sqrt{2^{n+k}} b_k \right\| + \left\| \sum_{k=0}^{p} \sqrt{2^{n+k}} e_k \right\| \leq 2 + \sum_{k=0}^{p} \sqrt{2^{n+k}} \|e_k\| \leq 2 + \rho \epsilon \leq 3.$$ 

This establishes statement (B). Given any $N$, it is clear that the intervals $I_k$ can be so chosen that the functions $y_k$ are all elements of $Z^N$. Then $y_p$ will also be an element of $Z^N$, and (A) is established.

REFERENCES


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