TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 184, October 1973

MEASURABLE TRANSFORMATIONS ON COMPACT GROUPS(1)

BY

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ABSTRACT. For an arbitrary finite Baire measure μ on an arbitrary compact group G, it is shown that every automorphism of the measure algebra of μ can be induced by an invertible completion Baire measurable point transformation of G. If μ is Haar measure, the point transformation is completion Borel measurable.

1. Introduction. Let μ be a finite Lebesgue-Stieltjes measure on the closed interval $I = [0, 1]$ or equivalently on the circle group $\mathbf{T}$. If ϕ is an automorphism of the measure algebra of (I, μ) it was shown by von Neumann [13] that ϕ is induced by an invertible Borel measurable point mapping T of I which necessarily preserves sets of measure zero (von Neumann made an additional unnecessary assumption that ϕ and therefore T are actually measure preserving). The theorem generalizes to any space which is point isomorphic to a Lebesgue-Stieltjes measure on the unit interval, e.g. to a finite Borel measure on a Polish (or more generally a Lusin) space, in particular to a finite Borel measure on $\Pi\, I_\alpha$, $\alpha \in A$, A countable, each $I_\alpha = I$, or on $\Pi\, \mathbf{T}_\alpha$, $\alpha \in A$, A countable, each $\mathbf{T}_\alpha = \mathbf{T}$.

Von Neumann's result was generalized by Dorothy Maharam [12] to the direct product $\Pi(I_\alpha, \mu_\alpha)$, $\alpha \in A$, A possibly uncountable, of normalized measures μ_α on I. Using many of the ideas of [12], plus others, the author ([3] and [4]) generalized it to an arbitrary finite measure μ on the completion of the product σ-algebra of $\Pi\, I_\alpha$, $\alpha \in A$, A possibly uncountable, which is the completion of its Baire σ-algebra. (The identical result obviously holds for $\Pi\, \mathbf{T}_\alpha$, $\alpha \in A$.) In each case ([12], [3], [4]) essentially the same proofs work for uncountable products of Polish spaces. (The argument in [4] is correct if all the Polish spaces have cardinal c, but incomplete otherwise. However the general case can easily be reduced to this one, the necessary additional argument is given below in §5, after the proof of Lemma 7.) Note that in [3] and [4], Baire and Borel mean completion Baire and completion Borel.

For any compact Hausdorff space X, if μ is a Radon measure on X (i.e. a finite regular measure on the Borel σ-algebra $\mathcal{B}_X$) and μ_0 is its restriction to the

Received by the editors October 31, 1972.

AMS (MOS) subject classifications (1970). Primary 28A65, 28A60; Secondary 28A70, 22C05.

(1) This research was supported by a grant from the National Research Council of Canada.

Baire σ-algebra $\mathfrak{B}_X^0$, then the measure algebras of μ and μ_0 are canonically isomorphic. One can therefore ask, for an arbitrary Radon measure μ on an arbitrary compact space X whether, given an automorphism ϕ of the measure algebra, it is induced by a $\mathfrak{B}_X$ (Borel) or $\mathfrak{B}_X^0$ (Baire) measurable invertible point transformation of X; or more generally one can ask whether ϕ can be induced by a $\overline{\mathfrak{B}}_{X,\mu}$ or $\overline{\mathfrak{B}}_{X,\mu_0}^0$ measurable invertible point transformation, where $\overline{\mathfrak{B}}_{X,\mu}$, $\overline{\mathfrak{B}}_{X,\mu_0}^0$ are the completions of $\mathfrak{B}_X$, respectively $\mathfrak{B}_X^0$, by μ, respectively μ_0. Such transformations will be called respectively Borel, Baire, completion Borel, completion Baire invertible point transformations. (In [12], [3] and [4] invertible point transformations are called point automorphisms. Since we are concerned mainly with groups and since such automorphisms are not necessarily group automorphisms, this convenient terminology seems inadvisable here.) It is easy to construct X for which the Baire results are false, see e.g. the introduction to [4]. However there are X for which the Borel results are also false, see Panzone and Segovia [14, §5, Example (c)]; a simpler example of this is given in §2 below. Using the Urysohn embedding procedure of a compact space in $\Pi\, I_\alpha$ (or $\Pi\, T_\alpha$) one sees that the Borel results must actually be false for arbitrary measures on these product spaces, which makes it more surprising that, as pointed out earlier, the completion Baire result is true. One is led to conjecture that the completion Baire result is true for measures on compact spaces which are "homogeneous" and can be "approximated" by compact metric spaces, in particular for compact groups. In this paper we show that this *is* true for compact groups (Theorem 1): for an arbitrary Radon measure m on a compact group G every automorphism ϕ of the measure algebra of m can be induced by a completion Baire invertible point transformation T of G. If m is Haar measure T is also completion Borel measurable. The proof consists in using the result of Weil and Pontrjagin that G is a projective limit of compact Lie groups, and then of generalizing the four key lemmas of [3] and [4]: Lemma 4 here is the generalization of Lemma 3 of [3], Lemma 7 that of Lemma C of [4], Lemma 8 that of Lemma 8 of [3] and the proof of Theorem 1 that of Lemma 7 of [3] or the theorem of [12].

In view of a result of C. T. Ionescu Tulcea [8, Chapter X, Theorem 2], one might be led to conjecture that a completion Borel point transformation inducing a given measure algebra automorphism always exists for $(X, \mathfrak{B}_X, \mu)$ if $(X, \mathfrak{B}_X, \mu)$ has the strong lifting property of [8]. That this is false is shown by the example in §2. The results of [14] and of [8] suggest that it may almost always be possible to realize a measure algebra automorphism of a Radon measure on a compact space X, by an invertible point transformation of a thick subset of X, but the topological character of the measure is almost completely lost in such a result. In connection with our contention that "homogeneity" of X should imply the Baire

point transformation property we note that Maharam [18] has recently shown that for any Radon measure m on a compact space X, $(X, m) \times \Pi_{\alpha \in A}(I_\alpha, m_\alpha)$ and $\Pi_{\alpha \in A}(I_\alpha, m_\alpha)$ are Borel point isomorphic, where each m_α is Lebesgue measure and A is "not too large".

For a compact group G and Haar measure $\overline{m}$ on G, it would be very interesting if one could show, not only that every automorphism of the measure algebra ϕ was induced by a completion Borel point transformation T, but also that T could further be chosen to be $\overline{m}$ Lusin measurable. We have no idea whether this is possible. Some remarks on Lusin measurability of point transformations are made in §7 of this paper.

My thanks are due to Kohur Gowrisankaran for his valuable criticism of the exposition of this paper, and to Dorothy Maharam, Shizuo Kakutani and Raghavan Narasimhan for their constant encouragement and advice during the preparation of this and the two earlier papers. Thanks are also due to Arthur Stone for pointing out several topological errors, and to Stelios Negrepontis for showing me the result given in Lemma 0.

2. A counterexample.

Lemma 0 (Gillman and Jerison [6, p. 137, Example 9H]). *Let S be an extremely disconnected compact Hausdorff space, in particular the Stone representation space of a measure algebra. Then every infinite closed (compact) subset of S contains a copy of $\beta\mathbf{N}$ (the Stone-Čech compactification of the positive integers $\mathbf{N}$), and so has cardinal at least 2^c.*

Note. S need only be basically disconnected. The proof given is that sketched by Gillman and Jerison and is included only for completeness.

Proof. Let F be the infinite closed set. Then F contains a copy D of N (i.e. D is a countably infinite discrete set). Let f be a bounded continuous function on D. Then we show that f can be extended to a continuous function on S. By Urysohn's extension theorem [6, p. 18, Theorem 1.17] it is enough to show that any pair of sets completely separated in D (i.e. separated by a continuous function on D) can be completely separated in S. Every pair of disjoint sets in D is completely separated in D; so let $D_1, D_2 \subset D$, $D_1 \cap D_2 = \emptyset$. Let $D_1 = \{p_{11}, p_{12}, \cdots\}$, $D_2 = \{p_{21}, p_{22}, \cdots\}$. There exist pairwise disjoint open-closed sets U_{ij} $(i = 1, 2;\ j = 1, 2, 3, \cdots)$ with $p_{ij} \in U_{ij}$. Let $U_i = \bigcup_j U_{ij}$, $i = 1, 2$. Then U_1 and U_2 are open and disjoint; since S is extremely disconnected, $\overline{U}_1$ is both open and closed and $\overline{U}_1 \cap U_2 = \emptyset$. Thus U_1, U_2 are separated by open-closed sets and so are completely separated in S, so $D_1 \subset U_1$ and $D_2 \subset U_2$ are completely separated in S.

Since $D \subset F \subset S$ and every bounded continuous function on D can be extended

to S, every bounded continuous function on D can certainly be extended to F, so $\beta D \subset \beta F$; further F is compact so $\beta F = F$ completing the proof.

Theorem 0. *Let I be the closed unit interval, m Lebesgue measure on I, S the Stone representation space of the measure algebra of (I, m), $\tilde{m}$ the measure induced by m on S. Let X be $I \cup S$ with the direct sum topology, so X is a compact Hausdorff space and $\mu = \frac{1}{2}(m + \tilde{m})$ is a Radon probability measure on X. Then*

(a) *there is a measure algebra automorphism ϕ of μ which cannot be induced by a Borel, completion Borel, Baire or completion Baire invertible point transformation,*

(b) *μ has the strong lifting property.*

Proof. (a) Let E denote the measure algebra of μ, E_I, E_S those of $\mu_I = \frac{1}{2}m$ and $\mu_S = \frac{1}{2}\tilde{m}$ respectively. If e, e_I, e_S denote their respective unit elements, $e_I \wedge e_S = 0$, $e_I \vee e_S = e$. Since (S, μ_S) is the Stone representation of (I, μ_I) there is a canonical isomorphism θ from E_I to E_S. For $a \in E_I$, let $\phi(a) = \theta(a)$, for $a \in E_S$, let $\phi(a) = \theta^{-1}(a)$; ϕ extends in an obvious way to an automorphism of E.

Suppose there exists a Borel, completion Borel, Baire or completion Baire invertible point transformation T of X inducing ϕ. Then there exist μ measurable sets $I_0 \subset I$, $S_0 \subset S$ such that

$$\mu(I - I_0) = \mu(S - S_0) = 0$$

and $TI_0 = S_0$, so that $\mu(S_0) = \mu(I_0) = \frac{1}{2} > 0$. Hence (since μ and its Baire restriction are inner regular) S_0 contains a closed set of positive measure, which is necessarily infinite (since $\tilde{m}$ has no point masses) and so by Lemma 0 has cardinal at least 2^c (in fact exactly 2^c). But I_0 has cardinal c which gives a contradiction.

(b) By a theorem of Ionescu Tulcea [8, Chapter VIII, Theorem 8] there is a strong lifting ρ_I of (I, μ_I). Since each element of $L^\infty(S, \mu_S)$ contains exactly one continuous function, the mapping ρ_S assigning this function to the element is a strong lifting of (S, μ_S). Every element f of $L^\infty(X, \mu)$ has a unique representation $f_I + f_S$, $f_I \in L^\infty(I, \mu_I)$, $f_S \in L^\infty(S, \mu_S)$; then ρ defined by $\rho(f) = \rho_I(f_I) + \rho_S(f_S)$ trivially defines a strong lifting of (X, μ).

Note. The argument in the proof of (b) actually shows that the strong lifting property is closed under direct sums. However, for both (I, μ_I) and (S, μ_S) every measure algebra automorphism *is* induced by an invertible Borel point transformation. Thus *this* property is not closed under direct sums.

3. Projective systems of compact groups. We begin with some remarks on projective systems of compact spaces. Let Λ be a directed set and $\{X_\lambda, \pi_{\lambda_1, \lambda_2};$

$\lambda_1, \lambda_2 \in \Lambda\}$ be a projective system of compact (i.e. compact Hausdorff) spaces, such that the $\pi_{\lambda_1,\lambda_2}$ are continuous surjections. Let $X = \text{proj lim}_\Lambda X_\lambda$. Then the compactness of X_λ implies that the maps $\pi_\lambda: X \to X_\lambda$ are surjections and in fact (see e.g. [2, p. 325, Note]) given any directed set $\Lambda' \subset \Lambda$, if $X^{\Lambda'} = \text{proj lim}_{\Lambda'} X_\lambda$, then the natural map from $X \to X^{\Lambda'}$ is a surjection. We call a directed set Λ *σ-directed* if given any *sequence* $\lambda_1, \lambda_2, \cdots$ in Λ, there exists $\lambda \in \Lambda$ with $\lambda_n < \lambda$ for all n. If $\{X_\lambda, \pi_{\lambda_1,\lambda_2}: \lambda_1, \lambda_2 \in \Lambda\}$ is a projective system with Λ σ-directed it is called a σ-directed projective system. In the case of compact spaces, by adding to Λ all equivalence classes of countable directed subsets $\Lambda_c \subset \Lambda$ (we say $\Lambda_{c_1}, \Lambda_{c_2}$ are equivalent if $\text{proj lim}_{\Lambda_{c_1}} X_\lambda = \text{proj lim}_{\Lambda_{c_2}} X_\lambda$), and to the system $\{X_\lambda\}$ the corresponding projective limits X^{Λ_c} and by noting, as above, that the maps $X \to X^{\Lambda_c}$ and $X^{\Lambda_c} \to X_\lambda$ for $\lambda \in \Lambda_c$ are surjective, we obtain a new projective system $\{X_\lambda, \pi_{\lambda_1,\lambda_2}: \lambda \in \bar{\Lambda}\}$ with $\bar{\Lambda} \supset \Lambda$, $\pi_{\lambda_1,\lambda_2}$ still surjective, $\bar{\Lambda}$ σ-directed, but with $\text{proj lim}_{\bar{\Lambda}} X_\lambda$ still X. We call this new directed set and system the *σ-saturation* of the original one.

For any compact space X, $\mathfrak{B}_X$ and $\mathfrak{B}_X^0$ denote its Borel and Baire σ-algebras respectively, and $\mathfrak{B}_X = \mathfrak{B}_X^0$ if X is metrisable. For any Radon (i.e. finite regular) measure $\bar{m}$ on $\mathfrak{B}_X$, let m denote its restriction to $\mathfrak{B}_X^0$. We recall that every set in $\mathfrak{B}_X$ differs from one in $\mathfrak{B}_X^0$, by a set in $\mathfrak{B}_X$ which is $\bar{m}$-null, and thus the measure algebras of $(X, \mathfrak{B}_X, \bar{m})$ and $(X, \mathfrak{B}_X^0, m)$ coincide. In the sequel we shall be concerned almost entirely with Baire measures (which is why we change notation from the introduction to denote the Baire rather than the Borel measure by an unembellished letter).

Next let $X = \text{proj lim } X_\lambda$, X_λ compact, and let m be a finite (necessarily regular) Baire measure on X. If $m_\lambda = \pi_\lambda(m)$, then m_λ is certainly a Baire measure. Moreover [2, Theorems 2.2 and 2.3] m is the projective limit in the sense of Bochner [1, pp. 118–120] of the m_λ. In fact $\mathfrak{B}_X^0 = \Sigma(\bigcup_{\lambda \in \Lambda} \pi_\lambda^{-1}(\mathfrak{B}_{X_\lambda}^0))$ [2, Theorem 2.3]. $\Sigma(\mathfrak{A})$ denotes the σ-algebra generated by $\mathfrak{A}$. If further Λ is σ-directed, then $\bigcup_{\lambda \in \Lambda} \pi_\lambda^{-1}(\mathfrak{B}_{X_\lambda}^0)$ is itself a σ-algebra and so must coincide with $\mathfrak{B}_X^0$.

Now let G be an arbitrary compact group. *All subgroups considered are closed.* Let

$\mathfrak{H}_0 = \{H$: H a normal subgroup of G such that G/H is Lie$\}$,

$\mathfrak{H} = \{H$: H a normal subgroup of G such that G/H is metrisable$\}$,

$\mathfrak{N} = \{H$: H a normal subgroup of $G\}$.

Then $\mathfrak{H}_0 \subset \mathfrak{H} \subset \mathfrak{N}$ and all three are directed by the relation $H_1 < H_2$ means $H_2 \subset H_1$. Clearly then H_2 is normal in H_1 and $G/H_1 = (G/H_2)/(H_1/H_2)$. We denote the projection map $G \to G/H$ by π_H, $G/H_2 \to G/H_1$ by π_{H_1,H_2}.

With the above ordering, given any H_1, H_2 in $\mathfrak{H}_0$, respectively $\mathfrak{H}$, respectively $\mathfrak{K}$, $H_1 \cap H_2$ gives a supremum in $\mathfrak{H}_0$, respectively $\mathfrak{H}$, respectively $\mathfrak{K}$; further $H_1H_2 = H_2H_1$ gives an infimum in $\mathfrak{H}_0$, respectively $\mathfrak{H}$, respectively $\mathfrak{K}$, thus each of $\mathfrak{H}_0$, $\mathfrak{H}$, $\mathfrak{K}$ is in fact a lattice. If $H_1, H_2, \cdots$ are in $\mathfrak{K}$, then $\bigcap_{n=1}^{\infty} H_n$ gives a supremum in $\mathfrak{K}$, thus $\mathfrak{K}$ is also σ-directed.

Thus $\{G/H: H \in \mathfrak{H}_0\}$ and $\{G/H: H \in \mathfrak{H}\}$ are both projective systems of compact metrisable groups. By the theorem of Weil-Pontrjagin ([17, p. 88], [15, pp. 323–331])

$$G = \operatorname{proj}\lim_{\mathfrak{H}_0} G/H.$$

Further it is easily seen that if $H_n \in \mathfrak{H}_0$ $(n = 1, 2, 3, \cdots)$ or if $H_n \in \mathfrak{H}$ $(n = 1, 2, 3, \cdots)$ and if $H_1 < H_2 < \cdots$ (i.e. if $H_1 \supset H_2 \supset \cdots$), then $G/\bigcap H_n = \operatorname{proj}\lim_n G/H_n$, so $G/\bigcap H_n$ is metrisable and $\bigcap H_n \in \mathfrak{H}$. Thus $\mathfrak{H}$ is also σ-directed ($\mathfrak{H}_0$ is not). It is now easily seen (again using the theorem of Weil-Pontrjagin) that $\mathfrak{H}$ is the σ-saturation of $\mathfrak{H}_0$ and so by the earlier remarks we also have

$$G = \operatorname{proj}\lim_{\mathfrak{H}} G/H.$$

We further have, for any $K \in \mathfrak{K}$,

$$\begin{aligned} G/K &= \operatorname{proj}\lim\{G/H: H \in \mathfrak{H} \text{ (or } \mathfrak{H}_0), H \supset K\} \\ &= \operatorname{proj}\lim\{(G/K)/(H/K): H \in \mathfrak{H} \text{ (or } \mathfrak{H}_0), H \supset K\}. \end{aligned}$$

The following elementary corollary of these remarks is used in the sequel.

Lemma 1. *If $K \in \mathfrak{K}$, $K \neq \{e\}$, then there exists $H \in \mathfrak{H}$ (or even $H \in \mathfrak{H}_0$) such that $H \not\supset K$.*

[$e = e_G$ will henceforth always denote the identity element of G and not, as in §2, of a measure algebra.]

We also need an elementary lemma in which the topology of G hardly enters. The proof is given only for completeness.

Lemma 2. *If H, K are normal subgroups of a group G such that $H \cap K = \{e\}$, then*

(i) *$\theta: g \to (\pi_H g, \pi_K g)$ defines an isomorphism of G into $G/H \times G/K$; if G is a compact group then θ is a homeomorphism;*

(ii) *cosets Hx and Ky of H and K meet in at most one element of G;*

(iii) *Hx and Ky meet if and only if they project into the same coset of the normal subgroup $HK = KH$, i.e. $(Hx) \cap (Ky) \neq \emptyset$ if and only if*

$$\pi_{HK,H}(Hx) = \pi_{HK,K} \circ T_K \circ \pi_K g$$

(iv) *if* T_H *and* T_K *are bijections of* G/H *and* G/K *respectively and* $\tilde{T}$ *is the bijection of* $G/H \times G/K$ *given by* $\tilde{T} = (T_H, T_K)$, *and if further* $\pi_{HK,H} \circ T_H \circ \pi_H = \pi_{HK,K} \circ T_K \circ \pi_K$ *and* $\pi_{HK,H} \circ T_H^{-1} \circ \pi_H = \pi_{HK,K} \circ T_K^{-1} \circ \pi_K$, *then* $\tilde{T}(\theta G) = \theta G$ *and so* $T = \theta^{-1} \circ \tilde{T} \circ \theta$ *gives a bijection of* G *onto itself.*

Proof. (i) The mapping θ is clearly a homomorphism with kernel $H \cap K = \{e\}$. If G is a compact group, the continuity of θ follows from that of its components, and a continuous bijection on a compact space is a homeomorphism.

(ii) If $(Hx) \cap (Ky)$ contains two distinct points, there exist $h_i \in H$, $k_i \in K$ $(i = 1, 2,)$ such that $h_1 x = k_1 y \neq h_2 x = k_2 y$. So $h_i = k_i y x^{-1}$ $(i = 1, 2)$; so $h_1 h_2^{-1} = k_1 y x^{-1} x y^{-1} k_2^{-1} = k_1 k_2^{-1}$ and since $H \cap K = \{e\}$, we have $h_1 = h_2$, $k_1 = k_2$, contradiction.

(iii) If $(Hx) \cap (Ky) \neq \emptyset$ then there exist $h \in H$, $k \in K$ such that $hx = ky$. We have

$$HK(hx) = KH(hx) = KHx = KH(Hx) = HK(Hx), \qquad HK(ky) = HKy = HK(Ky),$$

so $HK(Hx) = HK(Ky)$. Conversely if $HK(Hx) = HK(Ky)$ then $HKx = HKy$, so there exist $h_1, h_2 \in H$, $k_1, k_2 \in K$ such that $h_1 k_1 x = h_2 k_2 y$ and so

$$k_2 y = h_2^{-1} h_1 k_2 x = k_3 h_3 x$$

since $HK = KH$. So $k_3^{-1} k_2 y = h_3 x$, i.e. $(Hx) \cap (Ky) \neq \emptyset$.

(iv) If the condition is satisfied then for any $g \in G$,

$$\pi_{HK,H} \circ T_H \circ \pi_H g = \pi_{HK,K} \circ T_K \circ \pi_K g$$

and so by (ii) and (iii) the cosets $T_H \circ \pi_H g$ of H and $T_K \circ \pi_K g$ of K meet in exactly one element g' and $\theta g' = (T_H \circ \pi_H g, T_K \circ \pi_K g) = \tilde{T}\theta g$. Thus $\tilde{T}$ maps θG into θG, but as the condition also holds with T_H^{-1}, T_K^{-1} in place of T_H and T_K, $\tilde{T}^{-1}$ also maps θG into θG, and so $\tilde{T}$ maps θG onto θG and $\theta^{-1} \circ \tilde{T} \circ \theta$ defines a bijection of G onto itself proving (iv).

This completes the proof of the lemma.

Now let G be a compact group, $\mathfrak{H}_0$, $\mathfrak{H}$, $\mathfrak{K}$ as before. Let $\overline{m}$ be a Radon measure on G, m its Baire contraction. (Every finite Baire measure is, of course, the contraction of some Radon measure.) (If $\overline{m}$ is Haar measure the completions of m and $\overline{m}$ coincide [7, p. 287, Theorem H].) Put $m_H = \pi_H m$ for any $H \in \mathfrak{K}$, m_H is a measure on $\mathfrak{B}^0_{G/H}$ (which coincides with $\mathfrak{B}_{G/H}$ if $H \in \mathfrak{H}$). Put $\mathcal{G}^H =$

$\pi_H^{-1}(\mathfrak{B}^0_{G/H})$. Then since $\mathfrak{H}$ is σ-directed,

$$\mathfrak{B}^0 = \mathfrak{B}^0_G = \bigcup_{H\in\mathfrak{H}} \mathfrak{G}^H = \bigcup_{H\in\mathfrak{H}} \pi_H^{-1}(\mathfrak{B}^0_{G/H}) = \bigcup_{H\in\mathfrak{H}} \pi_H^{-1}(\mathfrak{B}_{G/H}).$$

Now let $E = E_G$ denote the measure algebra of $(G, \mathfrak{B}^0, m)$, $E_{G/K}$ that of $(G/K, \mathfrak{B}^0_{G/K}, m_K)$, $K \in \mathfrak{K}$. Let $\mathfrak{E}^K$ be the subalgebra of E consisting of equivalence classes of elements of $\mathfrak{G}^K$. π_K induces an isomorphism $\hat{\pi}_K$ of $E_{G/K}$ and $\mathfrak{E}^K$. Note that since $G/\{e\} = G$, $\mathfrak{E}^{\{e\}} = E$. Also note that $E = \bigcup_{H\in\mathfrak{H}} \mathfrak{E}^H$.

For any compact space X, and Radon measure $\bar{m}$, we denote the respective completions of $(X, \mathfrak{B}_X, \bar{m})$ and $(X, \mathfrak{B}^0_X, m)$ by $(X, \bar{\mathfrak{B}}_{X,\bar{m}}, \bar{m})$ and $(X, \bar{\mathfrak{B}}^0_{X,m}, m)$. When there is no danger of confusion we write $\bar{\mathfrak{B}}_X$, $\bar{\mathfrak{B}}^0_X$ in place of $\bar{\mathfrak{B}}_{X,\bar{m}}$, $\bar{\mathfrak{B}}^0_{X,m}$ respectively. The measure algebra of an incomplete measure and its completion obviously always coincide.

Between $\mathfrak{B}^0_G$ and its completion $\bar{\mathfrak{B}}^0_{G,m}$ lies a σ-algebra $\tilde{\mathfrak{B}}^0_{G,m}$ of great importance to us: it is the projective limit σ-algebra of the completions $\bar{\mathfrak{B}}_{G/H,m_H}$ of $\mathfrak{B}_{G/H}$ with respect to m_H, $H \in \mathfrak{H}$, i.e.

$$\tilde{\mathfrak{B}}^0_{G,m} = \bigcup_{H\in\mathfrak{H}} \tilde{\mathfrak{G}}^H_m = \bigcup_{H\in\mathfrak{H}} \pi_H^{-1}(\bar{\mathfrak{B}}_{G/H,m_H})$$

where $\tilde{\mathfrak{G}}^H_m = \pi_H^{-1}(\bar{\mathfrak{B}}_{G/H,m_H})$. More generally for $K \in \mathfrak{K}$

$$\tilde{\mathfrak{B}}^0_{G/K,m_K} = \bigcup\{\pi_{H,K}^{-1}(\bar{\mathfrak{B}}_{G/H,m_H}) : H \in \mathfrak{H}, H \supset K\}$$

and

$$\tilde{\mathfrak{G}}^K_m = \pi_K^{-1}(\tilde{\mathfrak{B}}^0_{G/K,m_K}).$$

If $H \in \mathfrak{H}$, then $\tilde{\mathfrak{B}}^0_{G/H,m_H} = \bar{\mathfrak{B}}^0_{G/H,m_H} = \bar{\mathfrak{B}}_{G/H,m_H}$, so the two definitions of $\tilde{\mathfrak{G}}^H_m$ coincide. Obviously, E_G is the measure algebra of $(G, \tilde{\mathfrak{B}}^0_{G,m}, m)$, $E_{G/K}$ of $(G/K, \tilde{\mathfrak{B}}^0_{G/K,m_K}, m_K)$ and $\mathfrak{E}^K$ of $(G, \tilde{\mathfrak{G}}^K_m, m)$ where $K \in \mathfrak{K}$. Since there is no real danger of confusion we hereafter write $\tilde{\mathfrak{B}}^0_{G/K}$, $\bar{\mathfrak{B}}^0_{G/K}$, $\bar{\mathfrak{B}}_{G/K}$, $\tilde{\mathfrak{G}}^K$ for $\tilde{\mathfrak{B}}^0_{G/K,m_K}$, $\bar{\mathfrak{B}}^0_{G/K,m_K}$, $\bar{\mathfrak{B}}_{G/K,\bar{m}_K}$, $\tilde{\mathfrak{G}}^K_m$ respectively and when $K = \{e\}$ i.e. $G/K = G$, simply $\tilde{\mathfrak{B}}^0$, $\bar{\mathfrak{B}}^0$, $\bar{\mathfrak{B}}$. Note that $\tilde{\mathfrak{G}}^K = \bigcup\{\tilde{\mathfrak{G}}^H : H \in \mathfrak{H}, H \supset K\}$.

Lemma 3. (i) *If $H, K \in \mathfrak{K}$ then $\pi^{-1}_{H,H\cap K}(\mathfrak{B}^0_{G/H})$ and $\pi^{-1}_{K,H\cap K}(\mathfrak{B}^0_{G/K})$ generate $\mathfrak{B}^0_{G/H\cap K}$. Hence $\mathfrak{G}^H$ and $\mathfrak{G}^K$ generate $\mathfrak{G}^{H\cap K}$.*

(ii) *If* $H_1, H_2 \in \mathfrak{H}$ *then* $\overline{\mathfrak{B}}_{G/H_1 \cap H_2}$ *is the completion of the* σ*-algebra generated by* $\pi^{-1}_{H_1, H_1 \cap H_2}(\overline{\mathfrak{B}}_{G/H_1})$ *and* $\pi^{-1}_{H_2, H_1 \cap H_2}(\overline{\mathfrak{B}}_{G/H_2})$.

(iii) *If* $H \in \mathfrak{H}$, $K \in \mathfrak{K}$, *then*

$$G/H \cap K = \text{proj lim}\{G/H \cap \overline{H}: \overline{H} \supset K, \overline{H} \in \mathfrak{H}\},$$

$$\mathfrak{B}^0_{G/H \cap K} = \bigcup\{\pi^{-1}_{H \cap \overline{H}, H \cap K}(\mathfrak{B}_{G/H \cap \overline{H}}): \overline{H} \supset K, \overline{H} \in \mathfrak{H}\},$$

$$\overline{\mathfrak{B}}^0_{G/H \cap K} = \bigcup\{\pi^{-1}_{H \cap \overline{H}, H \cap K}(\overline{\mathfrak{B}}_{G/H \cap \overline{H}}): \overline{H} \supset K, \overline{H} \in \mathfrak{H}\}.$$

Proof. (i) The map $g \to (\pi_H g, \pi_K g)$ defines a homomorphism of G into $G/H \times G/K$ with kernel $H \cap K$ and so defines an isomorphism θ of $G/H \cap K$ into $G/H \times G/K$. The homomorphism is continuous (since each factor is), so θ is continuous, since it is also a bijection on a compact space it is a homeomorphism. Further if p_H, p_K are the natural projections from $G/H \times G/K$ to G/H and G/K respectively, then $\pi_{H, H \cap K} = p_H \circ \theta$ and $\pi_{K, H \cap K} = p_K \circ \theta$. Now $\mathfrak{B}^0_{G/H} \otimes \mathfrak{B}^0_{G/K}$ is the Baire σ-algebra of $G/H \times G/K$ and this is generated by $p_H^{-1}(\mathfrak{B}^0_{G/H})$ and $p_K^{-1}(\mathfrak{B}^0_{G/K})$. Since

$$\mathfrak{B}^0_{\theta(G/H \cap K)} = \theta(G/H \cap K) \cap (\mathfrak{B}^0_{G/H} \otimes \mathfrak{B}^0_{G/K})$$

it follows that $\theta(G/H \cap K) \cap p_H^{-1}(\mathfrak{B}^0_{G/H})$ and $\theta(G/H \cap K) \cap p_K^{-1}(\mathfrak{B}^0_{G/K})$ generate $\mathfrak{B}^0_{\theta(G/H \cap K)}$ from which the first assertion follows at once by taking inverse images under θ. The second assertion follows from the first by taking inverse images under $\pi^{-1}_{H \cap K}$.

(ii) follows from (i) by noting that $H_1 \cap H_2$ also is in $\mathfrak{H}$ and that for $H \in \mathfrak{H}$, $\mathfrak{B}_{G/H} = \mathfrak{B}^0_{G/H}$ and so

$$\overline{\mathfrak{B}}_{G/H} = \overline{\mathfrak{B}}^0_{G/H} = \overline{\mathfrak{B}}^0_{G/H}.$$

(iii) Now $G/H \cap K = \text{proj lim}\{G/H_0: H_0 \supset H \cap K, H_0 \in \mathfrak{H}\}$.

If $H_0 \in \mathfrak{H}$, $H_0 \supset H \cap K$, then put $H_0' = H_0 \cap H$, so $H_0' \in \mathfrak{H}$, $H_0' \supset H \cap K$ and we have

$$H \cap H_0' K = H_0'(H \cap K) = H_0' \subset H_0.$$

{The only nontrivial assertion here is that $H \cap H_0' K \subset H_0'(H \cap K)$; but if $h \in H$ and $h = h_0' k$, $h_0' \in H_0'$, $k \in K$, then $h_0' \in H$ since $H_0' \subset H$, and so $k \in H$, i.e. $k \in H \cap K$ and so $h \in H_0'(H \cap K)$.} Now $H_0' K \in \mathfrak{H}$ and $H_0' K \supset K$ thus the set

$\{H \cap \bar{H}: \bar{H} \in \mathfrak{H}, \bar{H} \supset K\}$, which is trivially σ-directed, is cofinal in $\{H_0: H_0 \in \mathfrak{H}, H_0 \supset H \cap K\}$. The assertions of (iii) are now immediate.

4. Measurable transformations on projective limits: Invariance. Given a measure space $(X, \mathbf{M}, m)$ let T be an invertible, measurable transformation of X onto itself (i.e. T^{-1} is also a measurable transformation) such that $m(TZ) = 0$ if and only if $m(Z) = 0$. Such a T always induces an automorphism ϕ of the measure algebra of $(X, \mathbf{M}, m)$ by the rule $\phi\hat{Y} = \widehat{TY}$, for all $Y \in \mathbf{M}$, where $\hat{Y}$ denotes the equivalence class of Y. T need not be measure preserving, however if an invertible measurable T induces a measure algebra automorphism ϕ it necessarily preserves sets of measure zero and so does its inverse. *Accordingly in the sequel any invertible point transformation T mentioned will always be assumed to satisfy $m(Z) = 0$ if and only if $m(TZ) = 0$.* In [12], [3] and [4] such transformations are called point automorphisms, we avoid this convenient terminology since not every point automorphism of a compact group G in the above sense is necessarily a group automorphism!

Now let $G, \mathfrak{H}_0, \mathfrak{H}, \mathfrak{K}, E, \mathfrak{E}^K, \mathfrak{G}^K, \tilde{\mathfrak{G}}^K$ be as before.

Definition. (a) Let ϕ be an automorphism of E. $K \in \mathfrak{K}$ is said to be *invariant under* ϕ if $\phi(\mathfrak{E}^K) = \mathfrak{E}^K$; ϕ then induces an automorphism ϕ_K of $E_{G/K}$.

(b) Let T be a $\tilde{\mathfrak{B}}^0$ measurable invertible point transformation of G. $K \in \mathfrak{K}$ is said to be *invariant under* T if $T(\tilde{\mathfrak{G}}^K) = \tilde{\mathfrak{G}}^K$.

Note. If both H and K are invariant under ϕ or under T, then so is $H \cap K$.

If $K \in \mathfrak{K}$, and ϕ_K is an automorphism of $E_{G/K}$ respectively T_K is a $\tilde{\mathfrak{B}}^0_{G/K}$ measurable invertible point transformation of G/K, then for $K_1 < K$ (i.e. $K_1 \supset K$) we shall speak of K_1 being invariant under ϕ_K respectively T_K when we should strictly say that K_1/K is invariant under ϕ_K respectively T_K; this is because $G/K_1 = (G/K)/(K_1/K)$ and we think of G/K as being the projective limit of G/K_1, $K_1 \supset K$, rather than of $(G/K)/(K_1/K)$; further what either statement means is simply that $\phi_K(\mathfrak{E}^{K_1,K}) = \mathfrak{E}^{K_1,K}$ respectively $T_K(\pi^{-1}_{K_1,K}(\tilde{\mathfrak{B}}^0_{G/K_1})) = \pi^{-1}_{K_1,K}(\tilde{\mathfrak{B}}^0_{G/K_1})$, where $\mathfrak{E}^{K_1,K}$ is the subalgebra of $E_{G/K}$ consisting of equivalence classes containing elements of $\pi^{-1}_{K_1,K}(\tilde{\mathfrak{B}}^0_{G/K_1})$. Note that if ϕ is an automorphism of E, K is invariant under ϕ and ϕ_K is the induced automorphism of $E_{G/K}$, and $K_1 < K$ is invariant under ϕ_K, then K_1 is also invariant under ϕ; a similar remark holds for point transformations T.

It seems hardly necessary to point out that in the above definition ϕ is not in any sense a mapping of the group K. It is perhaps helpful here to forget momentarily that K is a subgroup and think of it merely as an index in a projective system.

We now prove an elementary but vital lemma.

Lemma 4. (a) *If ϕ is an automorphism of the measure algebra E of (G, m) then for any $H \in \mathfrak{H}$, there exists $\overline{H} \in \mathfrak{H}$ with $H < \overline{H}$ and $\overline{H}$ invariant under ϕ.*

(b) *If T is an invertible $\tilde{\mathfrak{B}}^0$ measurable point transformation of G, then for any $H \in \mathfrak{H}$, there exists $\overline{H} \in \mathfrak{H}$ with $H < \overline{H}$ and $\overline{H}$ invariant under T.*

Proof. (a) Let $H_0 = H$, let a_{0j}, $j = 1, 2, 3, \cdots$, be a countable basis of E_{G/H_0} (which exists since $H \in \mathfrak{H}$), then $b_{0j} = \hat{\pi}_{H_0}^{-1}(a_{0j})$, $j = 1, 2, 3, \cdots$, is a basis of $\mathfrak{E}^{H_0}$. For each positive integer j and each integer n, since $E = \bigcup_{H \in \mathfrak{H}} \mathfrak{E}^H$, there exists an $H(n, j) \in \mathfrak{H}$ such that $\phi^n(b_{0j}) \in \mathfrak{E}^{H(n,j)}$; we may take $H(0, j) = H_0$ for all j. Now

$$H_1 = \bigcap_{n=-\infty}^{\infty} \bigcap_{j=1}^{\infty} H(n, j)$$

also belongs to $\mathfrak{H}$, $H_0 < H_1$ and $\phi^n(b_{0j}) \in \mathfrak{E}^{H_1}$ for all j and n. Now apply the same argument to a countable basis b_{1j} of $\mathfrak{E}^{H_1}$ to get $H_2 > H_1$, $H_2 \in \mathfrak{H}$, such that $\phi^n(b_{1j}) \in \mathfrak{E}^{H_2}$ for all j and n. In this way we get a sequence $H_1 < H_2 < \cdots$ in $\mathfrak{H}$. Put $\overline{H} = \bigcap_{r=1}^{\infty} H_r$. Then $\overline{H} \in \mathfrak{H}$; also $\phi^n(a) \in \mathfrak{E}^{\overline{H}}$ if $a \in \mathfrak{E}^{H_r}$ $(r = 1, 2, \cdots)$ and $\bigcup_{r=1}^{\infty} \mathfrak{E}^{H_r}$ generates $\mathfrak{E}^{\overline{H}}$, so $\phi^n(a) \in \mathfrak{E}^{\overline{H}}$ if $a \in \mathfrak{E}^{\overline{H}}$; i.e. $\overline{H}$ is invariant under ϕ.

(b) Let $H_0 = H$, let A_{0j}, $j = 1, 2, 3, \cdots$, be a separating sequence of generators of $\mathfrak{B}_{G/H_0}$, put $B_{0j} = \pi_{H_0}^{-1}(A_{0j})$. As in the proof of (a), there exists $H_1 \in \mathfrak{H}$, $H_0 < H_1$ such that $T^n(B_{0j}) \in \tilde{\mathfrak{G}}^{H_1}$ for all $j = 1, 2, 3, \cdots$ and $n = 0, \pm 1, \pm 2, \cdots$. Since A_{0j} is a separating sequence it follows that $T^n(\pi_{H_0}^{-1} X) \in \tilde{\mathfrak{G}}^{H_1}$ for all $X \in \mathfrak{B}_{G/H_0}$ and that $T^n(\pi_{H_0}^{-1}(p)) \in \tilde{\mathfrak{G}}^{H_1}$ for all $p \in G/H_0$, hence, using standard properties of inverse maps, $T^n(\pi_{H_0}^{-1} X) \in \tilde{\mathfrak{G}}^{H_1}$ for all $X \in \overline{\mathfrak{B}}_{G/H_0}$; so $T^n(\tilde{\mathfrak{G}}^{H_0}) \subset \tilde{\mathfrak{G}}^{H_1}$. Now the proof proceeds as in (a).

Corollary. (a) *If $K \in \mathfrak{K}$ and if ϕ_K is an automorphism of $E_{G/K}$, then for any $H \in \mathfrak{H}$ with $H < K$, there exists $\overline{H} \in \mathfrak{H}$ with $H < \overline{H} < K$ and $\overline{H}$ invariant under ϕ_K.*

(b) *If $K \in \mathfrak{K}$ and if T_K is a $\tilde{\mathfrak{B}}^0_{G/K}$ measurable invertible point transformation of G/K then for any $H \in \mathfrak{H}$ with $H < K$, there exists $\overline{H} \in \mathfrak{H}$ with $H < \overline{H} < K$ and $\overline{H}$ invariant under T_K.*

Proof. The proofs are identical to those of the lemma itself, except that $\mathfrak{E}^{H,K}$, $E_{G/K}$ replace $\mathfrak{E}^H$, E in the proof of (a), $\pi_{H,K}^{-1}(\overline{\mathfrak{B}}_{G/H})$, $\tilde{\mathfrak{B}}^0_{G/K}$ replace $\tilde{\mathfrak{G}}^H$, $\tilde{\mathfrak{B}}^0$ in the proof of (b).

Note. If T is an invertible $\mathfrak{B}^0$ measurable transformation (preserving null sets), then it is also $\tilde{\mathfrak{B}}^0$ measurable; and if it is a $\tilde{\mathfrak{B}}^0$ measurable transformation

then it is also $\overline{\mathfrak{B}}^0$ measurable. (The converse statements are, of course, false.)

5. Lemmas on compact metric groups. In this section we give three lemmas each of which has some independent interest. The first of these is just von Neumann's theorem.

Lemma 5. *If X is a compact metric (or more generally a Polish) space, m is a finite Borel measure on X (necessarily Radon) then every automorphism of the measure algebra of $(X, \mathfrak{B}_X, m)$ is induced by an invertible Borel point transformation of X.*

We now mention a few facts on metric spaces and groups. A Lusin space is a continuous bijective image, a Suslin space a continuous image of a Polish space [16, pp. 94–96, Definitions 2 and 3]. A continuous image of a compact metric space is compact, Suslin and hence [16, p. 106, Proposition 4, Corollary 2] metrizable. A compact metric group is either finite or perfect and so of cardinal c. If B_1 and B_2 are two Borel subsets of a Polish space each of cardinal c, there exists an invertible Borel measurable bijection R of B_1 and B_2 (Kuratowski [9, p. 451, Remark (i)]).

Lemma 6. *Let G be a compact metric group, H a (closed) normal subgroup of cardinal c, m a finite Borel measure on G. Then there exists a Borel set $Z \subset G$, with $m(Z) = 0$ and such that Z meets every coset of H in a set of cardinal c.*

Proof. Let π_H, as usual, denote the natural projection $G \to G/H$. By a theorem of Mackey ([11, p. 102, Lemma 1.1], see also [5, Theorem 1 and Remark 3]) there exists a transversal (= section) S of G/H in G, which is Borel in G (and is such that the transversal map r: $G/H \to G$ which is the inverse of π_H on S is Borel measurable). S being a Borel set in a Polish space G is Lusin [16, p. 95, Theorem 2]. Hence (since H is also compact metric and so Polish) $S \times H$ is also Lusin [16, p. 94, Lemma 4]. Each element of G has a unique representation th, $t \in S$, $h \in H$ and so there exists a natural bijection θ: $S \times H \to G$ given by $(t, h) \to th$, $t \in S$, $h \in H$; since if h_n, $h \in H$, t_n, $t \in S$, $h_n \to h$, $t_n \to t$ implies $t_n h_n \to th$ in G, it follows that θ is continuous.

If $h_1 \neq h_2$, h_1, $h_2 \in H$, then $Sh_1 \cap Sh_2 = \emptyset$. Since H is compact metric of cardinal c and $\bigcup_{h \in H} Sh = G$, there exist $h_n \in H$ $(n = 1, 2, \cdots)$ such that $\bigcup_{n=1}^{\infty} \{h_n\}$ is dense in H, $m(Sh_n) = 0$ for all n and so $m(\bigcup_{n=1}^{\infty} Sh_n) = 0$. For the Sh are all disjoint Borel sets in G. If P is the set of $h \in H$ such that $m(Sh) > 0$, then P is countable, and by the Baire category theorem $H - P$ is an everywhere dense $\mathfrak{G}_\delta$ in H. $H - P$ is separable, let $h_n \in H - P$, $n = 1, 2, 3, \cdots$, be such that $\bigcup_{n=1}^{\infty} \{h_n\}$ is dense in $H - P$; then it is also dense in H, and this sequence

$\{h_n\}$ has all the required properties. Put $L_n = Sh_n$, $L = \bigcup_{n=1}^{\infty} L_n$, so $L = S\bigcup_{n=1}^{\infty}\{h_n\}$ and $L \cap H = \bigcup_{n=1}^{\infty}\{h_n\}$. L is Borel in G and $m(L) = 0$. There exists a (two-sided) invariant metric ρ_H on H. Let

$$V_{n,\delta} = \{h: \rho_H(h, h_n) < \delta\}, \qquad W_{n,\delta} = \bigcup_{h \in V_{n,\delta}} Sh = SV_{n,\delta}$$

$S \times V_{n,\delta}$ is open in the Lusin space $S \times H$ and so [16, p. 95, Theorem 2] it is Lusin; hence its image $W_{n,\delta}$ in G under the continuous bijective map is Lusin (Corollary to the definition of a Lusin space [16, p. 92, Definition 2]) and so it is Borel in G [16, p. 101, Theorem 5]. Now

$$\bigcap_{r=1}^{\infty} W_{n,1/r} = S\left(\bigcap_{r=1}^{\infty} V_{n,1/r}\right) = Sh_n = L_n \quad \text{and} \quad m(L_n) = 0,$$

so given $\epsilon > 0$, there exists W_{n,δ_n} such that $m(W_{n,\delta_n}) < \epsilon/2^n$. Put $W^\epsilon = \bigcup_{n=1}^{\infty} W_{n,\delta_n}$, $V^\epsilon = \bigcup_{n=1}^{\infty} V_{n,\delta_n}$. Then $W^\epsilon = SV^\epsilon$, $W^\epsilon \cap H = V^\epsilon$, which is a dense open subset of H since it contains $\bigcup_{n=1}^{\infty}\{h_n\}$, and $m(W^\epsilon) < \epsilon$. Put $Z = \bigcap_{k=1}^{\infty} W^{1/k}$, then Z is Borel in G and $m(Z) = 0$. Further $Z \cap H = \bigcap_{k=1}^{\infty} V^{1/k}$ is an everywhere dense $\mathcal{G}_\delta$ of H and so by the Baire category theorem has cardinal c. But $Z = S\bigcap_{k=1}^{\infty} V^{1/k}$, so Z has the same number of elements in each coset of H (since S is a transversal) and so Z meets each coset of H in a set of cardinal c.

Our last lemma in this section gives a vital extension property for measure algebra automorphisms and invertible point transformations of compact metric groups. We remark that for compact metric groups $\mathfrak{H} = \mathfrak{K}$.

Lemma 7. *Let G be a compact metric group, $H \in \mathfrak{H}$, m a finite Borel (i.e. Radon) measure on G. Let ϕ be an automorphism of the measure algebra E of $(G, \mathfrak{B}_G, m)$ such that H is invariant under ϕ and so ϕ induces an automorphism ϕ_H of $E_{G/H}$. Let T_H be an invertible completion Borel measurable point transformation of G/H inducing ϕ_H. Then there exists an invertible completion Borel point transformation T of G, inducing ϕ and such that $\pi_H \circ T = T_H \circ \pi_H$.*

Proof. We first show that there exists an invertible Borel point transformation T_1 on a Borel subset $\Omega \subset G$ with $m(G - \Omega) = 0$, such that T_1 induces ϕ and $\pi_H \circ T_1 = T_H \circ \pi_H$ on Ω. Let T_1 be *any* invertible Borel point transformation of G inducing ϕ (such a T_1 exists by Lemma 5). For any Borel set $X \subset G/H$,

$$\widehat{\phi(\pi_H^{-1}X)} = \hat{\pi}_H^{-1}(\phi_H\hat{X})$$

(again here $\hat{X}$ denotes the element of $E_{G/H}$ to which X belongs, $\hat{\pi}_H$ the canonical isomorphism $\mathfrak{E}^H$ to $E_{G/H}$ induced by π_H); so

$$m(T_1^n(\pi_H^{-1}(X)) \triangle \pi_H^{-1}(T_H^n(X))) = 0$$

for all integers $n = 0, \pm 1, \pm 2, \cdots$. Let $Z_{X,n}$ denote this last set, which is Borel. Let X_k $(k = 1, 2, 3, \cdots)$ be a separating sequence of generators of $\mathfrak{B}_{G/H}$. Put $Z = \bigcup_{k=1}^{\infty} \bigcup_{n=-\infty}^{\infty} Z_{X_k,n}$, and $N = \bigcup_{n=-\infty}^{\infty} T_1^n Z$, then N is Borel in G, $m(N) = 0$, $T_1^n N = N$, and

$$T_1^n(\pi_H^{-1}(X_k)) - N = \pi_H^{-1}(T_H^n(X_k)) - N,$$

for all $n = 0, \pm 1, \pm 2, \cdots$ and $k = 1, 2, \cdots$. Hence since X_k is a separating sequence of G/H, for all $g_1 \in G/H$,

$$T_1^n(\pi_H^{-1}(g_1)) - N = \pi_H^{-1}(T_H^n g_1) - N;$$

and so if $\Omega = G - N$, then Ω is Borel, $m(G - \Omega) = 0$ and for $g \in \Omega$, $\pi_H \circ T_1^n g = T_H^n \circ \pi_H g$, as required.

We now show how to extend T_1 from Ω to an invertible completion Borel point transformation of G so that $\pi_H \circ T = T_H \circ \pi_H$. H is a compact metric group, and so by one of the remarks at the beginning of this section, it is either finite or has cardinal c.

Case (i). *H is finite.* Let H have l elements, then for any $g_1 \in G/H$, $\pi_H^{-1}(g_1)$ has l elements. Since T_1 is bijective on Ω and maps $\Omega \cap \pi_H^{-1}(g_1)$ onto $\Omega \cap \pi_H^{-1}(T_H g_1)$ it follows that $\Omega \cap \pi_H^{-1}(g_1)$ and $\Omega \cap \pi_H^{-1}(T_H g_1)$ both contain the same number $l(g_1)$ of elements, $0 \leq l(g_1) \leq l$. Thus $N \cap \pi_H^{-1}(g_1)$ and $N \cap \pi_H^{-1}(T_H g_1)$ both contain $l - l(g_1)$ elements; let R_{g_1} be any bijection of $N \cap \pi_H^{-1}(g_1)$ onto $N \cap \pi_H^{-1}(T_H g_1)$. Define T on G by $Tg = T_1 g$ if $g \in \Omega$, $Tg = R_{\pi_H g} g$ if $g \in N$. T is an invertible completion Borel point transformation of G with the required properties.

Case (ii). *H has cardinal c.* By Lemma 6, there exists a Borel set $Z \subset G$ with $m(Z) = 0$ and such that $Z \cap \pi_H^{-1}(g_1)$ has cardinal c for all $g_1 \in G/H$. Let

$$\Omega' = \Omega - \bigcup_{n=-\infty}^{\infty} T_1^n(\Omega \cap Z);$$

Ω' is Borel, $m(G - \Omega') = 0$, $T_1 \Omega' = \Omega'$ and for $g \in \Omega'$, $\pi_H \circ T_1 g = T_H \circ \pi_H g$. Put

$N' = G - \Omega'$, then N' is Borel, $m(N') = 0$, and since $N' \supset Z$, $N' \cap \pi_H^{-1}(g_1)$ has cardinal c for every $g_1 \in G/H$. Thus for every $g_1 \in G/H$, $N' \cap \pi_H^{-1}(g_1)$ and $N' \cap \pi_H^{-1}(T_H g_1)$ are (Borel) subsets of G of cardinal c, so there exists a bijection R_{g_1} of $N' \cap \pi_H^{-1}(g_1)$ onto $N' \cap \pi_H^{-1}(T_H g_1)$. Define T on G by $Tg = T_1 g$ if $g \in \Omega'$, $Tg = R_{\pi_H g} g$ if $g \in N'$. T is an invertible completion Borel point transformation of G with the required properties.

Note that if X_1 and X_2 are compact metric spaces with X_2 *countably infinite*, the corresponding reformulation of Lemma 7 with G replaced by $X_1 \times X_2$, G/H replaced by X_1, may be *false*. Thus the modification to Polish spaces of Lemma C of [4] is actually false if the Polish spaces are countably infinite. Hence, as remarked in the introduction, the proof in [4], that for a measure μ on the product σ-algebra of an arbitrary product ΠX_α, $\alpha \in A$, of Polish spaces X_α, every measure algebra automorphism is induced by an invertible point transformation, is incomplete unless all the X_α have cardinal c. To overcome this, we reduce the general case to this special case as follows. We may assume that each X_α has at least two points, and that A is uncountable. Now note that we can partition A into disjoint subsets β, such that each β has cardinal $\aleph_0$. If B denotes the set of subsets β of this partition then cardinal (B) = cardinal (A); and if $X^\beta = \Pi X_\alpha$, $\alpha \in \beta$, then X^β is a Polish space of cardinal c (since each X_α has at least two points). Now the spaces ΠX_α, $\alpha \in A$, and ΠX^β, $\beta \in B$, are homeomorphic, the argument in [4] proves the theorem for the latter space and hence for the former.

If T_H in Lemma 7 is actually Borel measurable, can T be chosen Borel measurable, instead of just completion Borel measurable? Clearly T is Borel except on the Borel null set N'; and the fibres $N' \cap \pi_H^{-1}(g_1)$ and $N' \cap \pi_H^{-1}(T_H g_1)$ are, by the last remark before Lemma 6, Borel isomorphic, in either case: H finite or H of cardinal c. But this does not seem to produce a Borel isomorphism T of N' onto itself satisfying $\pi_H \circ T = T_H \circ \pi_H$. If one could produce such an isomorphism, then T could be chosen Borel measurable; this can easily be done in the case of Haar measure. One could then forget the σ-algebras $\tilde{\mathfrak{B}}^0_{G/K}$, and discuss invariance only for Baire measurable point transformations. The transformations in Lemmas 4, 8 and 9 would then be Baire measurable, the proofs of these would be simplified and shortened (in the case of Lemma 9 tremendously so), and one would be able to prove that measure algebra automorphisms were induced by Baire measurable transformations, not just completion Baire measurable ones. As remarked above, this *can* be done for Haar measure.

6. Main lemma and theorems. Our next and most important lemma is the general (nonseparable) analogue of Lemma 7. Note that if $H_1 \in \mathfrak{H}$, $H \in \mathfrak{K}$ then $H < H_1$ implies $H \in \mathfrak{H}$.

Lemma 8. *Let G be a compact group, $\mathfrak{K}$, $\mathfrak{H}$ as before, m a finite Baire measure on G, ϕ an automorphism of the measure algebra E of $(G, \mathfrak{B}^0, m)$. Suppose $K \in \mathfrak{K}$ is invariant under ϕ, so that ϕ induces an automorphism ϕ_K of $E_{G/K}$. Suppose further that ϕ_K is induced by an invertible $\tilde{\mathfrak{B}}^0_{G/K}$ measurable point transformation T_K of G/K. Suppose finally that there exists $H \in \mathfrak{H}$ such that $K \cap H = \{e\}$. Then there exists an invertible $\tilde{\mathfrak{B}}^0$ measurable point transformation T of G which induces ϕ and is such that $\pi_K \circ T = T_K \circ \pi_K$.*

Proof. By Lemma 4(a) there exists $H' \in \mathfrak{H}$ with $H < H'$ invariant under ϕ. Since $H' \subset H$, $H' \cap K = \{e\}$, there is thus no loss of generality in assuming that H itself is invariant under ϕ. Now $HK = KH \in \mathfrak{K}$ and $HK < K$, $HK < H$, so $HK \in \mathfrak{H}$. By Lemma 4, Corollary (b) there exists $H_1 \in \mathfrak{H}$, $HK < H_1 < K$, such that H_1 is invariant under T_K, and so also under ϕ_K and under ϕ. Put $H_2 = H \cap H_1$. Then $H_2 \in \mathfrak{H}$, and H_2 is invariant under ϕ, so ϕ induces an automorphism ϕ_{H_1} of E_{G/H_1} and ϕ_{H_2} of E_{G/H_2}.

We next show that T_K induces an invertible $\tilde{\mathfrak{B}}^0_{G/H_1} = \overline{\mathfrak{B}}_{G/H_1}$ measurable point transformation T_{H_1} of G/H_1. Since H_1 is invariant under T_K, $T_K(\pi^{-1}_{H_1,K}(\tilde{\mathfrak{B}}^0_{G/H_1})) = \pi^{-1}_{H_1,K}(\tilde{\mathfrak{B}}^0_{G/H_1})$, so for each $Y \in \tilde{\mathfrak{B}}^0_{G/H_1} = \overline{\mathfrak{B}}_{G/H_1}$, there exists $Y' \in \overline{\mathfrak{B}}_{G/H_1}$ such that $T_K(\pi^{-1}_{H_1,K}Y) = \pi^{-1}_{H_1,K}(Y')$. Applying this to a separating sequence of generators of $\mathfrak{B}_{G/H_1}$ (which exists because $H_1 \in \mathfrak{H}$, i.e. G/H_1 is compact metrisable), we get, using standard properties of inverse maps, that for each $p \in G/H_1$, there exists $Z \in \overline{\mathfrak{B}}_{G/H_1}$ such that

$$T_K(\pi^{-1}_{H_1,K}(p)) = \pi^{-1}_{H_1,K}(Z),$$

and so

$$\pi^{-1}_{H_1,K}(p) = T_K^{-1}(\pi^{-1}_{H_1,K}(Z)).$$

T_K^{-1} is also an invertible $\tilde{\mathfrak{B}}^0_{G/K}$ measurable point transformation of G/K, and H_1 is also invariant under T_K^{-1}. So if Z contains two distinct points q_1 and q_2, then $T_K^{-1}(\pi^{-1}_{H_1,K}(q_1))$ and $T_K^{-1}(\pi^{-1}_{H_1,K}(q_2))$ are disjoint sets of the form $\pi^{-1}_{H_1,K}(Z_1)$, $\pi^{-1}_{H_1,K}(Z_2)$ both contained in $\pi^{-1}_{H_1,K}(p)$, which is impossible. So Z consists of a single point $p' \in G/H_1$ and $T_K(\pi^{-1}_{H_1,K}(p)) = \pi^{-1}_{H_1,K}(p')$. Putting $p' = T_{H_1}p$ we get a point transformation T_{H_1} of G/H_1, which by an argument similar to that above is easily seen to be a bijection of G/H_1 onto itself, moreover $\pi_{H_1,K} \circ T_K = T_{H_1} \circ \pi_{H_1,K}$, so T_{H_1} and its inverse are clearly $\overline{\mathfrak{B}}_{G/H_1}$ measurable and T_{H_1} induces ϕ_{H_1}, the automorphism of E_{G/H_1} induced by ϕ.

We now see that G/H_2, with the finite Borel measure $\pi_{H_2}m$, H_1/H_2 (a normal subgroup of G/H_2), $G/H_1 = (G/H_2)/(H_1/H_2)$, ϕ_{H_2}, ϕ_{H_1} and T_{H_1} satisfy the conditions of Lemma 7 (since $H_1, H_2 \in \mathfrak{H}$ and $H_1 < H_2$). Hence there exists a completion Borel invertible point transformation T_{H_2} of G/H_2, inducing ϕ_{H_2} and such that

$$\pi_{H_1,H_2} \circ T_{H_2} = T_{H_1} \circ \pi_{H_1,H_2}.$$

We next show that $KH_2 = H_1$. Now $H_2 = H_1 \cap H$, so $H_1 \supset H_2$ and $H_1 \supset K$, so $H_1 \supset KH_2$. Further $H_1 \subset KH$. So if $h_1 \in H_1$, then $h_1 \in KH$ so $h_1 = kh$ with $k \in K$, $h \in H$, so $h = k^{-1}h_1 \in H_1$ (since $K \subset H_1$), so $h \in H_1 \cap H = H_2$. Thus $h_1 = kh \in KH_2$, i.e. $H_1 \subset KH_2$; thus $H_1 = KH_2$.

Now let $\tilde{T}$ be the bijection of $G/H_2 \times G/K$ given by

$$\tilde{T}(p, q) = (T_{H_2}p, T_K q), \qquad p \in G/H_2,\ q \in G/K.$$

Since $K \cap H_2 \subset K \cap H = \{e\}$, by Lemma 2(i), $\theta: g \to (\pi_{H_2}g, \pi_K g)$ is a (homeomorphic) isomorphism of G into $G/H_2 \times G/K$. Since $H_1 = KH_2$, and we have $\pi_{H_1,K} \circ T_K = T_{H_1} \circ \pi_{H_1,K}$, $\pi_{H_1,H_2} \circ T_{H_2} = T_{H_1} \circ \pi_{H_1,H_2}$; it follows that if $g \in G$,

$$\pi_{KH_2,K} \circ T_K \circ \pi_K g = T_{H_1} \circ \pi_{H_1,K} \circ \pi_K g = T_{H_1} \circ \pi_{H_1} g$$

and also

$$\pi_{KH_2,H_2} \circ T_{H_2} \circ \pi_{H_2} g = T_{H_1} \circ T_{H_1,H_2} \circ \pi_{H_2} g = T_{H_1} \circ \pi_{H_1} g$$

and thus

$$\pi_{KH_2,K} \circ T_K \circ \pi_K = \pi_{KH_2,H_2} \circ T_{H_2} \circ \pi_{H_2}.$$

The same relations hold with $T_{H_1}^{-1}$, $T_{H_2}^{-1}$, T_K^{-1} in place of T_{H_1}, T_{H_2} and T_K, so by Lemma 2(iv) $\tilde{T}$ maps $\theta(G)$ onto itself and so $T = \theta^{-1} \circ \tilde{T} \circ \theta$ defines a bijection of G onto itself which clearly satisfies $\pi_K \circ T = T_K \circ \pi_K$ and $\pi_{H_2} \circ T = T_{H_2} \circ \pi_{H_2}$.

It remains to show that T and T^{-1} are $\tilde{\mathfrak{B}}^0$ measurable (i.e. map $\tilde{\mathfrak{B}}^0$ into itself) and that T induces ϕ. Let $\bar{H} \in \mathfrak{H}$, $\bar{H} \supset K$, $\bar{H}$ invariant under T_K. Then $\bar{H}$, $\bar{H} \cap H_2$ are both invariant under ϕ, let $\phi_{\bar{H}}$, $\phi_{\bar{H} \cap H_2}$ denote the induced automorphisms of $E_{G/\bar{H}}$ respectively $E_{G/\bar{H} \cap H_2}$. As in the earlier part of the proof, T_K induces an invertible $\bar{\mathfrak{B}}_{G/\bar{H}}$ measurable point transformation $T_{\bar{H}}$ of $G/\bar{H}$,

which induces $\phi_{\bar{H}}$ and is such that $\pi_{\bar{H},K} \circ T_K = T_{\bar{H}} \circ \pi_{\bar{H},K}$ and so $\pi_{\bar{H}} \circ T = T_{\bar{H}} \circ \pi_{\bar{H}}$. So if $X \in \tilde{\mathfrak{G}}^{\bar{H}}$, i.e. $\pi_{\bar{H}} X \in \mathfrak{B}_{G/\bar{H}}$, then $\pi_{\bar{H}} \circ TX = T_{\bar{H}} \circ \pi_{\bar{H}} X \in \mathfrak{B}_{G/\bar{H}}$, i.e. $TX \in \tilde{\mathfrak{G}}^{\bar{H}}$ and similarly $T^{-1}X \in \tilde{\mathfrak{G}}^{\bar{H}}$. Further since for $X \in \tilde{\mathfrak{G}}^{\bar{H}}$,

$$\widehat{\pi_{\bar{H}} \circ TX} = \widehat{T_{\bar{H}} \circ \pi_{\bar{H}} X} = \phi_{\bar{H}}(\widehat{\pi_{\bar{H}} X}) = \hat{\pi}_{\bar{H}} \circ \phi(\hat{X})$$

(where $\hat{X}$ denotes the equivalence class to which X belongs in the measure algebra concerned, and $\hat{\pi}_{\bar{H}}$ denotes the canonical isomorphism $\mathfrak{E}^{\bar{H}} \to E_{G/\bar{H}}$), it follows that $\widehat{TX} = \phi\hat{X}$ for $X \in \tilde{\mathfrak{G}}^{\bar{H}}$. Similarly if $X \in \tilde{\mathfrak{G}}^{H_2}$ then TX, $T^{-1}X \in \tilde{\mathfrak{G}}^{H_2}$ and $\widehat{TX} = \phi\hat{X}$.

Now the set of $X \in \tilde{\mathfrak{G}}^{\bar{H}\cap H_2}$, such that TX, $T^{-1}X \in \tilde{\mathfrak{G}}^{\bar{H}\cap H_2}$ and $\widehat{TX} = \phi\hat{X}$, forms a σ-algebra which includes $\tilde{\mathfrak{G}}^{\bar{H}}$ and $\tilde{\mathfrak{G}}^{H_2}$ and so includes the σ-algebra generated by them: in particular (Lemma 3) it includes $\mathfrak{G}^{\bar{H}\cap H_2}$. Thus for any $p \in G/\bar{H} \cap H_2$, $\pi^{-1}_{\bar{H}\cap H_2}(p) \in \mathfrak{G}^{\bar{H}\cap H_2}$ and so $T(\pi^{-1}_{\bar{H}\cap H_2}(p))$ and $T^{-1}(\pi^{-1}_{\bar{H}\cap H_2}(p)) \in \tilde{\mathfrak{G}}^{\bar{H}\cap H_2}$. So if $N \in \mathfrak{B}_{G/\bar{H}\cap H_2}$ and $m_{\bar{H}\cap H_2}(N) = 0$, then $T(\pi^{-1}_{\bar{H}\cap H_2}(N))$ is of the form $\pi^{-1}_{\bar{H}\cap H_2}(Y)$, $Y \subset G/\bar{H} \cap H_2$; since there exists $N' \in \mathfrak{B}_{G/\bar{H}\cap H_2}$ with $N \subset N'$ and $m_{\bar{H}\cap H_2}(N') = 0$, and since $\pi^{-1}_{\bar{H}\cap H_2}(Y) \subset T\pi^{-1}_{\bar{H}\cap H_2}(N') \in \tilde{\mathfrak{G}}^{\bar{H}\cap H_2}$, it follows that $Y \in \mathfrak{B}_{G/\bar{H}\cap H_2}$ and $m_{\bar{H}\cap H_2}(Y) = 0$; similarly for T^{-1}. Thus for every $X \in \tilde{\mathfrak{G}}^{\bar{H}\cap H_2}$, TX, $T^{-1}X \in \tilde{\mathfrak{G}}^{\bar{H}\cap H_2}$ and $\widehat{TX} = \phi\hat{X}$.

Now by Lemma 3(iii), since $H_2 \cap K = \{e\}$,

$$\tilde{\mathfrak{B}}^0 = \bigcup\{\tilde{\mathfrak{G}}^{\bar{H}\cap H_2} : \bar{H} \in \mathfrak{H},\ \bar{H} \supset K\};$$

and since by Lemma 4, Corollary (b), $\{\bar{H} : \bar{H} \in \mathfrak{H},\ \bar{H} \supset K,\ \bar{H}$ invariant under $T_K\}$ is cofinal in $\{\bar{H} : \bar{H} \in \mathfrak{H},\ \bar{H} \supset K\}$ and further is σ-directed, we also have

$$\tilde{\mathfrak{B}}^0 = \bigcup\{\tilde{\mathfrak{G}}^{\bar{H}\cap H_2} : \bar{H} \in \mathfrak{H},\ \bar{H} \supset K,\ \bar{H} \text{ invariant under } T_K\}.$$

Hence if $X \in \tilde{\mathfrak{B}}^0$, then TX, $T^{-1}X \in \tilde{\mathfrak{B}}^0$ and $\widehat{TX} = \phi\hat{X}$, which shows that T has all the required properties and completes the proof of the lemma.

Corollary. *Suppose G, m, ϕ, K, ϕ_K, T_K satisfy the hypotheses of the lemma. Suppose $H \in \mathfrak{H}$ is invariant under ϕ and $H \not\supset K$, so that $H \cap K \neq K$; $H \cap K$ is invariant and ϕ induces an automorphism $\phi_{H\cap K}$ of $E_{G/H\cap K}$. Then there exists*

an invertible $\tilde{\mathfrak{B}}^0_{G/H\cap K}$ *measurable point transformation* $T_{H\cap K}$ *of* $G/H \cap K$ *which induces* $\phi_{H\cap K}$ *and is such that* $\pi_{K,H\cap K} \circ T_{H\cap K} = T_K \circ \pi_{K,H\cap K}$.

Proof. Consider the group $G/H \cap K$ with the finite, Baire measure $\pi_{H\cap K}m$ and the automorphism $\phi_{H\cap K}$ of $E_{G/H\cap K}$. The normal subgroup $K/H \cap K$ is invariant under $\phi_{H\cap K}$ and $\phi_{H\cap K}$ induces the automorphism ϕ_K of $E_{G/K}$, since $G/K = (G/H \cap K)/(K/H \cap K)$. Finally $H/H \cap K$ is also a normal subgroup of $G/H \cap K$, $(G/H \cap K)/(H/H \cap K) = G/H$ is compact metrisable and $(K/H \cap K) \cap (H/H \cap K) = \{e_{G/H\cap K}\}$. The corollary now follows by applying the lemma with $G/H \cap K$ in place of G, $\pi_{H\cap K}m$ in place of m, $\phi_{H\cap K}$ in place of ϕ, $K/H \cap K$ in place of K, ϕ_K and T_K unchanged, $H/H \cap K$ in place of H.

Note. Comparing our proofs with those in [3] and [4] for the product space $S = \prod I_\alpha$, $\alpha \in A$, we see that subgroups H in $\mathfrak{R}$ correspond to the subsets B of A. The intersection of two groups in $\mathfrak{R}$ corresponds to the union of the corresponding two subsets of A, the product of two groups in $\mathfrak{R}$ corresponds to the intersection of the corresponding two subsets of A.

We need one last lemma.

Lemma 9. *Let G be a compact group, $\mathfrak{R}$, $\mathfrak{H}$ as before, m a finite Baire measure on G, ϕ an automorphism of the measure algebra E of $(G, \mathfrak{B}^0, m)$. Let K_j, $j \in J$, be a totally ordered set of groups in $\mathfrak{R}$ such that*

(i) *each K_j is invariant under ϕ with ϕ_{K_j} the induced automorphism of E_{G/K_j},*

(ii) *each ϕ_{K_j} is induced by a $\tilde{\mathfrak{B}}^0_{G/K_j}$ measurable invertible point transformation T_{K_j} of G/K_j,*

(iii) *if $j_1, j_2 \in J$ with $K_{j_1} < K_{j_2}$, then $\pi_{K_{j_1},K_{j_2}} \circ T_{K_{j_2}} = T_{K_{j_1}} \circ \pi_{K_{j_1},K_{j_2}}$.*

If $K = \bigcap_{j\in J} K_j$, then K is invariant under ϕ, and if ϕ_K is the induced automorphism of $E_{G/K}$, then there exists a $\tilde{\mathfrak{B}}^0_{G/K}$ measurable invertible point transformation T_K of G/K inducing ϕ_K with $\pi_{K_j,K} \circ T_K = T_{K_j} \circ \pi_{K_j,K}$ for all $j \in J$.

Proof. $\{G/K_j : j \in J\}$ is a projective system. There is a natural continuous homomorphism from G to proj $\lim_J G/K_j$ whose kernel is $K = \bigcap_{j\in J} K_j$, so there is a continuous algebraic isomorphism of the compact group G/K to proj lim G/K_j, which must therefore be a homeomorphism. Thus $G/K =$ proj lim G/K_j. So by [2, Theorem 2.3], $\mathfrak{B}^0_{G/K} = \Sigma(\bigcup_{j\in J} \pi^{-1}_{K_j,K}\, \mathfrak{B}^0_{G/K_j})$, so $\mathfrak{G}^K = \Sigma(\bigcup_{j\in J} \mathfrak{G}^{K_j})$ and $\mathfrak{E}^K = \Sigma(\bigcup_{j\in J} \mathfrak{E}^{K_j})$. Now $\phi(\mathfrak{E}^{K_j}) = \mathfrak{E}^{K_j}$ for all $j \in J$, so $\phi(\mathfrak{E}^K) = \mathfrak{E}^K$, i.e. K is invariant under ϕ; let ϕ_K be the induced automorphism of $E_{G/K}$. Since if $K_{j_1} < K_{j_2}$, $\pi_{K_{j_1},K_{j_2}} \circ T_{K_{j_2}} = T_{K_{j_1}} \circ \pi_{K_{j_1},K_{j_2}}$, for each $g \in G/K$, the set

$\{T_{K_j} \circ \pi_{K_j,K} g : j \in J\}$ forms a consistent family and so corresponds to a point of $\operatorname{proj\,lim}(G/K_j) = G/K$. Call this point $T_K g$. If $g_1, g_2 \in G/K$, $g_1 \neq g_2$, then for at least one $j \in J$, $\pi_{K_j,K} g_1 \neq \pi_{K_j,K} g_2$ and so $T_{K_j} \circ \pi_{K_j,K} g_1 \neq T_{K_j} \circ \pi_{K_j,K} g_2$, i.e. $T_K g_1 \neq T_K g_2$. Thus T_K is injective and since T_K^{-1} is similarly defined by $T_{K_j}^{-1}$ it is bijective G/K to G/K. For all $j \in J$, $T_{K_j} \circ \pi_{K_j,K} = \pi_{K_j,K} \circ T_K$.

It remains to show that T_K, T_K^{-1} are $\tilde{\mathfrak{B}}^0_{G/K}$ measurable and that T_K induces ϕ_K. We first show that if $H \in \mathfrak{K}$ (or $\mathfrak{H}$), $H \supset K$, then $H = \bigcap_{j \in J} HK_j$. Clearly $H \subset \bigcap_{j \in J} HK_j$. Now if $g \in \bigcap_{j \in J} HK_j$, then for all $j \in J$, $g = h_j k_j$ with $h_j \in H$, $k_j \in K_j$. Since $G \times G$ is compact, the net $(\{h_j\}, \{k_j\})_{j \in J}$ has an adherent point (h, k) in $G \times G$, whose image, under the multiplication map $G \times G \to G$, must be an adherent point of $\{h_j k_j\} = \{g\}$. Thus $hk = g$; further h is an adherent point of H, so $h \in H$ since H is closed, similarly $k \in K_j$ for each $j \in J$, and so $k \in \bigcap_{j \in J} K_j = K$. Thus $g \in HK = H$.

We next consider the case when J is countable. Let $H \in \mathfrak{H}$, $H \supset K$, then as we have shown $H = \bigcap_{j \in J} HK_j$. For each $j \in J$, $HK_j \in \mathfrak{H}$ so by Lemma 4, Corollary (b) there exists $H_j' \in \mathfrak{H}$, invariant under T_{K_j} with $HK_j \supset H_j' \supset K_j$, so $H \supset H' = \bigcap_{j \in J} H_j' \supset K$, and since J is countable, $H' \in \mathfrak{H}$. Now, as in the proof of Lemma 8, if $X \in \pi^{-1}_{H_j',K}(\overline{\mathfrak{B}}_{G/H_j'})$, so do $T_K X$ and $T_K^{-1} X$ and $\widehat{T_K X} = \phi_K \hat{X}$ [$\hat{X}$ is again the element of $E_{G/K}$ to which X belongs]. For any finite subset $F \subset J$ let $H_F' = \bigcap_{j \in F} H_j' \in \mathfrak{H}$. Then $\{H_F'\}$ is directed and $\{G/H_F'\}$ is a projective system, whose projective limit is G/H' (the proof is identical with the one given for $\{G/K_j\}$, G/K, at the beginning of the proof of the lemma); so

$$\mathfrak{B}_{G/H'} = \sum \Big(\bigcup_F \pi^{-1}_{H_F',H'}(\mathfrak{B}_{G/H_F'}) \Big) = \sum \Big(\bigcup_{j \in J} \pi^{-1}_{H_j',H'}(\mathfrak{B}_{G/H_j'}) \Big),$$

since by Lemma 3(i) $\mathfrak{B}_{G/H_F'} = \Sigma(\bigcup_{j \in F} \pi^{-1}_{H_j',H_F'}(\mathfrak{B}_{G/H_j'}))$. Now the set of $X \in \pi^{-1}_{H',K}(\overline{\mathfrak{B}}_{G/H'})$ such that $T_K X$, $T_K^{-1} X \in \pi^{-1}_{H',K}(\overline{\mathfrak{B}}_{G/H'})$ and $\widehat{T_K X} = \phi_K \hat{X}$, forms a σ-algebra which includes $\pi^{-1}_{H_j',K}(\overline{\mathfrak{B}}_{G/H_j'})$ for each $j \in J$ and so includes the σ-algebra generated by these, which certainly includes $\pi^{-1}_{H',K}(\mathfrak{B}_{G/H'})$. Again, as in the proof of Lemma 8, it includes null subsets of $\pi^{-1}_{H',K}(\overline{\mathfrak{B}}_{G/H'})$ and so includes the whole of $\pi^{-1}_{H',K}(\overline{\mathfrak{B}}_{G/H'})$. Since $\pi^{-1}_{H,K}(\overline{\mathfrak{B}}_{G/H}) \subset \pi^{-1}_{H',K}(\overline{\mathfrak{B}}_{G/H'})$ we have for $X \in \pi^{-1}_{H,K}(\overline{\mathfrak{B}}_{G/H})$, $T_K X$, $T_K^{-1} X \in \tilde{\mathfrak{B}}^0_{G/K}$ and $\widehat{T_K X} = \phi_K \hat{X}$. H was an arbitrary element of $\mathfrak{H}$, with $H \supset K$, and $\tilde{\mathfrak{B}}^0_{G/K} = \bigcup \{\pi^{-1}_{H,K}(\overline{\mathfrak{B}}_{G/H}) : H \in \mathfrak{H}, H \supset K\}$, so T_K, T_K^{-1} are $\tilde{\mathfrak{B}}^0_{G/K}$ measurable and T_K induces ϕ_K.

We now consider the general case. Since J is totally ordered either it con-

tains a countable cofinal subset J_0, or it is σ-directed. If J contains a countable cofinal subset J_0 then $K = \bigcap_{j \in J_0} K_j$ and the existence of T_K with the required properties follows by the special case. It remains, therefore, only to consider the case when J is σ-directed. Let $H \in \mathfrak{H}$, $H \supset K$. Then $H = \bigcap_{j \in J} HK_j$. Again for each $j \in J$, there exists $H_j' \in \mathfrak{H}$ with $HK_j \supset H_j' \supset K_j$ and H_j' invariant under T_{K_j}. As before if $X \in \pi^{-1}_{H_j',K}(\overline{\mathfrak{B}}_{G/H_j'})$ then so do $T_K X$, $T_K^{-1} X$ and $\widehat{T_K X} = \phi_K \hat{X}$. Thus for $X \in \pi^{-1}_{HK_j,K}(\overline{\mathfrak{B}}_{G/HK_j})$, $T_K X$, $T_K^{-1} X \in \pi^{-1}_{H_j',K}(\overline{\mathfrak{B}}_{G/H_j'}) \subset \tilde{\mathfrak{B}}^0_{G/K}$ and $\widehat{T_K X} = \phi_K \hat{X}$. Now since J is σ-directed

$$\mathfrak{B}_{G/H} = \bigcup_{j \in J} \pi^{-1}_{HK_j,H}(\mathfrak{B}_{G/HK_j}).$$

But $\mathfrak{B}_{G/H}$ is countably generated, so (again since J is σ-directed) there exists $j_0 \in J$ such that $\mathfrak{B}_{G/H} = \pi^{-1}_{HK_{j_0},H}(\mathfrak{B}_{G/HK_{j_0}})$. Thus every $p \in \mathfrak{B}_{G/H}$ is of the form $\pi^{-1}_{HK_{j_0},H}(Z)$ with $Z \in \mathfrak{B}_{G/HK_{j_0}}$, so $\pi_{HK_{j_0},H}$ is a bijection which is continuous on the compact space G/H and so is a homeomorphism (and a group isomorphism) i.e. $G/H = G/HK_{j_0}$ and $\overline{\mathfrak{B}}_{G/H} = \overline{\mathfrak{B}}_{G/HK_{j_0}}$. So

$$\tilde{\mathfrak{B}}^0_{G/K} = \bigcup\{\pi^{-1}_{H,K}(\overline{\mathfrak{B}}_{G/H}) : H \in \mathfrak{H},\ H \supset K\}$$

$$= \bigcup\{\pi^{-1}_{HK_j,K}(\overline{\mathfrak{B}}_{G/HK_j}) : H \in \mathfrak{H},\ H \supset K,\ j \in J\};$$

and so for arbitrary $X \in \tilde{\mathfrak{B}}^0_{G/K}$, $T_K X$, $T_K^{-1} X \in \tilde{\mathfrak{B}}^0_{G/K}$ and $\widehat{T_K X} = \phi_K \hat{X}$. This concludes the proof of the lemma.

[*Note.* The proof of this lemma becomes much simpler if one assumes Baire measurability of the T_{K_j}. The proof of the analogous lemma for products of unit intervals is almost trivial.]

We are now in a position to prove our main result.

Theorem 1. *Let G be a compact group, m a finite Baire measure on G, necessarily the restriction of a Radon measure $\overline{m}$ on G to the Baire sets $\mathfrak{B}^0$. Then every automorphism ϕ of the measure algebra E of $(G, \mathfrak{B}^0, m)$ is induced by an invertible completion Baire point transformation T of G, in fact T and T^{-1} are $\tilde{\mathfrak{B}}^0$ measurable.*

Proof. Consider the family $\mathfrak{F}$ of ordered pairs (K, T_K) where (i) $K \in \mathfrak{K}$ and is invariant under ϕ, (ii) T_K is an invertible $\tilde{\mathfrak{B}}^0_{G/K}$ measurable (and so completion Baire measurable) point transformation of G/K, and (iii) the automorphisms

of $E_{G/K}$ induced by $\phi \mid \mathfrak{E}^K$ and by T_K are the same. We say that $(K, T_K) < (K', T_{K'})$ if $K < K'$, i.e. $K \supset K'$ and $\pi_{K,K'} \circ T_{K'} = T_K \circ \pi_{K,K'}$ on G/K'. This partial ordering is transitive. We next note that every linearly ordered subfamily $\{(K_j, T_{K_j}): j \in J\}$ of $\mathfrak{F}$ has an upper bound in $\mathfrak{F}$. For put $K = \bigcap_{j \in J} K_j$, then by Lemma 9, K is invariant under ϕ and if ϕ_K is the induced automorphism of $E_{G/K}$, there exists an invertible $\tilde{\mathfrak{B}}^0_{G/K}$ point transformation T_K of G/K inducing ϕ_K with $\pi_{K_j,K} \circ T_K = T_{K_j} \circ \pi_{K_j,K}$ for all $j \in J$; i.e. $(K, T_K) \in \mathfrak{F}$ and further $(K_j, T_{K_j}) < (K, T_K)$ for all $j \in J$.

Now by Lemma 4(a), there exist $K \in \mathfrak{H}$ invariant under ϕ, and, since by Lemma 5 these belong to $\mathfrak{F}$, it follows that $\mathfrak{F}$ is not empty. By Zorn's lemma, there exists a maximal member (K, T_K) of $\mathfrak{F}$. To complete the proof of the theorem it is enough to prove that $K = \{e\}$ and so $G/K = G$.

Suppose $K \neq \{e\}$, then by Lemma 1, there exists $H_0 \in \mathfrak{H}$ with $H_0 \not\supset K$. By Lemma 4(a) there exists $H \in \mathfrak{H}$ invarinat under ϕ and such that $H \subset H_0$ and so $H \not\supset K$, i.e. $H \cap K \neq K$. $H \cap K$ is invariant under ϕ, let $\phi_{H \cap K}$ be the automorphism of $E_{G/H \cap K}$ induced by ϕ. By Lemma 8, Corollary, it follows that there exists an invertible $\tilde{\mathfrak{B}}^0_{G/H \cap K}$ measurable point transformation $T_{H \cap K}$ of $G/H \cap K$ which induces $\phi_{H \cap K}$ and satisfies $\pi_{K,H \cap K} \circ T_{H \cap K} = T_K \circ \pi_{K,H \cap K}$. Thus $(H \cap K, T_{H \cap K}) \in \mathfrak{F}$, $K \neq H \cap K$ and $(K, T_K) < (H \cap K, T_{H \cap K})$, contradicting the maximality of (K, T_K). This completes the proof of the theorem.

Corollary. *If $\bar{m}$ is Haar measure on G, then T is also an invertible completion Borel measurable point transformation of G.*

[*Note.* As remarked at the end of §5, for Haar measure, T can be chosen Baire measurable and not just $\tilde{\mathfrak{B}}^0$ measurable.]

Proof. T, T^{-1} are measurable $\bar{\mathfrak{B}}^0_m$, but Haar measure is completion regular [7, p. 287, Theorem H] i.e. $\bar{\mathfrak{B}}^0_m$ coincides with $\bar{\mathfrak{B}}_{\bar{m}}$, the completion of $\mathfrak{B}$ by $\bar{m}$.

Combining Theorem 1 with the theorem of Lamperti [10, Theorem 3.1] we get

Theorem 2. *Let G be a compact group, m a finite Baire measure on G, i.e. the restriction of a Radon measure $\bar{m}$ to the Baire sets $\mathfrak{B}^0$. Let U be an invertible isometry of $L^p(G, \mathfrak{B}^0, m)$, $1 \leq p < \infty$, $p \neq 2$ or a positive invertible isometry of $L^2(G, \mathfrak{B}^0, m)$. [Note that $L^p(G, \mathfrak{B}^0, m)$ coincides with $L^p(G, \mathfrak{B}, \bar{m})$.] Then there exists an invertible completion Baire point transformation T of G such that $(Uf)(g) = f(T^{-1}g)\alpha(g)$ with $|\alpha(g)|^p = \omega_T(g)$ [for $p = 2$, $\alpha(g) = \omega_T^{1/2}(g)$], where $\omega_T(g)$ is defined by*

$$m(T^{-1}X) = \int_X \omega_T(g)\, m(dg)$$

for all $X \in \mathfrak{B}^0$.

Corollary. *If $\bar{m}$ is Haar measure on G, then T is also an invertible completion Borel point transformation of G.*

7. Some remarks on Lusin measurability. As remarked at the end of the introduction, it would be interesting to prove, in the case of Haar measure, that the invertible point transformation T of Theorem 1 could be chosen to be Lusin measurable, instead of just completion Borel measurable (T^{-1} is then automatically also Lusin measurable). In general, invertible Borel point transformations need not be Lusin measurable. Let A be an uncountable set, $S(A) = \Pi I_\alpha$, $\alpha \in A$, each $I_\alpha = [0, 1]$, μ the direct product of Lebesgue measures on I_α. $S(A)$ is compact, μ is a finite completion regular (Baire or Borel) measure on it. Let T be the bijection of $S(A)$ to itself, defined by

$$\begin{aligned} (Tx)_\alpha &= (x)_\alpha && \text{if } (x)_\alpha \neq 0 \text{ or } 1, \\ (Tx)_\alpha &= 1 && \text{if } (x)_\alpha = 0, \\ (Tx)_\alpha &= 0 && \text{if } (x)_\alpha = 1, \end{aligned}$$

where $(x)_\alpha$ is the αth coordinate of $x \in S(A)$. As remarked by Maharam [12], T is an invertible Borel or Baire measurable point transformation which induces the identity automorphism of the measure algebra of μ on $S(A)$, but differs from the identity map of $S(A)$ on a non μ measurable set. (Details of an argument for a similar example may be found in Ionescu Tulcea [8, pp. 162–164, Example 1].) It follows easily that T cannot be μ Lusin measurable, else it would differ from the identity only on a μ-null, and therefore μ measurable, set (see e.g. [8, p. 167, Assertion 3]). It has been pointed out to the author by S. Kakutani and G. E. F. Thomas that explicit examples of maps which are Borel, but not Lusin, measurable are rare in the literature and nonexistent in standard texts. We remark that the transformation mentioned above is one such example and the one referred to in the book of Ionescu Tulcea [8] is another.

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